Barycentric Coordinates on the Hyperbolic Plane

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Abstract

We describe a 'natural' set of coordinates for fundamental domains in the hyperbolic plane in the case when the fundamental domain is triangular. The metric, the measure and the Laplace-Beltrami operator are calculated in this new coordinate system. As a byproduct we give a hyperbolic analogue of the Euclidean expression of the area of a triangle in terms of its base and height.

1 Introduction

As is well known the hyperbolic plane $\mathbb{H}$ can be identified with the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. In this way a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ generates a discrete group action on the hyperbolic plane. The quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane by this group action can be identified with the so called fundamental domain F, a subset of $\mathbb{H}$ which covers the hyperbolic plane without any intersections under the action of Γ, $\mathbb{H} = \cup\{\gamma\bar{F};\ \gamma \in \Gamma\}$.
Ultimately we plan to study the spectral properties of the Laplace-Beltrami operator on these fundamental domains. A major problem which was encountered in this study is that the boundary arcs of the fundamental domain turn out to be quite complicated curves in terms of the usual coordinate systems used on the hyperbolic plane—here we are thinking of cartesian or geodesic polar coordinates in the Poincaré upper half plane or disc models. To overcome this we propose a barycentric coordinate system which is defined on any triangular domain. In terms of these barycentric coordinates a triangular domain is mapped onto a Euclidean right-angled, isoceles triangle (the side length of the isoceles triangle depends on the hyperbolic area of the domain).

In this report we begin by defining barycentric coordinates for an arbitrary triangular region in the hyperbolic plane. Then we derive the transformation equations from barycentric coordinates to one of the standard coordinate systems used on the hyperbolic plane (viz. the coordinates for the Poincaré disc model). From this it is easy to derive the metric and thence the Laplace-Beltrami operator for a given triangular domain in terms of barycentric coordinates—for each domain the form of the Laplace-Beltrami operator in barycentric coordinates will be different reflecting the geometry of the domain.

The main advantage of barycentric coordinates is that a triangular region in the hyperbolic plane is mapped onto a Euclidean (right-angled, isoceles) triangle. However, the coefficients of the Laplace-Beltrami operator are trigonometric functions in the coordinates. We will see that by making a trigonometric change of coordinates, and a suitable normalisation, the coefficients of the Laplace-Beltrami operator become algebraic expressions in the coordinates—while the fundamental domain is still mapped onto a Euclidean right-angled isoceles triangle. We illustrate this procedure with a simple example at the end of this report.

Although we stated above that the boundary arcs of the fundamental domain turn out to be quite complicated curves in terms of the usual coordinate systems, this is not entirely true. In the Klein disc model a triangular fundamental domain is mapped onto a Eucliden triangle. However this has the undesirable property that the shape of the Euclidean triangle depends on where on the Klein disc the fundamental domain is located. Of course, unlike barycentric coordinates, the image Euclidean triangle is not a right-angled isoceles triangle. Rather the shape of the image triangle reflects the geometry of the fundamental domain while the form of the Klein Laplacean is fixed regardless of the fundamental domain. We note that the coefficients of the Klein Laplacean are algebraic expressions in the coordinates. In future investigations we will also investigate the spectral properties of the Laplace-Beltrami operator using the Klein disc formulation of the Laplacean.

2 Barycentric coordinates on the hyperbolic plane

Consider a triangular region F in the hyperbolic plane as in figure 1. The boundary arcs of F are geodesic curves in the hyperbolic plane. We pick any point z in the interior of F and would like to define the barycentric coordinates for that point. As in the figure we join z to each of the three vertices of F with a geodesic arc. In doing so we have formed three hyperbolic triangles with areas A, A' and A''. This triple (A, A', A'') constitutes the barycentric coordinates for the point z. In practise, since $A + A' + A'' = \text{constant}$, one of these coordinates is redundant and when calculating the metric and Laplace-

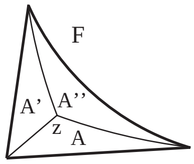

Figure 1: Barycentric Coordinates

Beltrami operator in barycentric coordinates we need to choose which two of these three numbers we are to work with.

Before we calculate the relationship between barycentric and Poincaré disc coordinates we need some simple geometric facts. First we consider a triangle inscribed inside a circle as in figure 2. It requires only a few lines of

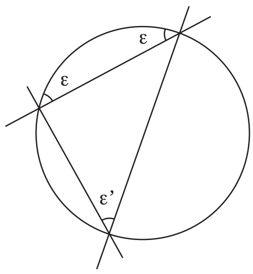

Figure 2:

work to show that the angles ε and ε' are equal.

Lemma 2.1 *The defect of the triangle ABC in figure 3 is given by*

$$\pi - (\alpha + \beta + \gamma) = 2\delta \,. \tag{1}$$

Proof: We see immediately from the previous observation and figure 3 that $\alpha = \alpha'$, $\beta = \beta'$ and $\delta = \delta'$. Then taking the sum of the angles of the triangle A′B′C we get

$$\begin{aligned} & \alpha' + \delta' + \beta' + \delta + \gamma = \pi \\ \Rightarrow \quad & 2\delta = \pi - (\alpha + \beta + \gamma) \,. \end{aligned}$$

□

We would now like to consider the triangle ABC as a hyperbolic triangle in the Poincaré disc model. Since the sides AC and BC are straight the only way to do this is to construct a circle, denoted $\mathcal{C}_0$, which intersects the circle

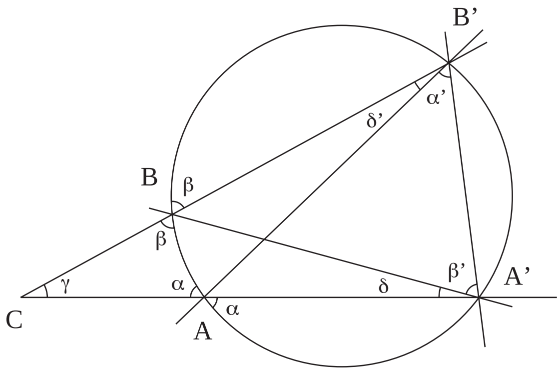

Figure 3:

ABB′A′ in figure 3 at right angles and has C as the centre,[1] see figure 4. Then normalising figure 4 so that the circle $\mathcal{C}_0$ has radius one, we can think of the interior of $\mathcal{C}_0$ as the Poincaré disc model of the hyperbolic plane and the triangle ABC as a bona fide hyperbolic triangle.

It will be convenient for us to think of the Poincaré disc as the unit disc in

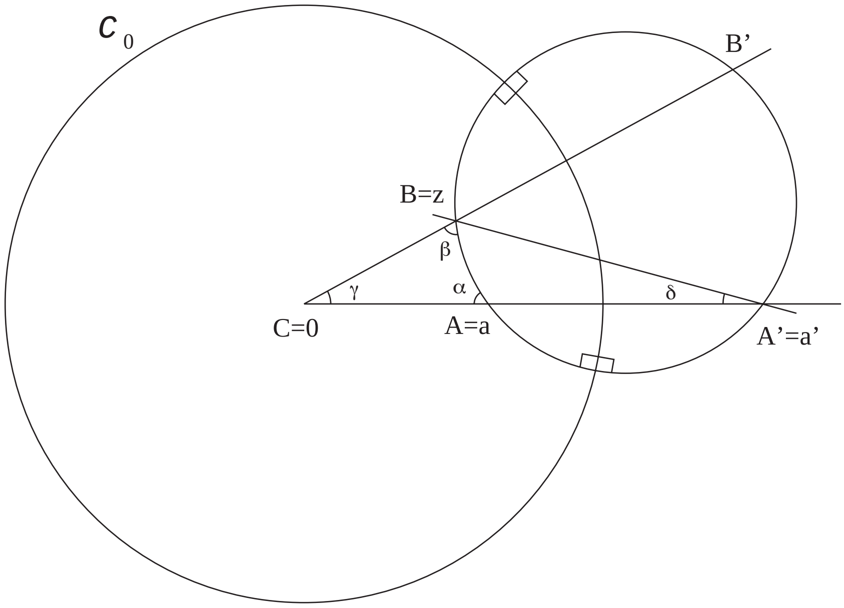

Figure 4:

the complex plane. In this way the point C is identified with the origin and we identify the points A and B with points a and z on the complex plane

[1]Given any circle $\mathcal{S}$ and any point C exterior to $\mathcal{S}$ we can construct a circle orthogonal to $\mathcal{S}$ with centre C in the following way: Take a tangent line, t, to $\mathcal{S}$ which intersects C (there are only two of them) and denote by P the point at which t intersects $\mathcal{S}$. Then the length |PC| is the radius of the circle centre C which is orthogonal to $\mathcal{S}$.

respectively.
It is a well known fact (see [2] pg 221, corollary 11) that—since the circles $\mathcal{C}_0$ and ABB′A′ are orthogonal to one another—the points A and A′ are related by inversion through the circle $\mathcal{C}_0$. To be precise this means that

$$|\mathrm{CA}| = \frac{r^2}{|\mathrm{CA}'|}$$

where r is the radius of $\mathcal{C}_0$. Using the identification of the Poincaré disc with the unit disc in the complex plane (i.e. $r = 1$) we see that A′ is identified with the point $a' = 1/\bar{a}$ on $\mathbb{C}$. From this we have the following theorem (but first a technical definition):

Definition 2.1 *Given $z, a \in \mathbb{C}$ we say that $\arg(z) > \arg(a)$ if z/a is in the upper half plane and $\arg(z) < \arg(a)$ if z/a is in the lower half plane. If z/a is real then clearly $\arg(z) = \arg(a)$.*

Theorem 2.1 *If we identify the Poincaré disc with the unit disc on the complex plane then for any hyperbolic triangle with vertices at the origin and at points z and a inside the unit disc*

$$e^{\pm iA} = \frac{\bar{z}a - 1}{z\bar{a} - 1}\,. \qquad (2)$$

Here A is the hyperbolic area of the triangle, the left hand side assumes the positive sign when $\arg(z) > \arg(a)$ and the negative sign when $\arg(z) < \arg(a)$.

Proof: From the fact that the area of a hyperbolic triangle is equal to its defect and the previous lemma we have that the left hand side of (2) is

$$e^{\pm iA} = e^{\pm i2\delta}\,.$$

On the other hand, from figure 4 and assuming that $\arg(z) > \arg(a)$, δ is given by

$$\delta = \arg\left(-\frac{1}{\bar{a}}\right) - \arg\left(z - \frac{1}{\bar{a}}\right)\,.$$

This means that

$$\begin{aligned}
\exp(iA) &= \exp(i2\delta) \\
&= \exp\left(i2\arg(-1/\bar{a})\right) \cdot \exp\left(i2\arg(\bar{z} - 1/a)\right) \\
&= \left(-\frac{1}{\bar{a}}\right) \Big/ \left(-\frac{1}{a}\right) \cdot \left(\bar{z} - \frac{1}{a}\right) \Big/ \left(z - \frac{1}{\bar{a}}\right) \\
&= \frac{\bar{z}a - 1}{z\bar{a} - 1}\,.
\end{aligned}$$

Likewise when $\arg(z) < \arg(a)$ the above formula gives $-\delta$ and we get the negative sign in the statement of the theorem. □

We speculate that the above theorem is the hyperbolic analogue of the Euclidean expression of the area of a triangle in terms of its base and height. It is now simple to define the transformation law between barycentric and Poincaré disc coordinates for a given hyperbolic triangle F. One of the vertices of F we choose to place at the origin of the Poincaré disc—this is in effect the choice, mentioned above, of which of the three barycentric coordinates (A, A', A'') to regard as redundant. The other two vertices of F are labelled by a and b in such a way that $\arg(b) > \arg(a)$, see figure 5. It

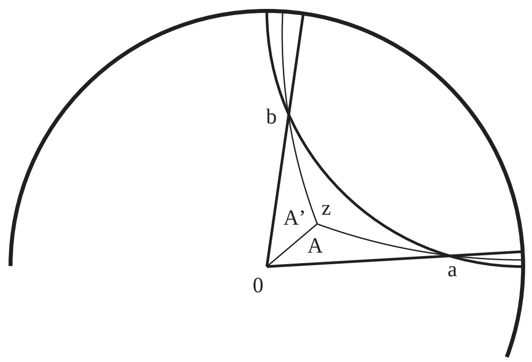

Figure 5:

then follows from the theorem that

$$e^{iA} = \frac{\bar{z}a - 1}{z\bar{a} - 1}, \quad e^{-iA'} = \frac{\bar{z}b - 1}{z\bar{b} - 1}. \tag{3}$$

Eliminating $\bar{z}$ and solving for z we get the inverse transformation

$$z = \frac{a(1 - e^{-iA'}) - b(1 - e^{iA})}{\bar{a}be^{iA} - a\bar{b}e^{-iA'}}. \tag{4}$$

3 The metric and the Laplace-Beltrami operator in barycentric coordinates

It appeared natural to use complex notation in the above discussion so we will continue with the derivation of the metric and the Laplace-Beltrami operator using complex notation.

Suppose we have two coordinate systems, $z = x + iy$ and $w = A' + iA$, and a non-holomorphic change of coordinates

$$z = z(w, \bar{w}) = \frac{a(1 - e^{-i(w+\bar{w})/2}) - b(1 - e^{(w-\bar{w})/2})}{\bar{a}be^{(w-\bar{w})/2} - a\bar{b}e^{-i(w+\bar{w})/2}}, \tag{5}$$

as follows from equation (4). Suppose also that we have the metric, the measure and the Laplace-Beltrami operator

$$ds^2 = \begin{pmatrix} dz & d\bar{z} \end{pmatrix} g_z \begin{pmatrix} d\bar{z} \\ dz \end{pmatrix} \tag{6}$$

$$d\mu = i\,\|g_z\|^{\frac{1}{2}}\,dz\wedge d\bar z \tag{7}$$

$$\Delta \equiv -\|g_z\|^{-\frac{1}{2}} \begin{pmatrix} \partial_{\bar z} & \partial_z \end{pmatrix} \|g_z\|^{\frac{1}{2}} g_z^{-1} \begin{pmatrix} \partial_z \\ \partial_{\bar z} \end{pmatrix} \tag{8}$$

in terms of the coordinates z. Here g_z is a 2×2 matrix (the metric tensor), $\|\cdot\|$ is the modulus of the determinant and dz, $d\bar z$, ∂_z, $\partial_{\bar z}$ are defined in the usual way

$$dz = dx + i dy\,,\quad d\bar z = dx - i dy$$
$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\,,\quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\,.$$

Then the metric in the w-coordinates is given by $g_w = J g_z J^\star$ where the Jacobian is

$$J = \begin{pmatrix} z_w & \bar z_w \\ z_{\bar w} & \bar z_{\bar w} \end{pmatrix}$$

and the derivatives follow from equation (5)

$$z_w = \frac{(i+1)(|a|^2 b - a|b|^2)e^{i(A-A')} - (a-b)(\bar a b e^{iA} + i a\bar b e^{-iA'})}{2\left(\bar a b e^{iA} - a\bar b e^{-iA'}\right)^2} \tag{9}$$

$$z_{\bar w} = \frac{(i-1)(|a|^2 b - a|b|^2)e^{i(A-A')} + (a-b)(\bar a b e^{iA} - i a\bar b e^{-iA'})}{2\left(\bar a b e^{iA} - a\bar b e^{-iA'}\right)^2}\,. \tag{10}$$

To get the measure and Laplace-Beltrami operator in terms of the new coordinates w we just make the substitution $g_z \to g_w$ and $z \to w$ in equations (7, 8).
In our case the metric g_z is a scalar matrix

$$g_z = \mathtt{g}_z\,\mathbb{I}\,,\quad \mathtt{g}_z = \frac{2}{(1-|z|^2)^2}$$

so that the above equations specialise to give the new metric, measure and Laplace-Beltrami operator

$$g_w = \frac{2}{(1-|z|^2)^2}\begin{pmatrix} |z_w|^2 + |z_{\bar w}|^2 & 2 z_w\overline{(z_{\bar w})} \\ 2\overline{(z_w)} z_{\bar w} & |z_w|^2 + |z_{\bar w}|^2 \end{pmatrix} \tag{11}$$

$$d\mu = i\,\frac{2\,\big||z_w|^2 - |z_{\bar w}|^2\big|}{(1-|z|^2)^2}\,dw\wedge d\bar w \tag{12}$$

$$\Delta \equiv -\frac{(1-|z|^2)^2}{2\,(|z_w|^2 - |z_{\bar w}|^2)} \begin{pmatrix} \partial_{\bar w} & \partial_w \end{pmatrix} \begin{pmatrix} \frac{|z_w|^2+|z_{\bar w}|^2}{|z_w|^2-|z_{\bar w}|^2} & -\frac{2 z_w\overline{(z_{\bar w})}}{|z_w|^2-|z_{\bar w}|^2} \\ -\frac{2\overline{(z_w)} z_{\bar w}}{|z_w|^2-|z_{\bar w}|^2} & \frac{|z_w|^2+|z_{\bar w}|^2}{|z_w|^2-|z_{\bar w}|^2} \end{pmatrix} \begin{pmatrix} \partial_w \\ \partial_{\bar w} \end{pmatrix}\,. \tag{13}$$

These plus equations (4, 9, 10) allows us (at least in principle) to write the Laplace-Beltrami operator in terms of A and A'.

It is clear that by making a trigonometric change of coordinates (to for instance $\cos(A)$, $\cos(A')$) the metric and the coefficients of the Laplace-Beltrami operator become algebraic functions of the coordinates. However, in terms of the coordinates $\cos(A)$ and $\cos(A')$ the triangular region in the hyperbolic plane is not mapped onto a Euclidean triangle as is the case with barycentric coordinates. To overcome this one can normalise the coordinates to A_n and A'_n so that they are in the interval $[0, \pi/2]$. Then in terms of $\cos(A'_n)$ and $\sin(A_n)$ the triangular region in the hyperbolic plane is mapped onto a Euclidean, right angled, isoceles triangle with side length one, and the metric and operator are expressed in terms of algebraic functions of the coordinates. We illustrate these ideas with the following example.

3.1 Example: $(\pi/2, \pi/n, \pi/n)$ triangular regions on the hyperbolic plane

Consider a triangle on the hyperbolic plane with angles $(\pi/2, \pi/n, \pi/n)$ at the vertices and n an integer strictly greater than 4. Then clearly the area of the triangle is

$$\text{Area} = \pi - (\pi/2 + \pi/n + \pi/n) = \frac{n-4}{n} \cdot \frac{\pi}{2}\,. \tag{14}$$

The two sides of the triangle of equal length have hyperbolic length α where [1]

$$\cosh(\alpha) = \frac{\cos(\pi/n)}{\sin(\pi/n)}\,.$$

If we place the vertex with the right angle at the origin of the Poincaré disc and the other two vertices on the *positive* real and imaginary axes then by construction

$$a = \tau\,, \quad b = i\tau$$

where ([1] pg 148) it is easily shown that

$$\tau = \tanh(\alpha/2) = \sqrt{\frac{\cos(\pi/n) - \sin(\pi/n)}{\cos(\pi/n) + \sin(\pi/n)}}\,.$$

Then equations (4), (9) and (10) give us

$$\begin{aligned}
z &= \frac{(1 - e^{-iA'}) - i(1 - e^{iA})}{i\tau\left(e^{iA} + e^{-iA'}\right)} \\
z_w &= \frac{\left(1 + (1+i)e^{iA}\right)\left(1 + (1-i)e^{-iA'}\right) - 1}{2\tau\left(e^{iA} + e^{-iA'}\right)^2} \\
z_{\bar{w}} &= i\frac{\left(1 - (1-i)e^{iA}\right)\left(1 - (1+i)e^{-iA'}\right) - 1}{2\tau\left(e^{iA} + e^{-iA'}\right)^2}
\end{aligned}$$

from which we get the metric g_w by equation (11).

We denote our final set of coordinates by

$$\zeta = u + iv = \cos\left(\frac{n \cdot A'}{n-4}\right) + i \sin\left(\frac{n \cdot A}{n-4}\right) . \qquad (15)$$

Then it is clear from the area formula (14) that the arguments of the cosine and sine take values in the interval $[0, \pi/2]$ and hence the $(\pi/2, \pi/n, \pi/n)$ triangular region is mapped onto the Euclidean triangle

$$\{u + iv;\ u, v \in [0,1],\ u \geq v\} .$$

It follows from the definition (15) that

$$\zeta_w = \frac{n}{2(n-4)} \left(\cos\left(\frac{n \cdot A}{n-4}\right) - \sin\left(\frac{n \cdot A'}{n-4}\right)\right) = \bar{\zeta}_{\bar{w}}$$
$$\zeta_{\bar{w}} = \frac{n}{2(n-4)} \left(-\cos\left(\frac{n \cdot A}{n-4}\right) - \sin\left(\frac{n \cdot A'}{n-4}\right)\right) = \bar{\zeta}_w$$

from which we can calculate the Jacobian

$$\mathcal{J} = \begin{pmatrix} w_\zeta & \bar{w}_\zeta \\ w_{\bar{\zeta}} & \bar{w}_{\bar{\zeta}} \end{pmatrix} = -\frac{n-4}{2n} \begin{pmatrix} (1-u^2)^{-\frac{1}{2}} - (1-v^2)^{-\frac{1}{2}} & (1-u^2)^{-\frac{1}{2}} + (1-v^2)^{-\frac{1}{2}} \\ (1-u^2)^{-\frac{1}{2}} + (1-v^2)^{-\frac{1}{2}} & (1-u^2)^{-\frac{1}{2}} - (1-v^2)^{-\frac{1}{2}} \end{pmatrix} .$$

We then find the new metric from

$$g_\zeta = \mathcal{J} g_w \mathcal{J}^\star$$

where we eliminate A and A' in g_w using

$$e^{iA} = \left(\sqrt{1-v^2} + iv\right)^{\frac{n-4}{n}} , \quad e^{-iA'} = \left(u - i\sqrt{1-u^2}\right)^{\frac{n-4}{n}} .$$

Then from g_ζ we can calculate the measure and Laplace-Beltrami operator from equations (7, 8) under the substitution $z \to \zeta$.

References

[1] A. F. Beardon. *The Geometry of Discrete Groups.* Springer-Verlag, Berlin, 1983.

[2] P. Kelly and G. Matthews. *The Non-Euclidean, Hyperbolic Plane.* Springer-Verlag, Berlin, 1981.