COHOMOLOGY OF REAL LIE ALGEBRAS

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Abstract. We show how to describe the cohomology of a nilpotent part of some parabolic subalgebra of a semisimple Lie algebra with values in its irreducible representation. The situation in the complex case is well–known, the Kostant's result (see below) gives an explicit description of a representation of a proper reductive subalgebra on the space of the complex cohomology. The aim of this work is to read the structure of the real cohomology from the structure of the complex one. We will use the notation of Dynkin and Satake diagrams for the description of semisimple and parabolic real and complex Lie algebras and their representations.

Keywords: semisimple Lie algebra, Lie algebra cohomologies, parabolic subalgebra, real form, real cohomology.

0. Introduction. The base of the description of the real cohomology is the structure of the complex one. Each standard parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{f}$ of the complex semisimple Lie algebra $\mathfrak{f}$ induces a decomposition $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$. (Each nilpotent part of the algebra $\mathfrak{f}$ is isomorphic to $\mathfrak{q}_+$ for some $\mathfrak{q} \subseteq \mathfrak{f}$.) Given a representation $\pi : \mathfrak{q}_+ \longrightarrow \mathfrak{gl}(V)$, we define the differential d : Hom $(\bigwedge^n \mathfrak{q}_+; V) \longrightarrow$ Hom $(\bigwedge^{n+1} \mathfrak{q}_+; V)$ in the usual way. The corresponding cohomology will be denoted by $H^n(\mathfrak{q}_+, V)$. We will be interested only in cases where π is a restriction of some irreducible representation of $\mathfrak{f}$.

Following Kostant (see [Ko]), we define the appropriate representation $\mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H^n(\mathfrak{q}_+, V))$ of the reductive subalgebra $\mathfrak{f}_0$ on the cohomology space. The main result of [Ko] is the description of highest weights of irreducible components of this representation. The construction of these weights yields the ordering on the cohomology and the corresponding Hasse diagram. The algorithmic description of these weights (which uses the notation of Dynkin diagrams) is well known, cf. [Si], which also includes the description of cohomologies $H^n(\mathfrak{f}_-, V)$ as a dual to $H^n(\mathfrak{q}_+, V^*)$.

Let us consider the real semisimple Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$. The algebra $\mathfrak{g}$ is called a real form of $\mathfrak{f}$. The real forms (up to isomorphisms) are in 1–1 correspondence with involutive automorphisms of $lief$ (up to conjugacy), see e.g. [On, OV]. Such an automorphism induces a symmetry of the Dynkin diagram of $\mathfrak{f}$ which is important for the description of irreducible representations of $\mathfrak{g}$ by the help of their complexification. Therefore we will be interested in involutive automorphisms more then in real forms.

Parabolic subalgebras of $\mathfrak{f}$ can be given by crossing out some vertices of the Dynkin diagram of $\mathfrak{f}$. Similarly, parabolic subalgebras of $\mathfrak{g}$ can be given by crossing

[1]This research has been supported by the grant 'Mathematical structures of Algebra and Geometry', CEZ:J07/98:143100009". The writing of the paper was finished at the University of Auckland with partial support by Marsden Fund and the postgraduate student scholarship of New Zealand Institute of Mathematics & its Applications.

out of some vertices of the Satake diagram of $\mathfrak{g}$. It is described in [Ya] which vertices can be crossed out in this case.

It follows from the structure of the representation on the complex cohomology (see [Ko]) that the cohomology of the complexification is isomorphic to the complexification of the real cohomology. Thus the problem is how to describe the representation on the real cohomology from the known complexification of this representation. The description of real representations by the help of its complexifications is well–known for semisimple Lie algebras. We generalize it to the reductive algebras and, moreover, we show the connection with Satake diagrams. Then we describe the relation between the real and complex cohomologies (we will see that Hasse diagrams of real and complex cohomologies are often the same).

It is useful to compute the results from [Ko] by computers. The web implementation, which computes both real and complex cohomologies, is available on the address `www.math.muni.cz/~silhan/lac`. It is based on the software package *LiE* (see [LCL]) which offers the data structures and corresponding procedures for the computation with semisimple Lie algebras.

Acknowledgments. This paper has been influenced by the lectures by Arkadiy Onishchik on real forms (Masaryk University in Brno, 2001), see [On]. The research has been supported by the grant 'Mathematical structures of Algebra and Geometry', CEZ:J07/98:143100009". The writing of the paper was finished at the University of Auckland with partial support by the Marsden Fund and the postgraduate student scholarship of New Zealand Institute of Mathematics & its Applications. Further I would like to thank to Andreas Čap for comments which simplified some ideas.

1. Known results: complex cohomology and real algebras

1.1. Weyl group and weights. Let us consider a complex semisimple Lie algebra $\mathfrak{f}$ with a Cartan subalgebra $\mathfrak{h}$, sets of simple roots, positive roots and roots $\Pi \subseteq \Delta_+ \subseteq \Delta$ and the Weyl group W. The group W is generated by *simple reflections* i.e. the reflections corresponding to the simple roots. The number of positive roots $\alpha \in \Delta_+$ which are transformed to $w(\alpha) \in \Delta_- = -\Delta$ is called the *length* of w, we write $|w|$. Equivalently (see [FH]), the length of w is the minimal number of simple reflections in any expression for w in terms of simple reflections.

The weights of $\mathfrak{f}$ can be described by labeling the nodes of the Dynkin diagram by the integer coefficients refering to the linear combination of fundamental weights. The weight is dominant for $\mathfrak{f}$ if and only if all the coefficients are non–negative (such Dynkin diagram describes an irreducible representation of $\mathfrak{f}$).

The *affine action* of the Weyl group is defined by

$$w.\Lambda = w(\Lambda + \mathrm{R}) - \mathrm{R}$$

for the weight Λ where $\mathrm{R} = \frac{1}{2}\sum_{\alpha\in\Delta} \alpha$ is the lowest strictly dominant weight of $\mathfrak{f}$. It means (in the terms of the Dynkin diagram) to add one over each node, then act with w and finally subtract one over each node.

1.2. Parabolic subalgebras. The standard parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{f}$ is defined by some set of simple roots $\Sigma \subseteq \Pi$ and is generated by the Cartan subalgebra, root spaces corresponding to the positive roots and root spaces corresponding to the negative roots which can be expressed as a negative sum of roots from $\Pi \setminus \Sigma$. The corresponding Dynkin diagram is obtained from the Dynkin diagram for $\mathfrak{f}$

by crossing out nodes corresponding to the simple roots from Σ. Further we will describe parabolic subalgebras by corresponding diagrams with crosses in both real and complex cases. Each parabolic subalgebra is conjugated to some standard parabolic subalgebra so we are interested only in the standard cases. It induces the decomposition $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$. (The reductive part $\mathfrak{f}_0$ includes the semisimple part of $\mathfrak{q}$ and the rest of the Cartan subalgebra and $\mathfrak{q}_+$ is the remaining nilpotent part of $\mathfrak{q}$.)

Irreducible representations of $\mathfrak{q}$ are irreducible representations of $\mathfrak{f}_0$ with the trivial action of $\mathfrak{q}_+$. Thus, the weights of $\mathfrak{q}$ can be described by the labeled Dynkin diagram, where coefficients over non–crossed nodes are integers. This weight is dominant for $\mathfrak{q}$ if and only if the coefficients over non–crossed nodes are non–negative (such Dynkin diagrams describe irreducible representations of the reductive part $\mathfrak{f}_0$ of $\mathfrak{q}$).

For each set $\Sigma \subseteq \Pi$ and the corresponding parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{f}$ we define $W^{\mathfrak{q}} \subseteq W$ as a subset of all elements, which map the weights dominant for $\mathfrak{f}$ into the weights dominant for $\mathfrak{q}$. Equivalently, $W^{\mathfrak{q}}$ is the set of all elements w for which the set $\Phi_w = w(\Delta_-) \cap \Delta_+$ contains only roots corresponding to $\mathfrak{q}_+$ i.e. the positive roots of $\mathfrak{f}$ which are not roots of the semisimple part of $\mathfrak{f}_0$ (see [Ko]).

1.3**. Cohomology of Lie algebras.** For a representation $\pi : \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{a}$ we define the differential $d : \mathrm{Hom}\,(\bigwedge^n \mathfrak{a}; V) \longrightarrow \mathrm{Hom}\,(\bigwedge^{n+1} \mathfrak{a}; V)$ by the formula

$$(dp)(X_0, \dots, X_n) = \sum_{i<j} (-1)^{i+j} p([X_i, X_j], X_0, \dots \hat{X}_i \dots \hat{X}_j \dots, X_n)$$
$$+ \sum_i (-1)^i \pi(X_i) p(X_0, \dots \hat{X}_i \dots, X_n).$$

The differential d induces the cohomology $H^n(\mathfrak{a}; V)$, called the cohomology of $\mathfrak{a}$ with the coefficients in V because $d^2 = 0$. (We set $\mathrm{Hom}\,(\bigwedge^n \mathfrak{a}; V) = 0$ for $n < 0$ and $n > \dim \mathfrak{a}$.)

On the complex level, we are interested only in the case, where $\mathfrak{a} = \mathfrak{q}_+$ and $\pi = \lambda'|\mathfrak{q}_+$ for some representation $\lambda' : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ on a complex vector space V. We have a natural representation $\beta' : \mathfrak{q} \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+; V))$ on the cohomology (for details see 1.5). This representation is completely reducible and thus we are interested only in the restriction $\beta' : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+; V))$.

1.4**. Theorem.** [Ko] **Kostant's result.** *For a finite dimensional representation $\lambda' : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ with the highest weight Λ and the restriction $\pi = \lambda'|\mathfrak{q}_+$, the irreducible components of β' are in bijective correspondence with the set $W^{\mathfrak{q}}$ and the multiplicity of each component is one. The highest weight of the irreducible component of the representation β' corresponding to $w \in W^{\mathfrak{q}}$ is $w.\Lambda = w(\Lambda + \mathrm{R}) - \mathrm{R}$ and it occurs in degree $|w|$.*

1.5**. Complexification of the real cohomology.** Now we will consider a real semisimple algebra $\mathfrak{g}$ with the complexification $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ with a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$. It means that the complexification $\mathfrak{q} = \mathfrak{p}(\mathbb{C})$ is a parabolic subalgebra of $\mathfrak{f}$. We have the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$ such that the complexification respects this decomposition. Let us consider the irreducible representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a real vector space

V with the complexification $\lambda(\mathbb{C}) : \mathfrak{f} \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$. Our aim is to describe the cohomology $H(\mathfrak{p}_+; V)$ with respect to the representation $\lambda|\mathfrak{p}_+$.

It follows from the structure of parabolic subalgebras that we have the natural action of the elements from $\mathfrak{q}$ on $\mathfrak{q}_+^*$ (the dual of the adjoint action) and on $V(\mathbb{C})$ (the restriction of $\lambda(\mathbb{C})$). This induces the representation of $\mathfrak{q}$ on Hom $(\bigwedge^n \mathfrak{q}_+; V(\mathbb{C}))$, $n \in \mathbb{N}$ which factorizes (see [Ko]) to the representation $\beta' : \mathfrak{q} \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+, V(\mathbb{C})))$ on the cohomology. It is completely reducible a thus we can consider $\beta' : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{q}_+, V(\mathbb{C})))$. Moreover, there exists (see [Ko]) a subspace $H' \subseteq \bigwedge \mathfrak{q}_+^* \otimes V(\mathbb{C})$ isomorphic to $H(\mathfrak{q}_+, V(\mathbb{C}))$ as an $\mathfrak{f}_0$–module (we consider a tensor product of the dual of the adjoint action and $\lambda(\mathbb{C})|\mathfrak{f}_0$ on H'). This isomorphism $i : H' \longrightarrow H(\mathfrak{q}_+, V(\mathbb{C}))$ is given by the projection to the factor space (i.e. the cohomology space).

Similarly, we can define the completely reducible representation of $\mathfrak{p}$ on $H(\mathfrak{p}_+; V)$ i.e. the representation $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+, V))$. Due to the isomorphism i, we can consider $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H)$ where $H = H' \cap \bigwedge \mathfrak{p}_+^* \otimes V$. Since β is the restriction of β' and $H' = H(\mathbb{C})$, we get $\beta' = \beta(\mathbb{C})$. Therefore, we have shown that the complexification of the real cohomology is the cohomology of the complexified algebra and its representation.

1.6**. Satake diagrams and parabolic subalgebras.** We describe the real form $\mathfrak{g}$ of the complex semisimple Lie algebra $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ in a more detailed way. Let us note that the classification of real simple algebras consists of real forms of complex simple algebras and from complex simple algebras understood as the real ones.

There is a 1–1 correspondence between the classes of real forms of a complex semisimple algebra $\mathfrak{f}$ (up to isomorphism), the classes of involutive antiautomorphisms of $\mathfrak{f}$ (up to conjugacy) and the classes of involutive automorphisms of $\mathfrak{f}$ (up to conjugacy). Involutive antiautomorphisms are called the *real structures* and they are just the complex conjugations given by real forms. Let us denote the involutive antiautomorphism (automorphism) of $\mathfrak{f}$ corresponding to $\mathfrak{g}$ by σ (θ). They can be chosen in such a way that there exists a *compact structure* (i.e. a real structure corresponding to a compact real form) τ such that $\theta = \sigma\tau$ and θ, σ, τ commute. See [On] for details. The involutive automorphisms θ is then called the *Cartan involution*. Clearly $\theta(\mathfrak{g}) = \mathfrak{g}$ and it induces the *Cartan decomposition* $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$ where $\mathfrak{l}$ is the $+1$–eigenspace and $\mathfrak{r}$ the -1–eigenspace of $\theta|\mathfrak{g}$. The Killing form of $\mathfrak{f}$ is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{r}$. It implies that $\mathfrak{l}$ is a compact Lie algebra.

One way of a diagram–like description of real semisimple Lie algebras are so called *Satake diagrams*, see [Sat, OV]. There exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{h}_\mathfrak{r} = \mathfrak{h} \cap \mathfrak{r}$ is a maximal abelian subspace of $\mathfrak{r}$. It yields the decomposition $\mathfrak{h} = \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{h}_\mathfrak{r}$ to the compact part $\mathfrak{h}_\mathfrak{l}$ and the real part $\mathfrak{h}_\mathfrak{r}$. The Cartan subalgebra $\mathfrak{h}(\mathbb{C}) \subseteq \mathfrak{f}$ yields the system of roots Δ and one can show that the mapping $\sigma^* : \Delta \longrightarrow \Delta$ given by the formula $\sigma^*\alpha(H) = \overline{\alpha(\sigma H)}$ for $\alpha \in \Delta$ and $H \in \mathfrak{h}(\mathbb{C})$ is an involutive automorphism of Δ. Roots satisfying $\sigma^*\alpha = -\alpha$ are called *compact* and remaining *non–compact* roots. Let us denote the set of compact roots by Δ_c; clearly $\Delta_c = \{\alpha \in \Delta \mid \alpha|\mathfrak{h}_\mathfrak{r} = 0\}$. The system of positive roots Δ_+ can be found in such a way that $\sigma^*(\alpha) \in \Delta_+$ for each non–compact root $\alpha \in \Delta_+$. To obtain such a system Δ_+, we consider the lexicographical ordering with respect to the base $H_1, \ldots, H_n$ of $\mathfrak{h}$ such that the first elements $H_1, \ldots, H_p$ constitute the base of $\mathfrak{h}_\mathfrak{r}$. The set of simple roots Π then has the following property: if $\alpha \in \Pi$ is a non–compact root then there exists a unique non–compact root $\alpha' \in \Pi$ such that $\sigma^*\alpha - \alpha'|\mathfrak{h}_\mathfrak{r} = 0$. (It is equivalent to the property $\sigma^*\alpha' - \alpha|\mathfrak{h}_\mathfrak{r} = 0$). The Satake diagram of $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{f}$ given by Π where the compact roots are denoted by a black dot $\bullet$, non–compact roots by a white dot $\circ$ and there is an arrow between α and a unique $\alpha' \in \Pi$ such that $\sigma^*\alpha - \alpha'|\mathfrak{h}_\mathfrak{r} = 0$ for each non–compact root $\alpha \in \Pi$ satisfying $\alpha \neq \alpha'$.

Parabolic subalgebras of $\mathfrak{g}$ can be again described by crossing out some vertices of the Satake diagram but it is not arbitrary, see [Ya]: we cannot cross out the compact roots and if we cross out some non–compact root α, we must cross out the non–compact root α' connected with α by an arrow (if there is such an arrow).

1.7. **Representations of real semisimple Lie algebras.** Facts about representations of real (semisimple) Lie algebras can be found in [On, OV], we will use the notation from [On]. First we consider an arbitrary real algebra $\mathfrak{g}$. The *complex structure* on the real vector space V is an automorphism $J : V \longrightarrow V$ such that $J^2 = -\mathrm{id}$. The *real (quaternionic) structure* on the complex vector space V is an antiautomorphism $J : V \longrightarrow V$ such that $J^2 = \mathrm{id}$ ($J^2 = -\mathrm{id}$). Having a complex vector space V, we will denote the set V understood as a real vector space by $V_\mathbb{R}$ (the underlying real vector space of V). Let us consider a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on the complex vector space V understood as a space $V_\mathbb{R}$ with the complex structure J. We define a complex space $\bar{V}$ as the space $V_\mathbb{R}$ with the complex structure $-J$ and the *complex–conjugated* representation $\bar{\rho} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\bar{V})$ on the complex space $\bar{V}$ such that $\rho = \bar{\rho}$ on the space $V_\mathbb{R} = \bar{V}_\mathbb{R}$. Let us fix a base on V and denote $x \mapsto C(x)$, $x \in \mathfrak{g}$ the matrix form of ρ. Then the same base can be regarded as a base of $\bar{V}$ and the corresponding matrix form of $\bar{\rho}$ is given by the complex conjugated matrix $x \mapsto \overline{C(x)}$, $x \in \mathfrak{g}$. For a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a real vector space V, we will denote its extension to the (complex) space $V(\mathbb{C})$ by $\rho^\mathbb{C} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$ and its complexification by $\rho(\mathbb{C}) : \mathfrak{g}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$. For a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a complex vector space V, we will denote its extension to the (complex) algebra $\mathfrak{g}(\mathbb{C})$ by $\rho_\mathbb{C} : \mathfrak{g}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$. The *complex–conjugated* representation of $\rho : \mathfrak{f} = \mathfrak{g}(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ on a complex space V with respect to the real form $\mathfrak{g}$ is the representation $\bar{\rho} = (\overline{\rho|\mathfrak{g}})_\mathbb{C}$ on a complex space $\bar{V}$.

Now we classify representations of $\mathfrak{g}$ according to the possible scalars. Let us consider a representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a real vector space V. The representation λ is

- *quaternionic* (or of the *quaternionic type*) if there exists a complex structure on V and a quaternionic structure on V (understood as a complex space), both commuting with the action of $\mathfrak{g}$
- *complex* (or of the *complex type*) if there exists a complex structure on V commuting with the action of $\mathfrak{g}$ and λ is not quaternionic
- *real* (or of the *real type*) if there is no complex structure on V commuting with the action of $\mathfrak{g}$.

Let us suppose that λ is irreducible. The complexification depends on the type in the following way. If λ is real then $\lambda(\mathbb{C})$ is irreducible too and $\lambda(\mathbb{C}) \sim \overline{\lambda(\mathbb{C})}$. If λ is complex (quaternionic) then the space V can be understood as a complex vector space and $\lambda(\mathbb{C}) \sim \lambda_\mathbb{C} \oplus \bar{\lambda}_\mathbb{C}$ and $\lambda_\mathbb{C} \not\sim \bar{\lambda}_\mathbb{C}$ ($\lambda_\mathbb{C} \sim \bar{\lambda}_\mathbb{C}$).

The self–conjugacy condition appears in both real and quaternionic representationand so we need some other tool to distinguish these two types. The irreducible self–conjugated representation $\gamma : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ on the complex vector space V with the highest weight $\Gamma = \bar{\Gamma}$ admits an antiautomorphism $J : V \longrightarrow V$ commuting with $\gamma|\mathfrak{g}$ such that $J^2 \in \{+\mathrm{id}, -\mathrm{id}\}$. We define the *index* $\varepsilon(\mathfrak{g}, \gamma)$ as the sign. This definition is correct, see [On]. (The index $+1$ says that γ can be obtained as the complexification of some real irreducible representation of $\mathfrak{g}$ and -1 says that γ is a part of the complexification of some quaternionic irreducible representation of $\mathfrak{g}$.) In the case of semisimple (reductive) algebras, we will sometimes write $\varepsilon(\mathfrak{g}; \Gamma)$ where Γ is the highest weight of γ.

Further we will suppose that $\mathfrak{g}$ is a real form of a complex semisimple algebra $\mathfrak{f}$. Each involutive automorphism of $\mathfrak{f}$ induces a symmetry of the Dynkin diagram of $\mathfrak{f}$ and thus we can consider the symmetry s induced by the Cartan involution θ corresponding to $\mathfrak{g}$. We will describe irreducible representations of $\mathfrak{f}$ by the help of their highest weights given by the vector of coefficients over the Dynkin diagram. Furthermore, we will denote the symmetry of the Dynkin diagram which realizes dual weights by ν (we will consider symmetries of diagrams on highest weights too). Section 2 shows how to see s and ν on the Satake diagram. These two symmetries commute and allow us to describe the relation between the irreducible representations λ and $\bar{\lambda}$ of $\mathfrak{g}$ on complex spaces V and $\bar{V}$. Denoting the highest weights of $\lambda_{\mathbb{C}}$ ($\bar{\lambda}_{\mathbb{C}}$) by Λ ($\bar{\Lambda}$), it holds $\bar{\Lambda} = s\nu(\Lambda)$. These facts and formulas for indices of semisimple Lie algebras are can be found in [On, OV], for details see Section 5.

1.8**. Hasse graph on the cohomology.** In the case of the complex cohomology given by a parabolic subalgebra $\mathfrak{q}$ of the semisimple algebra $\mathfrak{f}$, the set of vertices is $W^{\mathfrak{q}}$ (or equivalently, the set of irreducible components in the cohomology, see 1.4). Further, there is an arrow $w_1 \longrightarrow w_2$, $w_1, w_2 \in W^{\mathfrak{q}}$ if and only if $w_2 = s_\alpha w_1$ where s_α is the reflection corresponding to the root $\alpha \in \Delta$ and $|w_2| = |w_1| + 1$.

In the case of the real cohomology given by a parabolic subalgebra $\mathfrak{p}$ of a semisimple algebra $\mathfrak{g}$, the Hasse graph depends furthermore on the representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, precisely speaking on its type (see Section 6 for details). We define the set of vertices as the set of irreducible components in the cohomology. Futhermore, there is an arrow $\beta_1 \longrightarrow \beta_2$ where β_1, β_2 are irreducible components of the representation $\beta : \mathfrak{g} \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V))$ if and only if
(a) λ is complex or quaternionic and there is an arrow between $(\beta_1)_{\mathbb{C}}$ and $(\beta_2)_{\mathbb{C}}$ in the complex cohomology $H(\mathfrak{p}_+(\mathbb{C}); V)$ in the given direction or
(b) λ is real and there is some arrow between the component(s) $\beta_1(\mathbb{C})$ and $\beta_2(\mathbb{C})$ in the complex cohomology $H(\mathfrak{p}_+(\mathbb{C}); V(\mathbb{C}))$ in the given direction.
(The correctness of this definition follows from 1.5 for (b) and from 6.2 for (a).)

2. Symmetries of diagrams

Now we show how to see the needful symmetries on the Satake diagrams. Let us consider a real form $\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{f}$ and the corresponding Cartan involution θ which induces the symmetry s of the Dynkin diagram. Further, let us denote the symmetry which realizes the dual weights of $\mathfrak{f}$ by ν. We can consider the system of simple roots Π in such a way that Π induces both the Satake diagram of $\mathfrak{g}$ and the Dynkin diagram of $\mathfrak{f}$ (i.e. Π satisfies the properties from 1.6). Thus we can see all these symmetries on both diagrams. Moreover, let us denote the symmetry induced by arrows of the Satake diagram by a. We will show that very often $a = s\nu$. Let us note that each symmetry of the Dynkin diagram induces an involution of $\mathfrak{f}$, see [On].

2.1. We will use the notation from 1.6. Further we denote the set of non–compact roots $\Delta \setminus \Delta_c$ by Δ_{nc} and the corresponding sets of positive roots by Δ^+_{nc} and Δ^+_c. It gives the decomposition on the set of simple roots to $\Pi_c = \Pi \cap \Delta_c$ and $\Pi_{nc} = \Pi \cap \Delta_{nc}$. Let us consider the compact subalgebra $\mathfrak{l}_c \subseteq \mathfrak{l}$ corresponding to the root system Δ_c. The Weyl group W_c of $\mathfrak{l}_c(\mathbb{C})$ can be understood as a subgroup of the Weyl group W of $\mathfrak{f}$. There is a unique element $w_0 \in W$ ($w_0^c \in W_c$) such that $w_0(\Pi) = -\Pi$ ($w_0^c(\Pi_c) = -\Pi_c$). The dual of the involution $\hat{\nu}$ induced by ν satisfies $\hat{\nu}^* = -w_0$, see [On]. Similarly, $\hat{\nu}_c^* = -w_0^c$ where ν_c is the corresponding symmetry of Π_c.

Now we "improve" the involution θ to see the induced symmetry s on the system of simple roots Π. The automorphism θ induces a dual mapping on Δ defined as $(\theta^*\alpha)(H) = \alpha(\theta H)$ for $\alpha \in \Delta$ and $H \in \mathfrak{h}(\mathbb{C})$. Using the definition of Δ_+ by the help of a proper base of $\mathfrak{h}$, it is easy to see that $\theta^*(\Delta^+_{nc}) \subseteq \Delta^-_{nc}$ and $\theta^*|\Delta_c =$ id. The symmetry induced on Π by inner automorphisms of $\mathfrak{f}$ is the identity. Since the elements of the Weyl group are induced by inner automorphism of $\mathfrak{f}$, the composition $w_0^c\theta^*$ induces the same symmetry as θ^*. The form of the Weyl reflections corresponding to the root α is $S_\alpha(\beta) = \beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ for $\beta \in \Delta$ and thus $S_\alpha(\beta) \in \Delta^+_{nc}$ for $\alpha \in \Delta_c$ and $\beta \in \Delta^+_{nc}$. It implies that $w_0^c(\Delta^+_{nc}) = \Delta^+_{nc}$. Since $w_0^c(\Delta^+_c) = \Delta^-_c$, we have shown that $w_0^c\theta^*(\Delta_+) = \theta^* w_0^c(\Delta_+) = \Delta_-$. It follows that $w_0 w_0^c \theta^*$ fixes the set Π and induces the symmetry $w_0 w_0^c \theta^* | \Pi = s$.

Now we describe the relation between $w_0 w_0^c \theta^*$ and σ^*. It is easy to see that $\sigma^* = -\theta^*$. We have $w_0^c\sigma^* = -w_0^c\theta^* = (\hat{\nu}^* w_0) w_0^c \theta^*$ and thus $w_0^c\sigma^*|\Pi = \nu s$. Considering a simple root $\alpha \in \Pi_{nc}$, it is easy to see that $w_0^c(\alpha) = \alpha + \sum_{\beta\in\Pi_c} c_\beta \beta$ where $c_\beta \geq 0$ for $\beta \in \Pi_c$. Further, it follows from the construction of Satake diagrams that $\sigma^*(\alpha) = \alpha' + \sum_{\beta\in\Pi_c} d_\beta\beta$, where $\alpha' \in \Pi_{nc}$ and $d_\beta \geq 0$ for $\beta \in \Pi_c$; α and α' are connected by an arrow in the Satake diagram if $\alpha \neq \alpha'$. Since $w_0^c\sigma^*(\alpha) = \nu s(\alpha)$ is a simple root, we have $w_0^c\sigma^*(\alpha) = \alpha' = \nu s(\alpha)$. It shows that the symmetry νs coincides with the arrows of the Satake diagram on non–compact simple roots. Further we consider a simple root $\alpha \in \Pi_c$. Since $\sigma^*(\alpha) = -\alpha$, we have $w_0^c\sigma^*(\alpha) = -w_0^c(\alpha) = \nu_c(\alpha) = \nu s(\alpha)$.

2.2. **Theorem.** *Let us have a real semisimple Lie algebra $\mathfrak{g}_0$ and the corresponding symmetries s, ν, a on its Satake diagram.*
(i) It holds $s\nu = a$ on the non-compact roots and $s\nu = \nu_c$ on the compact roots.
(ii) If $\mathfrak{g}_0$ is simple, then $a = s\nu$ in all cases with except of $\mathfrak{su}_n$, $\mathfrak{so}_{k,2n-k}$ where k, n have the different parity and the compact form of $E6$. (Let us note that the Satake diagrams of $\mathfrak{su}_n$ and the compact form of $E6$ have only compact roots and in the case of $\mathfrak{so}_{n,2n-k}$ where n, k have the different parity, the number of compact roots is even.)

Proof. It remains to prove (ii). Let us suppose that $\mathfrak{g}$ is a real simple Lie algebra. If $s\nu$ is non–trivial on non–compact roots then $a = s\nu$ because there is the only possibility how to extend this symmetry from non–trivial roots to the whole Satake diagram. (If $\mathfrak{f}$ is simple too then it is easy to check it case by case. If $\mathfrak{f}$ is not simple then it follows from the definition of the Satake diagrams that $\Pi_{nc} = \Pi$.) If $s\nu$ is trivial on non–compact roots (i.e. $a =$ id), there are some exceptions satisfying $a \neq s\nu$: $\mathfrak{su}_n$, $\mathfrak{so}_{k,2n-k}$ where k, n have the different parity and the compact form of $E6$. ☐

3. Formulation of the problem

3.1. **Notation.** Further we will use the following notation. We will consider sets of roots $\Pi \subseteq \Delta_+ \subseteq \Delta$ constructed in 1.6 and a parabolic subalgebra given by the set $\Sigma \subseteq \Pi$ (see 1.2 and 1.6). As in 1.5, we will consider a real semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ with the complexification $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{q}_+$ which respect the decomposition where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{q} = \mathfrak{f}_0 \oplus \mathfrak{q}_+$ are parabolic subalgebras. The decompositions $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}$ and $\mathfrak{f}_0 = \mathfrak{f}_0^{ss} \oplus \mathfrak{z}(\mathbb{C})$ give the semisimple part and the center of $\mathfrak{g}_0$ and $\mathfrak{f}_0$. The construction of Satake diagrams from 1.6 gives the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the complexification $\mathfrak{h}(\mathbb{C}) \subseteq \mathfrak{f}$. The Weyl group of f will be denoted by W.

Let us denote σ (θ) the real structure (Cartan involution) corresponding to the real form $\mathfrak{g}$ of $\mathfrak{f}$ and s and ν symmetries of a diagram where s is the symmetry induced by θ and ν is the symmetry which realizes dual weights of $\mathfrak{f}$, see Section 2. Let us denote s', ν' the corresponding symmetries of diagrams of $\mathfrak{g}_0^{ss}$ and $\mathfrak{f}_0^{ss}$ corresponding to the real form $\mathfrak{g}_0^{ss}$ of $\mathfrak{f}_0^{ss}$. It is easy to see that $s'\nu'$ is the restriction of $s\nu$. (We have $s\nu = a$ in many cases and the symmetry a satisfies this condition. The exceptions must be discussed case by case.) Further we will not distinguish between $s\nu$ and $s'\nu'$ and both these symmetries will be denoted by $s\nu$. Let us note that the symmetries s, s' and ν, ν' do not satisfy this condition separately.

3.2. **Complexification and realification of the cohomology for complex and quaternionic $\mathfrak{g}$–representations.** Let us start with an arbitrary real Lie algebra $\mathfrak{g}$ with a representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a complex vector space V and consider the representations $\lambda^{\mathbb{R}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_{\mathbb{R}})$ and $\lambda_{\mathbb{C}} : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$. Our aim is to compare all three induced cohomologies. The mapping $\_{}_{\mathbb{C}} :$ Hom $(\bigwedge \mathfrak{a}; V) \longrightarrow$ Hom $(\bigwedge \mathfrak{a}(\mathbb{C}); V)$ which extends a given multiplinear mapping from $\mathfrak{a}$ to $\mathfrak{a}(\mathbb{C})$, is an isomorphism of complex vector spaces. Similarly, the induced mapping $\hat{}^{\mathbb{R}}$: Hom $(\bigwedge \mathfrak{a}; V) \longrightarrow$ Hom $(\bigwedge \mathfrak{a}; V_{\mathbb{R}})$ is an isomorphism of real vector spaces. Moreover, both these mappings commute with the differentil d, i.e. $d(p_{\mathbb{C}}) = (dp)_{\mathbb{C}}$ and $d(p^{\mathbb{R}}) = (dp)^{\mathbb{R}}$ for $p \in$ Hom $(\bigwedge \mathfrak{a}; V)$, see 1.3. It shows the isomorphism of cohomologies $H(\mathfrak{a}; V)$ and $H(\mathfrak{a}(\mathbb{C}); V)$ as complex spaces and cohomologies $H(\mathfrak{a}; V)$ and $H(\mathfrak{a}; V_{\mathbb{R}})$ as real spaces.

Now we consider the case $\mathfrak{a} = \mathfrak{p}_+$ with the representations $\pi = \lambda|\mathfrak{p}_+$, $\pi_{\mathbb{C}} = \lambda_{\mathbb{C}}|\mathfrak{p}_+$ and $\pi^{\mathbb{R}} = \lambda^{\mathbb{R}}|\mathfrak{p}_+$. Since the actions of $X \in \mathfrak{g}_0$ on spaces Hom $(\bigwedge \mathfrak{p}_+; V_{\mathbb{R}})$, Hom $(\bigwedge \mathfrak{p}_+; V)$ and Hom $(\bigwedge \mathfrak{p}_+(\mathbb{C})+(\mathbb{C}); V)$ commute with isomorphisms $\_{}_{\mathbb{C}}$ and $\hat{}^{\mathbb{R}}$ (see 1.5), the induced representations of $\mathfrak{g}_0$ on $H(\mathfrak{p}_+; V_{\mathbb{R}})$, $H(\mathfrak{p}_+; V)$ and $H(\mathfrak{p}_+; V_{\mathbb{C}})$ are equivalent. Thus, denoting $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V))$ the representations on the cohomology corresponding to π and understanding $H(\mathfrak{p}_+; V)$ as the complex space, the representations on the remaining cohomologies are of the form $\beta^{\mathbb{R}} : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V_{\mathbb{R}}))$ and $\beta_{\mathbb{C}} : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+(\mathbb{C}); V))$.

The Kostant's result 1.4 gives the explicit description of $\beta_{\mathbb{C}}$ and we need to describe $\beta^{\mathbb{R}} = (\beta_{\mathbb{C}}|\mathfrak{g}_0)^{\mathbb{R}}$. The following simple lemma says when the restriction $(\_|\mathfrak{g})^{\mathbb{R}}$ preserves the irreducibility. (It is the opposite result to the complexification of an irreducible $\mathfrak{g}$-representation in 1.7.)

3.3. **Lemma.** *Let us consider an arbitrary real Lie algebra $\mathfrak{g}$ with the complexification $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ and an irreducible representation $\gamma : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$. Then the following hold:*

(i) $(\gamma|\mathfrak{g})^{\mathbb{R}}$ *is irreducible if and only if* $\gamma \not\sim \bar{\gamma}$ *or* $\gamma \sim \bar{\gamma}$ *and* $\varepsilon(\mathfrak{g},\gamma) = -1$
(ii) $(\gamma|\mathfrak{g})^{\mathbb{R}}$ *is reducible if and only if* $\gamma \sim \bar{\gamma}$ *and* $\varepsilon(\mathfrak{g},\gamma) = 1$.

Proof. It is sufficient to prove only the assertion (ii). The implication $\Longleftarrow$ is clear. If $(\gamma|\mathfrak{g})^{\mathbb{R}}$ is reducible then exists the $\gamma|\mathfrak{g}$-invariant subspace $W \subsetneq V_{\mathbb{R}}$. The subspace iW is invariant too and so $W \cap iW$ and $W + iW$ are γ-invariant subspaces. It shows that $W \cap iW = 0$ and $V = W(\mathbb{C})$. □

3.4. **Summarization.** The previous observations give the way how to describe the real cohomology. In the case of complex and quaternionic $\mathfrak{g}$–representation we use the last but one paragraph and the previuos lemma. Therefore we need to know complex–conjugated representations and indices for reductive algebras $\mathfrak{g}_0$. Moreover, we show the relation between $\varepsilon(\mathfrak{g};\Lambda)$ and $\varepsilon(\mathfrak{g}_0; w.\Lambda)$ for a self–conjugated $\mathfrak{f}$–dominant weight Λ and a self–conjugated $\mathfrak{f}_0$–dominant weight $w.\Lambda$ where w is an element of the Weyl group of $\mathfrak{f}$. In the case of a real $\mathfrak{g}$–representation, we describe the real cohomology by the help of its complexification, see 1.5. It means to show which irreducible components (couples of irreducible components) in the complex cohomology correspond to an irreducible component in the real cohomology. We will again need to know the complex–conjugation for algebras $\mathfrak{g}_0$.

4. Conjugated representations of reductive algebras

Considering an irreducible representation $\gamma : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(V)$ on a complex vector space V, our aim is to describe the complex–conjugated representation $\bar{\gamma} : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(\bar{V})$ with respect to the real form $\mathfrak{g}_0$. We denote the highest weight of γ and $\bar{\gamma}$ by Γ and $\bar{\Gamma}$ and we will understand them as vectors of coefficients over diagrams. (Let us remind that for $\mathfrak{f}_0$–dominant weights, the coefficients over non–crossed nodes are non–negative integers and the coefficients over crosses are arbitrary real numbers.) Futher we will work with fundamental representations (weights) of the algebra $\mathfrak{f}_0$. It denotes the representation of $\mathfrak{f}_0$ generated by the vector of the highest weight in the corresponding represenation of $\mathfrak{f}$. In particular, the fundamental representation of $\mathfrak{f}_0$ corresponding to crosses are one dimensional. Let us remind that in irreducible representations, the center of $\mathfrak{g}_0$ acts by some (possibly complex) scalar.

4.1. First we claim that we can restrict to the fundamental representations of $\mathfrak{f}_0$. It follows from the relation $\overline{\Gamma_1 + \Gamma_2} = \bar{\Gamma}_1 + \bar{\Gamma}_2$ for $\mathfrak{f}_0$–dominant weights Γ_1 and Γ_2. Coefficients over non-crossed nodes satisfy this relation because the complex conjugation is given be the symmetry $s\nu$ for them. Coefficients over crosses are given by the action of the center. This action on the vector of the highest weight is given directly by the highest weigt understood as a form on the Cartan subalgebra of $\mathfrak{f}$ (which involves the center). The relation above now follows from the fact that the scalar action of the center $\mathfrak{g}_0$ in a complex–conjugated representation is given by the complex–conjugation in $\mathbb{C}$ (see 1.7). The same argument implies that if ρ_i is a fundamental representation corresponding to a cross then $\overline{r\rho_i} = r\bar{\rho}_i$, $r \in \mathbb{R}$. Summarizing, we have shown that for the description of $\bar{\gamma}$, it is sufficient to know the complex–conjugated representations for all fundamental represenations of $\mathfrak{f}_0$.

The description of the complex–conjugation for the fundamental weights corresponding to the non–crossed vertices is given by the symmetry $s\nu$ so we need to consider only croses. In this case, we use the structure of the zero cohomology. Let us consider a fundamental representation $\rho_i : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(V_i)$ with a highest weight Γ_i corresponding to the i-th cross. Understanding Γ_i as an $\mathfrak{f}$–dominant weight, we get corresponding fundamental representation $\rho' : \mathfrak{f} \longrightarrow \mathfrak{gl}(V_i')$ with this highest weight. The i-th vertex in the Satake diagram of $\mathfrak{f}$ is non–compact (because it was crossed out) and we have two possibilities according to the arrows.

I. First let us suppose that the i-th vertex has not an arrow. Then the representation ρ_i' is self–conjugated and has the index $+1$ so it is real. (All "quaternionic" vertices are compact and these have the coefficient 0. Now it follows from formulas in 5.1 that the index is $+1$). It implies that ρ_i' is the complexification of some representation of $\mathfrak{g}$ and according to 1.5, the same holds for the representation of $\mathfrak{f}_0$ on the zero cohomology $H^0(\mathfrak{q}_+; V)$. But this is just the representation ρ_i, see 1.4. It shows that ρ_i is self–conjugated.

II. Now let us suppose that the i-th vertex has an arrow. Then the representation ρ_i' is not self–conjugated because in this case, the symmetry $s\nu$ (for $\mathfrak{f}$) coincides with arrows, see Theorem 2.2. We use the (reducible) representation $\rho_i' \oplus \overline{\rho_i'}$ with the highest weights Γ_i and $\bar{\Gamma}_i = s\nu(\Gamma_i) \neq \Gamma_i$ which is the complexification of an irreducible representation of $\mathfrak{g}$, see 1.7. In the zero cohomology $H^0(\mathfrak{q}_+; V \oplus \bar{V})$, which is again the complexification of some representation of $\mathfrak{g}_0$, we have the components with the highest weights Γ_i and $s\nu(\Gamma_i)$ understood as weights of $\mathfrak{f}_0$. This implies that the representation ρ_i is either self–conjugated or $\bar{\rho}_i$ has the highest weight $s\nu(\Gamma_i)$. But we will show below that there is an element in $\mathfrak{g}_0$ which acts by some non–zero non–real scalar in ρ_i. In $\bar{\rho}_i$, this element must act by the complex–conjugated i.e. different scalar. It implies that $\bar{\rho}_i \not\simeq \rho_i$ and thus $\bar{\rho}_i$ has the highest weigth $s\nu(\Gamma_i)$.

Let us denote the root corresponding to the discussed cross by α and the root connected by an arrow by α'. The corresponding root elements will be denoted by H_α and $H_{\alpha'}$. We will show that $\sigma(H_\alpha - H_{\alpha'}) = -(H_\alpha - H_{\alpha'})$ where σ denotes the complex conjugation corresponding to the real form $\mathfrak{g} \subseteq \mathfrak{f}$ i.e. $i(H_\alpha - H_{\alpha'}) \in \mathfrak{g}$; this element acts by the scalar $-i$ in ρ_i. Denoting $(,)$ the Killing form on $\mathfrak{f}$, we can compute for any simple root $\tilde{\alpha} \in \Pi$ that $\tilde{\alpha}(H_\alpha - H_{\alpha'}) = (\tilde{\alpha}, \alpha) - (\tilde{\alpha}, \alpha')$ and

$$\tilde{\alpha}(\sigma(H_\alpha - H_{\alpha'})) = (\sigma^*\tilde{\alpha})(H_\alpha - H_{\alpha'}) = (\sigma^*\tilde{\alpha}, \alpha) - (\sigma^*\tilde{\alpha}, \alpha') = (\tilde{\alpha}, \sigma^*\alpha) - (\tilde{\alpha}, \sigma^*\alpha')$$

because σ^* is an involutive automorphism of the root system. From the construction of the Satake diagrams follows that $\sigma^*\alpha = \alpha' + \sum_{\beta \in \Pi_c} d_\beta \beta$ where Π_c denotes the system of compact simple roots and the coefficients d_β are non–negative integers. It follows that $\sigma^*\alpha' = \alpha + \sum_{\beta \in \Pi_c} d_\beta \beta$ because $\sigma^*\beta = -\beta$ for each compact root β and σ^* is an involution. It follows that $\tilde{\alpha}(\sigma(H_\alpha - H_{\alpha'})) = -\tilde{\alpha}(H_\alpha - H_{\alpha'})$. Since it holds for each simple root $\tilde{\alpha} \in \Pi$, it shows $\sigma(H_\alpha - H_{\alpha'}) = -(H_\alpha - H_{\alpha'})$. Summarizing, we have prooved the following theorem.

4.2**. Theorem.** *Let us consider a semisimple Lie algebra $\mathfrak{g}$ with the complexification $\mathfrak{f} = \mathfrak{g}(\mathbb{C})$ and their reductive subalgebras $\mathfrak{g}_0 \subseteq \mathfrak{f}_0 = \mathfrak{g}_0(\mathbb{C})$ given by the Satake diagram with crosses. If an irreducible representation of $\mathfrak{f}_0$ has a highest weight Γ understood as a vector of coefficients over the diagram, then the complex–conjugated representation (with respect to $\mathfrak{g}_0$) has the highest weight $s\nu(\Gamma)$ where the latter symmetries are given by $\mathfrak{g} \subseteq \mathfrak{f}$, see Section 2.* $\square$

4.3. *Remark.* In this section, we have considered only reductive algebras given by Satake diagrams with crosses and their complex–conjugated representation were described by using of the structure of the zero cohomology. Other possibility is to evaluate the action of the center. It can be used for arbitrary reductive algebras.

5. Indices

5.1. **Indices of semisimple algebras.** Using the notation from 3.1, we show how to compute indices $\varepsilon(\mathfrak{g}, \Lambda)$ for an irreducible self–conjugated $\mathfrak{f}$–representation with the highest weight Λ. First let us note that indices of semisimple cases are products of indices of simple parts (and corresponding restrictions of Λ). Further, there is a general result how to compute the indices which gives formulas for all (semi)simple Lie algebras. All these facts can be found in [On]. Another way how to compute the indices (which we will need in the proof of the next theorem) is to begin with the fundamental weights (i.e. representations with these weights). The set of vertices of the Dynkin diagram of $\mathfrak{f}$ can be understood as the set of the fundamental weights F and let us denote $Q \subseteq F$ the set of quaternionic fundamental weights. Then the self–conjugated representation with the highest weight $\Lambda = (\Lambda_i)$, $i \in F$ is real (quaternionic) if the sum $\sum_{i\in Q} \Lambda_i$ is an even (odd) number. This fact and indices of fundamental weights can be found in [Ti].

We are going to show both the resulting formulas and the types of fundamental representations in the summary below. This case–by–case discussion follows that all quaternionic fundamental representation correspond to the compact roots of the Satake diagram. In all cases, the parameter n denotes the number of nodes of a diagram. The fundamental representation ρ_i corresponds to the coefficient Λ_i i.e. it denotes the representation with the highest weight $(0, \dots, 1, \dots, 0)$ with the unique one on the position corresponding to $i \in F$. Simple real forms not mentioned below have index $+1$ for each self–conjugated $\mathfrak{f}$–representation i.e. they do not admit quaternionic representations. The denotation of real forms and corresponding Satake diagrams is used as in [OV].

- $\mathfrak{su}_{k,n-k+1}$, n odd, $n+1 = 2m$, $s\nu \neq \mathrm{id}$ .... $(-1)^{(m-k)\Lambda_m}$
  The unique self–conjugated representation is ρ_m. This representation is real (quaternionic) if $m-k$ is even (odd). Let us note that $m-k$ is the "bigger half" of the (odd) number of compact nodes in the Satake diagram.
- $\mathfrak{sl}_{n+1}(\mathbb{H})$, n odd, $s\nu = \mathrm{id}$ .... $(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_n}$
  All fundamental representations are self–conjugated. The representation ρ_i is real (quaternionic) if i is even (odd). In the other words, representations corresponding to compact nodes are quaternionic and the remaining are real.
- $\mathfrak{so}_{k,2n-k+1}$, $s\nu = \mathrm{id}$, $k = 2l$, ... $(-1)^{(k+\frac{n(n-1)}{2})\Lambda_n}$
  First let us note the the assumption $k = 2l$ is no restriction due to the isomorphism $\mathfrak{so}_{p,q} \simeq \mathfrak{so}_{q,p}$. All fundamental representations are self–conjugated. Representations $\rho_1, \dots, \rho_{n-1}$ are real. The representation ρ_n is real (quaternionic) if $n-k \equiv 0$ or 3 ($n-k \equiv 1$ or 2), all cases modulo 4. Let us note that $n-k$ is the number of compact nodes in the Satake diagram.
- $\mathfrak{sp}_{k,n-k}$, $s\nu = \mathrm{id}$ .... $(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2[\frac{1}{2}(n+1)]-1}}$
  All fundamental representations are self–conjugated. The representation ρ_i is real (quaternionic) if i is even (odd).
- $\mathfrak{so}_{k,2n-k}$. The fundamental representations $\rho_1, \dots, \rho_{n-2}$ are self–conjugated and real. Futher we distinguish two possibilities according to the parity of $n-k$. Let us note that $n-k$ is the number of compact nodes in the Satake diagram except the case $\mathfrak{so}_{k-1,k+1}$. In this case (the diagram with an arrow and no compact node) is $n-k = 2$.

  – $n-k$ even, $s\nu = \mathrm{id}$ .... $(-1)^{\frac{n-k}{2}(\Lambda_{n-1}+\Lambda_n)}$
  The fundamental representations ρ_{n-1} and ρ_n are self–conjugated. They are both real (quaternionic) if $4 \mid n-k$ ($4 \nmid n-4$).
  – $n-k$ odd, $s\nu \neq \mathrm{id}$ .... $+1$
  The fundamental representations ρ_{n-1} and ρ_n are not self–conjugated (they are mutually conjugated).
- $\mathfrak{u}^*_n(\mathbb{H})$. The fundamental representations $\rho_1, \dots, \rho_{n-2}$ are self–conjugated; the representation ρ_i, $i \leq n-2$ is real (quaternionic) if i is even (odd). Further we distinguish two possibilities according to the parity of n. Let us note that if n is even then one of "legs" of the Satake diagram is compact and the second is non–compact and if n is odd then both "legs" are non–compact and connected by an arrow.
  – n even, $s\nu = \mathrm{id}$ .... $(-1)^{\Lambda_1+\Lambda_3+\dots+\Lambda_{n-1}}$
  The fundamental representations ρ_{n-1} and ρ_n are self–conjugated. The representation corresponding to the (unique) compact node is quaternionic and the second is real. (In our numeration, Λ_{n-1} correpsonods to the compact root.)
  – n odd, $s\nu \neq \mathrm{id}$ .... $(-1)^{\Lambda_1+\Lambda_3+\dots+\Lambda_{n-2}}$
  The fundamental representations ρ_{n-1} and ρ_n are not self–conjugated (they are mutually conjugated).

  In the other words, representations corresponding to the compact nodes are quaternionic and representations corresponding to the non–compact nodes without arrows are real.
- The compact form and the real form EVI of E_7, $s\nu = \mathrm{id}$ .. $(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$
  All fundamental representations are self–conjugated. The quaternionic representations ρ_1, ρ_3, ρ_7 correspond to the compact roots of the Satake digram of EVI.

5.2. **Indices of reductive algebras.** Let us consider an irreducible representation $\gamma : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(V)$ on the complex vector space V which is self–conjugated i.e. the highest weight of γ satisfies $\Gamma = \bar{\Gamma}$. Then there exists a γ–invariant antiautomorphism J on V such that $J^2 \in \{+\mathrm{id}, -\mathrm{id}\}$. It implies that each element of the center of $\mathfrak{g}_0$ must act by some real scalar (non–real scalars do not commute with any antiautomorphism). But real scalars commute with each antiautomorphism and thus the action of the center has no effect for the index. Denoting $\mathfrak{g}_0^{ss}$ the semisimple part of $\mathfrak{g}_0$, we have shown that $\varepsilon(\mathfrak{g}_0, \gamma) = \varepsilon(\mathfrak{g}_0^{ss}, \gamma|\mathfrak{g}_0^{ss})$. The index on the right hand side can be easily computed using the formulas above.

If the representation γ is a self–conjugated component in the cohomology, we can determine the index more closely. Let us suppose that the highest weight $\Gamma = \bar{\Gamma}$ of γ is of the form $\Gamma = w.\Lambda$ for a self–conjugated $\mathfrak{f}$–dominant weight Λ and an element $w \in W$. The following theorem says that indices $\varepsilon(\mathfrak{g}_0, \Gamma)$ and $\varepsilon(\mathfrak{g}, \Lambda)$ are the same. Denoting $F_0 \subseteq F$ the set of fundamental weights corresponding to the non–crossed roots, it follows from the above list and corresponding Satake diagrams that all quaterionic fundamental representations correspond to the compact roots and thus $Q \subseteq F_0$. Therefore we observe $\varepsilon(\mathfrak{g}, \Lambda) = \varepsilon(\mathfrak{g}_0, \Lambda)$ where Λ is understood as an $\mathfrak{f}_0$–dominant weight on the right hand side.

5.3. **Theorem.** *Let us consider a real semisimple Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{g}_0$ with the complexification $\mathfrak{f}_0 \subseteq \mathfrak{f}$. Having a self–conjugated $\mathfrak{f}$–dominant weight Λ and a self–conjugated $\mathfrak{f}_0$–dominant weight $\Gamma = w(\Lambda)$ for $w \in W$, it holds*

$\varepsilon(\mathfrak{g},\Lambda) = \varepsilon(\mathfrak{g}_0,\Gamma)$. *The same assertion holds if we replace the Weyl action by the affine Weyl action i.e.* $\Gamma = w.\Lambda$.

Proof. Using the list above, it is easy to see that $\varepsilon(\mathfrak{g}_0;\Lambda_1+\Lambda_2) = \varepsilon(\mathfrak{g}_0;\Lambda_1)\varepsilon(\mathfrak{g}_0;\Lambda_2)$ for $\mathfrak{g}_0$–dominant weights Λ_1 and Λ_2. It follows the last note. We prove the theorem case by case for simple real Lie algebras. It is sufficient to consider only the real forms discussed in 5.1. Most of them can be shown easily if we observe the Weyl actions on the Dynkin diagram. The simple reflection $w_i \in W$ corresponding to $\alpha_i \in \Pi$ acting on a weight Λ' of $\mathfrak{f}$ has the following form:

Let a be the coefficient over the i-th node in the expression of Λ'. In order to get the coefficients over the nodes corresponding to $w_i(\Lambda')$, add a to the adjacent coefficients, with the multiplicity if there is a multiple edge directed towards the adjacent node, and replace a by $-a$. (This algorithmic background of computing with the Dynkin diagrams was established in [BE]).

Let us begin with the real form $\mathfrak{sl}_{n+1}(\mathbb{H})$ for n. The index of Λ is given by the parity of the sum $\Lambda_1+\Lambda_3+\ldots+\Lambda_n$. It is easy to see that the simple reflections do not change the parity of this sum. The index of $\mathfrak{so}_{k,2n-k+1}$ is either always $+1$ (for both Λ and Γ) or depends on the parity of the last coefficient. But first $n-2$ reflections do not change the last coefficient and the last two reflections do not change its parity. A similar consideration proves the theorem for algebras $\mathfrak{sp}_{k,n-k}$, $\mathfrak{so}_{k,2n-k}$, $\mathfrak{u}^*_n(\mathbb{H})$ for n even and the compact form and the real form EVI of E_7.

The remaining cases $\mathfrak{su}_{p,n+1-p}$, $n = 2m-1$ and $\mathfrak{u}^*_n(\mathbb{H})$ for n odd must be discussed more carefully. We will consider the usual matrix presentation (see e.g. [Sam]) of the (complex) algebras $\mathfrak{sl}_{2m}(\mathbb{C})$ and $\mathfrak{so}_{2n}(\mathbb{C})$. We will express the weight Λ in the base from simple roots as Λ_s and in the "matrix" base as Λ_e. (In the case of $\mathfrak{sl}_n(\mathbb{C})$, the matrix base is $e^1,\ldots,e^{n+1}$ where e^i extracts the i–th element of the diagonal. In the case of $\mathfrak{so}_{2n}(\mathbb{C})$, the matrix base is $e^1,\ldots,e^n$ where e^i extracts the $2i$–th element of the diagonal.) Similarly, we will consider the vectors Γ, Γ_s and Γ_e. The Cartan matrices will be denoted by $C(A_n)$ and $C(D_n)$.

I. Let us start with the algebra A_n, $n = 2m-1$ with the real form $\mathfrak{su}_{p,n+1-p}$ (i.e. the non–trivial symmetry $s\nu$). We will consider $\Lambda_s = \Lambda\cdot C^{-1}(A_n)$ with respect to the system of simple roots $\Pi = \{e^1-e^2,\ldots,e^n-e^{n+1}\}$. The structure of the matrix $C^{-1}(A_n)$ implies that the symmetry of vectors Λ and Λ_s is equivalent. If we denote $\Lambda_s = (a_1,\ldots,a_n)$ we get $\Lambda_e = (a_1, a_2-a_1,\ldots,a_n-a_{n-1},-a_n)$. It implies that the symmetric vector Λ_s corresponds to the antisymmetric vector Λ_e. We will further consider the form $\Lambda_e = (b_1,\ldots,b_m,-b_m,\ldots,-b_1)$. Starting with a (symmetric) highest weight $\Lambda = (\Lambda_1,\ldots,\Lambda_m,\ldots,\Lambda_n)$, a short computations reveals that $b_k = \Lambda_k+\cdots+\Lambda_{m-1}+\frac{1}{2}\Lambda_m$ for $1 \le k \le m$.

In this point of view, the Weyl group W is isomorphic to Sym $_{2m}$. Considering the form of elements of Λ_e we see that the parity of Λ_m is the same as the parity of the double of an arbitrary element of Λ_e. But the last parity is not changing by the permutations. Thus, we have shown that Λ_m and Γ_m have the same parity. (We do not need to do the backward transformation if the weight Γ is symmetric which is the case according to our assumptions.) Since indexes of Λ and Γ depend on this parity in the same way, we have proved the theorem for the real form $\mathfrak{su}_{p,n+1-p}$ for n odd.

II. The case of the algebra D_n, n odd with the real form $\mathfrak{u}^*_n(\mathbb{H})$ (i.e. the non–trivial symmetry $s\nu$) will be similar. We will consider $\Lambda_s = \Lambda\cdot C^{-1}(D_n)$ with respect to the system of simple roots $\Pi = \{e^1-e^2,\ldots,e^{n-1}-e^n,e^{n-1}+e^n\}$. The structure of the matrix $C^{-1}(D_n)$ implies that the symmetry of vectors Λ and Λ_s is equivalent (and this symmetry means the equality of the last two elements). If we denote $\Lambda_s = (a_1, \ldots, a_n)$ we get $\Lambda_e = (a_1, a_2 - a_1, \ldots, a_{n-2} - a_{n-1}, a_{n-1} + a_n - a_{n-2}, a_n - a_{n-1})$. It implies that the symmetric vector Λ_s corresponds to the vector Λ_e with zero at the last position. We will further consider the form $\Lambda_e = (b_1, \ldots, b_{n-1}, 0)$. Starting with a (symmetric) highest weight $\Lambda = (\Lambda_1, \ldots, \Lambda_{n-1}, \Lambda_{n-1})$, a short computations reveals that $b_k = \Lambda_k + \cdots + \Lambda_{n-1}$ for $1 \leq k \leq n-1$.

In this point of view, the Weyl group W is generated by permutations (i, j) and permutations (i, j) with the sign change of elements at positions i and j. Considering the form of elements of Λ_e we see that the parity of the sum $\Lambda_1 + \Lambda_3 + \ldots \Lambda_{l-2}$ is the same as the parity of the sum $\sum_{i=1}^{n-1} (\Lambda_e)_i = \Lambda_1 + 2\Lambda_2 + \cdots + (l-2)\Lambda_{l-2} + (l-1)\Lambda_{l-1}$. But the last parity is not changed by the permutations and sign changes, if the resulting vector Γ has zero as the last component. Thus, we have shown that the sums $\Lambda_1 + \Lambda_3 + \ldots \Lambda_{l-2}$ and $\Gamma_1 + \Gamma_3 + \ldots \Gamma_{l-2}$ have the same parity. Since the indices of Λ and Γ depend on this parity in the same way, we have proved the theorem for the real form $\mathfrak{u}_n^*(\mathbb{H})$ for n odd. □

6. Relation between real and complex cohomologies

Now we know how to identify the couples of conjugated representations of $\mathfrak{f}_0$ (with respect to $\mathfrak{g}_0$) and how to compute the index of a self–conjugated representation of $\mathfrak{f}_0$ (with respect to $\mathfrak{g}_0$) and thus we can finish the considerations from the Section 3. Using the same notation, we describe the real cohomology for all types of the irreducible representation $\lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. First we show that we can consider symmetries of diagrams as isomorphisms of the Weyl group.

6.1. Symmetries as isomorphisms of the Weyl group. For an arbitrary symmetry a' of the Dynkin diagram of $\mathfrak{f}$, we define the isomorphism $a' : W \longrightarrow W$ in the following way: the image of the simple reflection w_i corresponding to the simple root $\sigma_i \in \Pi$ will be simple reflection $w_{a'(i)}$ corresponding to the simple root $a'(\sigma_i)$. In the other words, an element $w = w_{i_1} \ldots w_{i_k}$ given by the sequence $i_1, \ldots, i_k$ with respect to the (ordered) set Π is mapped to the element $a'(w)$ given by the same sequence with respect to the set $a'(\Pi)$. It also shows the correctness of the definition of a'.

6.2. Real cohomology for complex and quaternionic $\mathfrak{g}$–representations. Let us suppose that λ is complex or quaternionic i.e. V is a complex vector space and let us denote the highest weight of $\lambda_{\mathbb{C}} : \mathfrak{f} \longrightarrow \mathfrak{gl}(V)$ by Λ. Then we can consider the representation on the complex cohomology $\beta_{\mathbb{C}} : \mathfrak{f}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+(\mathbb{C})); V)$ and the representation $\beta^{\mathbb{R}} : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V_{\mathbb{R}}))$ on the real cohomology is given by the relation $\beta^{\mathbb{R}} = (\beta_{\mathbb{C}}|\mathfrak{g}_0)^{\mathbb{R}}$, see Section 3. We will show that irreducible components of $\beta_{\mathbb{C}}$ and $\beta^{\mathbb{R}}$ are in 1–1 correspondence. Let us remind that the highest weights of components of $\beta_{\mathbb{C}}$ are of the form $w.\Lambda$, $w \in W^{\mathfrak{q}}$, see Theorem 1.4.

I. First let us suppose that λ is complex i.e. $\Lambda \neq s\nu(\Lambda)$. We claim the following useful property. For each $w \in W$, an arbitrary symmetry a' of the Dynkin diagram and a highest weight Λ', the following relation holds:

$$(*) \qquad a'(w.\Lambda') = w.\Lambda' \iff \Lambda' = a'(\Lambda') \wedge w = a'(w).$$

The equality $w.\Lambda = a'(w.\Lambda') = a'(w).a'(\Lambda')$ implies that the irreducible component with the highest weight $a'(w.\Lambda')$ is a cohomology component in the cohomology induced by $\mathfrak{k}$–representations with highest weights Λ' and $a'(\Lambda')$. It implies $\Lambda = a'(\Lambda')$ and thus $w = a'(w)$. Now putting $a' = s\nu$ and $\Lambda' = \Lambda$, it follows from Theorem 4.2 that each component in $\beta_{\mathbb{C}}$ is complex

II. If λ is quaternionic then $\Lambda = s\nu(\Lambda)$ and $\varepsilon(\mathfrak{g}, \Lambda) = -1$. If there is a component in $\beta_{\mathbb{C}}$ with the highest weight $w.\Lambda$, $w \in W^{\mathfrak{q}}$ such that $s\nu(w.\Lambda) = w.\Lambda$ then $\varepsilon(\mathfrak{g}_0, w.\Lambda) = \varepsilon(\mathfrak{g}, \Lambda) = -1$, see Theorem 5.3. It says that each component in $\beta_{\mathbb{C}}$ is complex or quaternionic.

Summarizing, the space of the real cohomology $H(\mathfrak{p}_+; V)$ for a complex or quaternionic representation λ is just the space of the complex cohomology $H(\mathfrak{p}_+(\mathbb{C}); V)$ understood as an real vector space. Moreover, the Lemma 3.3 implies that the irreducible components of $\beta_{\mathbb{C}}$ and $\beta^{\mathbb{R}}$ are the same and from the definition of arrows in 1.8 follows that the Hasse graphs are the same too.

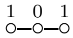
6.3. **Real cohomology for real $\mathfrak{g}$–representations.** If the representation λ is real then we start with its (irreducible) complexification $\lambda(\mathbb{C}) : \mathfrak{k} \longrightarrow \mathfrak{gl}(V(\mathbb{C}))$ with the highest weight Λ. It follows from 1.5 that the complexification of the required representation $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+; V))$ on the real cohomology is just the representation $\beta(\mathbb{C}) : \mathfrak{k}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+(\mathbb{C}); V(\mathbb{C})))$ on the complex cohomology for the $\mathfrak{k}$–representation $\lambda(\mathbb{C})$. Now, our aim is to say which irreducible components of $\beta(\mathbb{C})$ correspond to irreducible components in β and which couples of irreducible components of $\beta(\mathbb{C})$ correspond to irreducible components in β.

The description of highest weights of $\beta(\mathbb{C})$ is given by the Theorem 1.4. Moreover, each highest weight has the multiplicity one. Using the Theorem 4.2, it implies that there is no quaternionic component in β. Thus, the type of components in β is fully determined by the symmetry $s\nu$. In particular, the existence of complex components in β is conditioned by the non–triviality of this symmetry.

(a) $s\nu = \mathrm{id}$. All components in $H(\mathfrak{p}_+, V)$ are clearly of the real type and the Hasse graphs of $H(\mathfrak{p}_+(\mathbb{C}), V(\mathbb{C}))$ and $H(\mathfrak{p}_+, V)$ are isomorphic. Considering simple Lie algebras, it includes all real forms of algebras B_n, C_n, E_7, E_8, F_4, G_2 and real algebras $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_{2m}(\mathbb{H})$, $\mathfrak{so}_{k,2n-k}$ for $k - n$ even, $u^*_{2p}(\mathbb{H})$, EI and EIV.

(b) $s\nu \neq \mathrm{id}$. There can be real and complex components in $H(\mathfrak{p}_+, V)$ (and one can easily show that there are really both these types of representations in the cohomology for each choice of the parabolic subalgebra). A component in $\beta(\mathbb{C})$ with the highest weights $w.\Lambda$, $w \in W^{\mathfrak{q}}$ corresponds to a component in β if and only if $s\nu(w.\Lambda) = w.\Lambda$ (or equivalently $s\nu(w) = w$) and a couple of components in $\beta(\mathbb{C})$ with the highest weights $w_1.\Lambda \neq w_2.\Lambda$, $w_1, w_2 \in W^{\mathfrak{q}}$ corresponds to a component in β if and only if $s\nu(w_1.\Lambda) = w_2.\Lambda$ (or equivalently $s\nu(w_1) = w_2$). The Hasse graph on $H(\mathfrak{p}_+, V)$ is obtained from the Hasse graph on $H(\mathfrak{p}_+(\mathbb{C}), V(\mathbb{C}))$ by connecting these couples of components which correspond to one component in the real case. Considering simple Lie algebras, it includes cases $\mathfrak{su}_{k,n-k}$, $\mathfrak{so}_{k,2n-k}$ for $k - n$ odd, $u^*_{2p+1}(\mathbb{H})$, EII, EIII and the compact form of the algebra E_6 and all simple complex Lie algebras understood as real ones.

6.4. *Example.* **Cohomology for $\mathfrak{su}_{3,1}$ and the adjoint representation.** The Satake diagram of $\mathfrak{su}_{3,1}$ is  and it admits the only parabolic subalgebra given by the diagram  . The adjoint representation of $\mathfrak{su}_{3,1}$ is clearly real.

The adjoint representation in the complex case is described by the Dynkin diagram with coefficients  . Irreducible components of the complex cohomology with the structure of the Hasse graph are shown in the Figure 1. (The way from

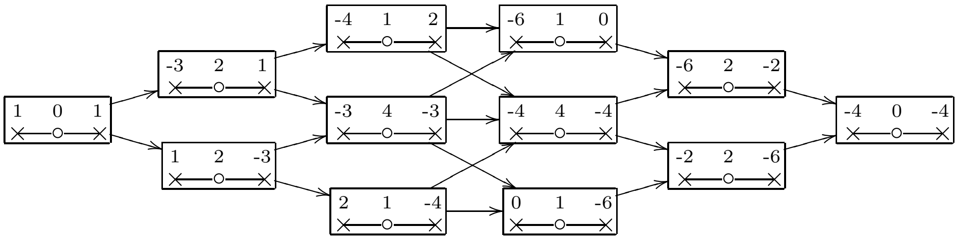


FIGURE 1. The complexificated cohomology of $\mathfrak{su}_{3,1}$.

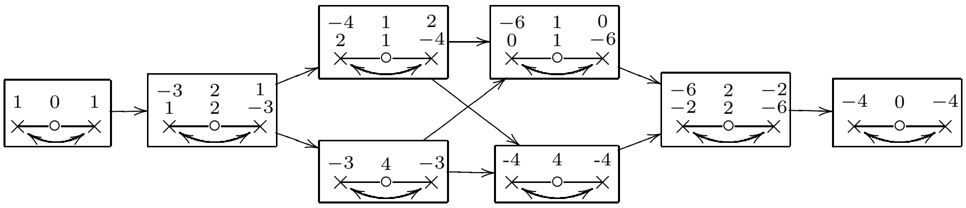


FIGURE 2. The real cohomology of $\mathfrak{su}_{3,1}$.

the Kostant's result to the algorithmic computation of the cohomology is described in [Si].)

Components of the real cohomology are described by the Satake diagram with the highest weight of the complexification (for the components of the real type) or by the Satake diagram with a couple of the highest weights of the complexification (for the components of the complex type). Since the complex Hasse graph is symmetric (with respect to the highest weights of its components) according to the middle line, the real Hasse graph is just the "upper part" of the complex one, see Figure 2.

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