Conformal Mappings from the Upper Half Plane to Fundamental Domains on the Hyperbolic Plane

Mark Harmer and Gaven Martin
Department of Mathematics
University of Auckland
New Zealand

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Abstract

We use the classical theory of Schwarz-Christoffel mappings to find conformal maps from the upper half plane to triangular regions in the hyperbolic plane. We then find the pullback of the (hyperbolic) Laplace-Beltrami operator to the upper half plane.

1 Introduction

As is well known the hyperbolic plane $\mathbb{H}$ can be identified with the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. In this way a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ generates a discrete group action on the hyperbolic plane. The quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane by this group action can be identified with the so called fundamental domain F, a subset of $\mathbb{H}$ which covers the hyperbolic plane without any intersections under the action of Γ, $\mathbb{H} = \cup\{\gamma\bar{F};\ \gamma \in \Gamma\}$.
Ultimately we plan to study the spectral properties of the Laplace-Beltrami operator on these fundamental domains. One approach to studying this problem is to map the fundamental domain onto a region which is simpler to analyse—for instance the unit disc or the upper half plane. By the Riemann mapping theorem such a mapping can always be found which is, moreover, conformal.
Although, we know the existence of a conformal map from our fundamental domain into the upper half plane, in practise it is only possible to write out this conformal map explicitly in a number of specific cases. One of these cases is when the region is enclosed by three circular arcs, then the conformal map is given in terms of hypergeometric functions. By a judicious choice of parameters this three sided region can be made to correspond to a bona fide

hyperbolic triangle which may be thought of as the fundamental domain F for some discrete group action Γ. This is the problem we will discuss here. From this mapping we then derive the expression for the pullback of the Laplace-Beltrami operator to the upper half plane.

2 Schwarz-Christoffel Mappings to Regions enclosed by Circular Arcs

A detailed discussion of the theorey of Schwarz-Christoffel mappings may be found in [4]. We give a brief summary here.
In the case of a region, $R \subset \mathbb{C}$, bounded by three circular arcs we seek a function $f(z)$ which maps the upper half plane $\mathbb{C}_+$ onto R such that the 'vertices' (A_1, A_2 and A_3) of R are the image of distinguished points a_1, a_2 and a_3 on the real axis—as noted above, for regions with more than three sides an explicit expression for the appropriate conformal map is not, in general, known. If the angle at the vertex A_i is $\pi\alpha_i$ then it is clear that in the neighbourhood of a_i f must have the form

$$f(z) = (z - a_i)^{\alpha_i} f_i(z)$$

where f_i is regular near a_i and real on the real axis. Repeating this for each vertex and substituting the ansatz into the Schwarzian derivative

$$\{f, z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

we get that $\{f, z\}$ is equal to a rational expresion plus a function $g(z)$ which is analytic in $\mathbb{C}_+$ and regular on the real axis. In fact, since the Schwarzian derivative is invariant under fractional linear transformations, in particular under the transformation which maps a circular boundary arc of R onto the real axis, we see that $\{f, z\}$ is real on the real axis. Consequently $g(z)$ must be real on the real axis and since it is analytic in $\mathbb{C}_+$ it reduces to a constant. This gives us a third order differential equation satisfied by $f(z)$. This can be further simplified by noting that if F_1, F_2 are two linearly independant solutions of

$$F'' + P(z)F' + Q(z)F = 0 \tag{1}$$

then

$$f(z) = \frac{F_1(z)}{F_2(z)}$$

satisfies

$$\{f, z\} = 2Q - P^2/2 - P' \,. \tag{2}$$

In our case if we set $a_1 = 0$, $a_2 = \infty$ and $a_3 = 1$ and use (2) to solve for P and Q we see that (1) becomes the hypergeometric equation

$$z(1 - z)F'' + [c - (a + b + 1)z]F' - abF = 0 \,. \tag{3}$$

The constants a, b, c are given in terms of the angles α_i by ([4] pg 206)

$$a = \tfrac{1}{2}(1 - \alpha_1 + \alpha_2 - \alpha_3) \tag{4}$$
$$b = \tfrac{1}{2}(1 - \alpha_1 - \alpha_2 - \alpha_3) \tag{5}$$
$$c = 1 - \alpha_1 \, . \tag{6}$$

2.1 Schwarz-Christoffel Mappings to Hyperbolic Triangles

Given that the region R is bounded by circular arcs with angles α_i at each of the vertices we have to choose the correct solutions (F_1 and F_2) of the hypergeometric equation so that the image of $f = F_1/F_2$ is in some sense a genuine hyperbolic triangle. We start by choosing F_2 to be the usual hypergeometric function

$$F_2(z) = F(z; a, b, c) \, . \tag{7}$$

Then it is known ([4] pg 314) that

$$F_1(z) = z^{1-c} F(z; a', b', c') \tag{8}$$

where

$$a' = a - c + 1 = \tfrac{1}{2}(1 + \alpha_1 + \alpha_2 - \alpha_3) \tag{9}$$
$$b' = b - c + 1 = \tfrac{1}{2}(1 + \alpha_1 - \alpha_2 - \alpha_3) \tag{10}$$
$$c' = 2 - c = 1 + \alpha_1 \tag{11}$$

is a linearly independant solution of the hypergeometric equation with the same parameters (a, b and c) as F_2. We would like to find the vertices $A_1 = f(0)$, $A_2 = f(\infty)$ and $A_3 = f(1)$ of the image, R, of $f = F_1/F_2$. Since the hypergeometric function is bounded near the origin and $c < 1$, $A_1 = f(0) = 0$. We choose the branch of z^{1-c} so that z^{1-c} is real on the positive real axis and

$$z^{1-c} = |z| e^{i\pi(1-c)}$$

on the negative real axis. This means that

$$\begin{aligned} A_3 &= f(1) = \frac{1 \cdot F(1; a', b', c')}{F(1; a, b, c)} \\ &= \frac{\Gamma(1+\alpha_1)}{\Gamma(1-\alpha_1)} \frac{\Gamma\left(\frac{1}{2}(1 - \alpha_1 + \alpha_2 + \alpha_3)\right)}{\Gamma\left(\frac{1}{2}(1 + \alpha_1 + \alpha_2 + \alpha_3)\right)} \frac{\Gamma\left(\frac{1}{2}(1 - \alpha_1 - \alpha_2 + \alpha_3)\right)}{\Gamma\left(\frac{1}{2}(1 + \alpha_1 - \alpha_2 + \alpha_3)\right)} \, , \end{aligned} \tag{12}$$

where we have used the identity [3]

$$F(1; a, c, b) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, .$$

To evaluate the limit of $f(z)$ as $z \to \infty$ we use the fact that the analytic continuation of the hypergeometric function to points near infinity on the plane cut along the real axis is given by ([2], pg 63)

$$\begin{aligned} F(z;a,b,c) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(1/z;a,1-c+a,1-b+a) + \\ & \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(1/z;b,1-c+b,1-a+b)\,. \end{aligned} \tag{13}$$

From the fact that the α_i are positive it follows that

$$\begin{aligned} A_2 &= \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z^{1-c}F(z;a',b',c')}{F(z;a,b,c)} \\ &= e^{i\pi(1-c)}\frac{\Gamma(c')\Gamma(a'-b')}{\Gamma(a')\Gamma(c'-b')}\frac{\Gamma(a)\Gamma(c-b)}{\Gamma(c)\Gamma(a-b)} \\ &= e^{i\pi\alpha_1}\frac{\Gamma(1+\alpha_1)\Gamma\left(\frac{1}{2}(1-\alpha_1+\alpha_2+\alpha_3)\right)\Gamma\left(\frac{1}{2}(1-\alpha_1+\alpha_2-\alpha_3)\right)}{\Gamma(1-\alpha_1)\Gamma\left(\frac{1}{2}(1+\alpha_1+\alpha_2+\alpha_3)\right)\Gamma\left(\frac{1}{2}(1+\alpha_1+\alpha_2-\alpha_3)\right)}\,. \end{aligned} \tag{14}$$

In summary A_1 lies at the origin, A_3 lies on the positive real axis and the arcs $\overline{A_1A_2}$, $\overline{A_1A_3}$ are straight lines.
Consider the circular arc joining A_2 and A_3. This is part of a circle which we denote $\mathcal{S}$. Since $\alpha_1+\alpha_1+\alpha_3<\pi$, A_1 is exterior to $\mathcal{S}$. It is easy to show that given a circle $\mathcal{S}$ and a point A_1 not inside $\mathcal{S}$ we can construct another circle $\mathcal{C}_0$ which has centre A_1 and intersects $\mathcal{S}$ at right angles, see figure 1. If we rescale $f(z)$ so that the radius of $\mathcal{C}_0$ is unity then the region R can be thought of as a hyperbolic triangle in the Poincaré disc model where the hyperbolic plane is identified with the interior of $\mathcal{C}_0$. Using the cosine rule for hyperbolic triangles [1] we see that the hyperbolic length of the side joining A_1 and A_2, $\rho(A_1,A_2)$ satisfies

$$\cosh(\rho(A_1,A_2)) = \frac{\cos(\pi\alpha_1)\cos(\pi\alpha_2)+\cos(\pi\alpha_3)}{\sin(\pi\alpha_1)\sin(\pi\alpha_2)}\,.$$

On the other hand, the Euclidean length of this arc in the Poincaré disc model (assuming one end is at the origin, [1] pg 148) is $\tanh\left(\rho(A_1,A_2)/2\right)$ where

$$\begin{aligned} \tanh\left(\frac{\rho(A_1,A_2)}{2}\right) &= \sqrt{\frac{\cosh\left(\rho(A_1,A_2)\right)-1}{\cosh\left(\rho(A_1,A_2)\right)+1}} \\ &= \sqrt{\frac{\cos(\pi\alpha_1+\pi\alpha_2)+\cos(\pi\alpha_3)}{\cos(\pi\alpha_1-\pi\alpha_2)+\cos(\pi\alpha_3)}}\,. \end{aligned}$$

So normalising the radius of $\mathcal{C}_0$ is equivalent to rescaling $f(z)$ so that the distance (equation 14) of A_2 from the origin is given by the above quantity.

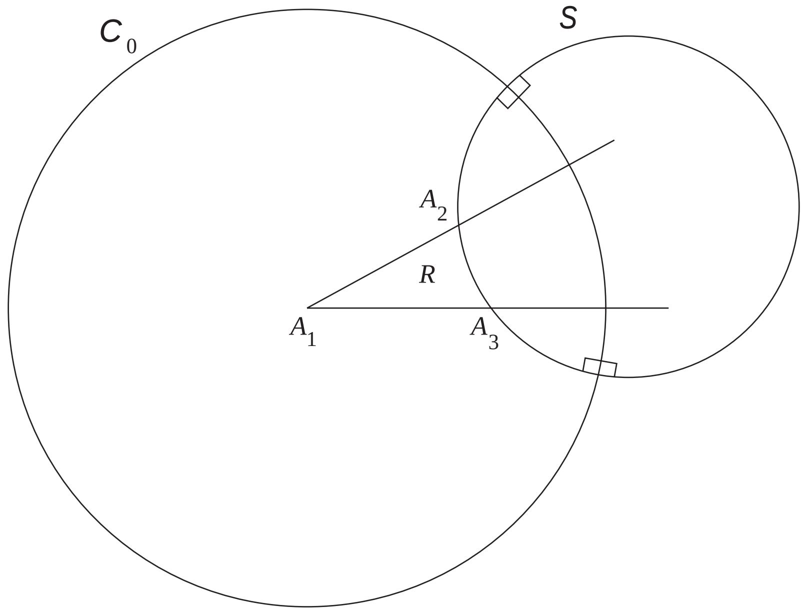

Figure 1:

That is

$$\hat{f}(z) = \gamma \frac{F_1(z)}{F_2(z)} = \gamma \frac{z^{1-c} F(z; a', b', c')}{F(z; a, b, c)}$$

where

$$\gamma = \sqrt{\frac{\cos(\pi\alpha_1 + \pi\alpha_2) + \cos(\pi\alpha_3)}{\cos(\pi\alpha_1 - \pi\alpha_2) + \cos(\pi\alpha_3)}} \frac{\Gamma(1-\alpha_1)\Gamma\left(\frac{1}{2}(1+\alpha_1+\alpha_2+\alpha_3)\right)\Gamma\left(\frac{1}{2}(1+\alpha_1+\alpha_2-\alpha_3)\right)}{\Gamma(1+\alpha_1)\Gamma\left(\frac{1}{2}(1-\alpha_1+\alpha_2+\alpha_3)\right)\Gamma\left(\frac{1}{2}(1-\alpha_1+\alpha_2-\alpha_3)\right)}, \tag{15}$$

maps the upper half plane to a hyperbolic triangle R in the Poincaré disc model.

Remark 2.1 *Since γ is unique we can carry out the same normalisation procedure as above using the arc from A_1 to A_3 and set the two values of γ equal to one another. This gives us*

$$\frac{\Gamma\left(\frac{1}{2}+\theta\right)\Gamma\left(\frac{1}{2}-\theta\right)}{\Gamma\left(\frac{1}{2}+\varphi\right)\Gamma\left(\frac{1}{2}-\varphi\right)} = \left[\frac{\cos(\pi\alpha_1 + \pi\alpha_2) + \cos(\pi\alpha_3)}{\cos(\pi\alpha_1 - \pi\alpha_2) + \cos(\pi\alpha_3)} \cdot \frac{\cos(\pi\alpha_1 - \pi\alpha_3) + \cos(\pi\alpha_2)}{\cos(\pi\alpha_1 + \pi\alpha_3) + \cos(\pi\alpha_2)}\right]^{\frac{1}{2}},$$

where

$$\theta = \tfrac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3), \quad \varphi = \tfrac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3).$$

All that remains is to calculate how the Laplace-Beltrami operator on the Poincaré disc transforms to an operator on the upper half plane.

3 The Laplace-Beltrami Operator

We seek to study the eigenvalue problem $\Delta\varphi(w) = \lambda\varphi$ where

$$\Delta \equiv -\left(1-|w|^2\right)^2 \frac{\partial^2}{\partial w \partial \bar{w}}$$

is the Laplace-Beltrami operator on the Poincaré disc ($\partial/\partial w = 1/2(\partial/\partial x - i\partial/\partial y)$ and similarly for $\partial/\partial \bar{w}$). Consider $w = \hat{f}(z)$ which maps the upper half plane to a subset of the Poincaré disc. We use w to denote the coordinate on the Poincaré disc and z to denote the coordinate on the upper half plane. Since $\hat{f}$ is holomorphic

$$\frac{\partial}{\partial z}\varphi(\hat{f}(z)) = \varphi_w \hat{f}_z \;\Rightarrow\; \frac{\partial^2}{\partial z \partial \bar{z}}\varphi = |\hat{f}_z|^2 \frac{\partial^2 \varphi}{\partial w \partial \bar{w}}\,,$$

where subscripts denote differentiation. Denoting the wronskian of two solutions F_1 and F_2 by

$$\mathrm{W}\{F_1, F_2\} \equiv F_{1,z}F_2 - F_1 F_{2,z}$$

it is clear that

$$\hat{f}_z = \frac{\mathrm{W}\{\gamma F_1, F_2\}}{F_2^2}\,.$$

Consequently, the Laplace-Beltrami operator on the upper half plane becomes

$$\Delta \equiv -\frac{\left(1-|\hat{f}|^2\right)^2}{|\hat{f}_z|^2}\frac{\partial^2}{\partial z \partial \bar{z}} = -\frac{\left(|\gamma F_1|^2 - |F_2|^2\right)^2}{|\mathrm{W}\{\gamma F_1, F_2\}|^2}\frac{\partial^2}{\partial z \partial \bar{z}}\,.$$

It is not difficult to show that the wronskian $\mathrm{W} = \mathrm{W}\{F_1, F_2\}$ of two solutions of the same hypergometric equation (with parameters a, b and c) satisfies the following differential equation

$$\frac{d\mathrm{W}}{dz} = -\frac{c-(a+b+1)z}{z(1-z)}\mathrm{W}\,.$$

This has solution

$$\mathrm{W} = C \cdot \frac{(1-z)^{c-a-b-1}}{z^c} = Cz^{\alpha_1 - 1}(1-z)^{\alpha_3 - 1}\,,$$

where C is a constant. Using this we see that

$$\mathrm{W}\{\gamma F_1, F_2\} = \gamma\alpha_1(1-z)^{\alpha_3-1}z^{\alpha_1-1}\,.$$

In summary the Laplace-Beltrami operator on the upper half plane is written

$$\Delta \equiv -\frac{\rho^2(x,y)}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

where

$$\rho(x,y) = \frac{1}{\alpha_1}|z|^{1-\alpha_1}|1-z|^{1-\alpha_3}\left(\gamma|F_1|^2 - \gamma^{-1}|F_2|^2\right) ,$$

γ is given by equation (15) and F_1 and F_2 by equations (8) and (7) along with (9–11) and (4–6). Furthermore, the weight function $\rho^2(x,y)$ is real and positive in the upper half plane with the following behaviour in the neighbourhood of infinity

$$\rho(x,y) = c_0|z|^{1+\alpha_2} + \mathrm{O}(|z|) .$$

Here we have again used equation (13) and the constant, assuming $\alpha_2 \neq 0$, is

$$c_0 = \frac{2\sin(\pi\alpha_1)\sin(\pi\alpha_2)}{[(\cos(\pi\alpha_1-\pi\alpha_2)+\cos(\pi\alpha_3))(\cos(\pi\alpha_1+\pi\alpha_2)+\cos(\pi\alpha_3))]^{\frac{1}{2}}} \times$$

$$\frac{\Gamma(1-\alpha_1)\Gamma(1+\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_2)}{\Gamma\left(\frac{1}{2}(1-\alpha_1+\alpha_2-\alpha_3)\right)\Gamma\left(\frac{1}{2}(1+\alpha_1+\alpha_2-\alpha_3)\right)\Gamma\left(\frac{1}{2}(1+\alpha_1+\alpha_2+\alpha_3)\right)\Gamma\left(\frac{1}{2}(1-\alpha_1+\alpha_2+\alpha_3)\right)} .$$

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