VOLTERRA SPACES ARE RE-VISITED

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Abstract. In this paper, we investigate weakly Volterra spaces and relevant topological properties. New characterizations of weakly Volterra spaces are provided. An analogy of the well-known Banach category theorem in terms of Volterra properties is achieved. It is shown that every weakly Volterra homogeneous space is Volterra, and there exists a metrizable Baire space whose hyperspace of nonempty compact subsets endowed with the Vietoris topology is not weakly Volterra.

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1. Introduction

Let $f : X \to Y$ be a function from one topological space X into another topological space Y. We shall denote by $C(f)$ (resp. $D(f)$) the set of points at which f is continuous (resp. discontinuous). Recall that f is said to be *pointwise discontinuous*, abbreviated as PWD, if $C(f)$ is dense in X. This class of functions was originally introduced by Hankel [7] in 1870, and uesd to be the main object of studies in the classical real function theory until the appearance of the works of Lebesgue. It can be shown that a function of a Baire space to a metric space is PWD if and only if $D(f)$ is of first category. In 1881, Volterra proved the following interesting result in [11]: *Let $f : \mathbb{R} \to \mathbb{R}$ be a PWD function. Then there exists no other PWD function $g : \mathbb{R} \to \mathbb{R}$ with $C(g) = D(f)$.* This idea and its generalizations have been studied in the last ten years by Gauld, Greenwood and Piotrowski in [2], [3], [4] and [5] respectively. Their work leads to the following definitions of Volterra and weakly Volterra spaces.

Definition 1.1. A topological space X is called *Volterra* (resp. *weakly Volterra*) if for each pair of real-valued PWD functions $f, g : X \to \mathbb{R}$, the set $C(f) \cap C(g)$ is dense (resp. nonempty) in X.

We notice that the range space $\mathbb{R}$ in Defintion 1.1 can be replaced by any developable space. Although Volterra and weakly Volterra spaces are defined in terms of "external" functions on them, there are some "internal" characterizations for these two classes of spaces as well, namely, *a space X is Volterra (resp. weakly Volterra) if and only if the intersection of any two*

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dense G_δ-sets in X is dense (resp. nonempty) [3]. Recall that a space is *Baire* (resp. *of second category*) if the intersection of any countably many dense open subsets is dense (resp. nonempty). Now, it is clear that every Baire space is Volterra, and every space of second category is weakly Volterra. Of course, all nonempty Baire spaces are of second category, and all nonempty Volterra spaces are weakly Volterra. In general, these four classes of spaces are all distinct, and relevant examples can be found in [3], [4], [5] and [6]. In answering a question in [4], Gruenhage and Lutzer [6] provided some natural classes of topological spaces in which a space is Volterra if and only if it is Baire. However, it is still an open question whether it is true that every Volterra Moore space is Baire (cf. Question 2.11 of [6]).

In this paper, we shall continue the study of Volterra and weakly Volterra spaces. In Section 2, new characterizations of weakly Volterra spaces are given, and an error in a result of [4] is corrected. In Section 3, an analogy of the well-known Banach category theorem is established. This enables us to discover a decomposition for an arbitrary topological space in terms of Volterra properties, and further prove that any weakly Volterra homogeneous space is Volterra. In the last section, we use a classical example of Fleissner and Kunen in [1] to show that in general, the property of being (weakly) Volterra is not preserved by the hyperspace of nonempty compact subsets of a given space endowed with the Vietoris topology.

All topological spaces are assumed T_1, although it is not always necessary. As usual, $\overline{A}$ and $\mathrm{int}A$ will denote the closure and interior of a subset A in a space X respectively. When X is a subspace of a topological space Y, we shall use $\overline{A}^X$ and $\mathrm{int}_X A$ to denote the closure and interior of a subset A in the subspace X respectively. For a cardinal κ, $\mathrm{cf}(\kappa)$ denotes the cofinality of κ, and κ^+ will represent the next cardinal after κ. The symbol ${}^A B$ stands for the set of all functions from a set A to a set B. We refer the readers to [8] for basic facts and undefined notation about Baire spaces.

2. A question related to weakly Volterra spaces

In this section, we first correct an error in a resut of Gauld, Greenwood and Piotrowski in [4], and then provide some new characterizations of weakly Volterra spaces. The following result can be found in [4].

Theorem 2.1. [4] *If X is a Volterra space, $Y_1, ..., Y_n$ ($n \in \mathbb{N}$) are developable spaces and $f_i : X \to Y_i$ ($i \leq n$) are PWD functions, then $\bigcap\{C(f_i) : 1 \leq i \leq n\}$ is dense in X.*

In the light of Theorem 2.1, it is natural and also interesting to consider the following question.

Question 2.2. *Is it true that for any weakly Volterra space X, any developable spaces $Y_1, ..., Y_n$ ($n \geq 3$) and any PWD functions $f_i : X \to Y_i$ ($1 \leq i \leq n$), $\bigcap\{C(f_i) : 1 \leq i \leq n\} \neq \emptyset$?*

This question has been already considered in [4] and a negative answer was provided there. More precisely, a weakly Volterra space X and three real-valued functions $f, g, h : X \to \mathbb{R}$ such that $C(f)$, $C(g)$ and $C(h)$ are dense G_δ-sets of X, but $C(f) \cap C(g) \cap C(h) = \emptyset$, were constructed in Example 3 of [4]. Unfortunately, this example is false.

Example 2.3. *The space X in Example 3 of* [4] *is not weakly Volterra.*

Proof. First, we shall briefly describe the space presented in [4]. Let $A = \{\langle x, y\rangle \in \mathbb{R}^2 : y \geq 0\}$. For each $r \geq 0$, let $A_r = \{\langle x, y\rangle \in \mathbb{R}^2 : y + r > 0\}$. Define B, B_r to be the sets obtained by rotating A, A_r $120°$ about $\langle 0, 0\rangle$ anti-clockwise, and C, C_r by a similar rotation clockwise. Let

$$D = (A_0 \cap B_0) \cup (B_0 \cap C_0) \cup (C_0 \cap A_0), \text{ and}$$

$$E = (A_0 \setminus (B \cup C)) \cup (B_0 \setminus (C \cup A)) \cup (C_0 \setminus (A \cup B)).$$

Furthermore, let us define

$$\mathcal{B}_1 = \{(A_r \cap B_s \cap C_t \cap D) \setminus F : r, s, t > 0 \text{ and } F \subseteq \mathbb{R}^2 \text{ is finite}\},$$

$$\mathcal{B}_2 = \{(A_r \cap B_s \cap C_t) \setminus F : r, s, t > 0 \text{ and } F \subseteq \mathbb{R}^2 \text{ is finite}\}, \text{ and}$$

$$\mathcal{B}_3 = \{(A_r \cap B_s \cap C_t \cap E) \setminus F : r, s, t > 0 \text{ and } F \subseteq \mathbb{R}^2 \text{ is finite}\}.$$

Then the space X considered in Example 3 of [4] is $\mathbb{R}^2$ endowed with the topology generated by $\bigcup\{\mathcal{B}_i : 1 \leq i \leq 3\}$ as a base.

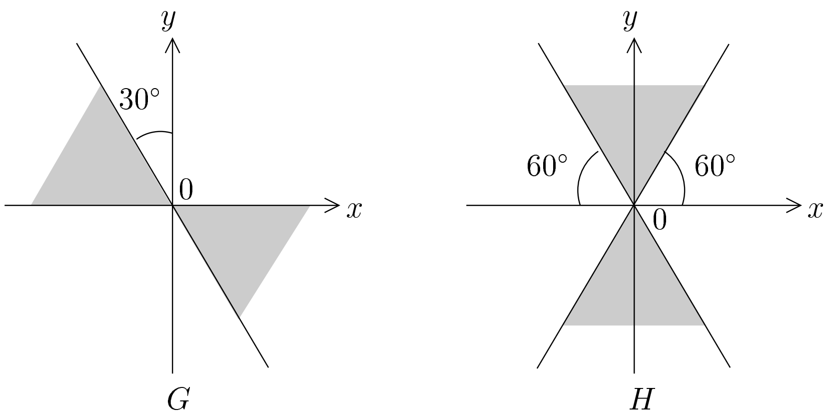

FIGURE 1

Figure 2.1.

It is clear that A, B, C are dense G_δ-sets of X. In addition, it can be checked easily that both $A_0 \cap B_0$ and $C \setminus (A \cup B)$ are G_δ-sets of X (But, they are not dense in X). Now, consider the two subsets G and H of X

shown in Figure 2.1 above as the two shaded regions without including their boundaries. These two sets can be defined by the following formulae

$$G = (A_0 \cap B_0) \cup (C \setminus (A \cup B)), \text{ and } H = (B_0 \cap C_0) \cup (A \setminus (B \cup C)).$$

It is not difficult to see that G is dense in X. Being the union of two G_δ-sets in X, G is also a G_δ-set of X. Thus, G is a dense G_δ-set in X. Similarly, H is also a dense G_δ-set of X. However, it is obvious that $G \cap H = \emptyset$. Therefore, we have verified that the space X is not weakly Volterra. □

To answer Question 2.2, we shall first provide some new characterizations of weakly Volterra spaces.

Theorem 2.4. *The following statements are equivalent for a space X:*

(a) X is a weakly Volterra space.

(b) The intersection of any finitely many dense G_δ-sets of X is somewhere dense in X.

(c) The intersection of any finitely many dense G_δ-sets of X is not empty.

Proof. It is clear that $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$.

We shall prove $(a) \Rightarrow (b)$ by induction. Suppose X is weakly Volterra. First, for any two dense G_δ-sets A_1, A_2 of X, we define $B_1 = A_1 \setminus \overline{A_1 \cap A_2}$ and $B_2 = A_2 \setminus \overline{A_1 \cap A_2}$. It is obvious that $B_1 \cap B_2 = \emptyset$. Since A_1 and A_2 are dense in X, we have $\overline{B_1} = X \setminus \mathrm{int}\overline{A_1 \cap A_2}$, and $\overline{B_2} = X \setminus \mathrm{int}\overline{A_1 \cap A_2}$. If $\mathrm{int}\overline{A_1 \cap A_2} = \emptyset$, then B_1 and B_2 are two dense G_δ-sets of X which are disjoint. This is a contradiction. Therefore, we have shown that the intersection of any two dense G_δ-sets of X is somewhere dense in X.

Next, suppose it has been shown that the intersection of any i many dense G_δ-sets of X is somewhere dense in X, where $1 \le i \le n$ and $n \ge 3$. Let $A_1, ..., A_{n+1}$ be $n+1$ many dense G_δ-sets of X. Then, by our induction hypothesis, $\mathrm{int}\overline{\bigcap\{A_i : 1 \le i \le n\}} \ne \emptyset$. For each $1 \le j \le n-1$, let us define the subset $C_j \subset X$ by

$$C_j = (A_j \setminus \overline{\bigcap\{A_i : 1 \le i \le n\}}) \cup (\bigcap\{A_i : 1 \le i \le n\}).$$

Furthermore, we define the set $C_n \subset X$ by the following

$$C_n = (A_n \setminus \overline{\bigcap\{A_i : 1 \le i \le n\}}) \cup (A_{n+1} \cap \mathrm{int}\overline{\bigcap\{A_i : 1 \le i \le n\}}).$$

Now for every $1 \le j \le n-1$, since A_j is dense in X, we have

$$\begin{aligned}\overline{C_j} &= \overline{X \setminus \overline{\bigcap\{A_i : 1 \le i \le n\}}} \cup \overline{\bigcap\{A_i : 1 \le i \le n\}} \\ &= (X \setminus \mathrm{int}\overline{\bigcap\{A_i : 1 \le i \le n\}}) \cup \overline{\bigcap\{A_i : 1 \le i \le n\}} \\ &= X.\end{aligned}$$

Thus, all the sets C_j $(1 \le j \le n-1)$ are dense G_δ-sets of X. Similarly, one can check C_n is also a dense G_δ-set in X. Moreover, it is easy to see that

$$\bigcap\{C_j : 1 \leq j \leq n\} \subset \bigcap\{A_i : 1 \leq i \leq n+1\}.$$

By our induction hypothesis again, $\bigcap\{C_j : 1 \leq j \leq n\}$ is somewhere dense in X, then so is $\bigcap\{A_i : 1 \leq i \leq n+1\}$. □

Our next result shall provide an affirmative answer to Question 2.2.

Corollary 2.5. *Let X be a weakly Volterra space, $Y_1, ..., Y_n$ $(n \in \mathbb{N})$ developable spaces and $f_i : X \to Y_i$ $(1 \leq i \leq n)$ PWD functions. Then $\bigcap\{C(f_i) : 1 \leq i \leq n\} \neq \emptyset$.*

Proof. It is easy to see that each $C(f_i)$ $(1 \leq i \leq n)$ is a dense G_δ-set of X. Hence, by Theorem 2.4, we obtain $\bigcap\{C(f_i) : 1 \leq i \leq n\} \neq \emptyset$. □

3. Homogeneous spaces

A space X is said to be *homogeneous* if for any two distinct points $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$. In this section, the following main theorem shall be proved.

Theorem 3.1. *Let X be a homogeneous space. Then X is Volterra if and only if it is weakly Volterra.*

To achieve this goal, we shall first study non-weakly Volterra subspaces in a given space. It is shown that the rôle of non-weakly Volterra subspaces in the theory of Volterra spaces is somehow similar to that of first category sets in the theory of Baire spaces. Then the proof of Theorem 3.1 follows from a number of lemmas, which are interesting for their own sake.

Lemma 3.2. *If a space X contains a nonempty weakly Volterra open subspace Y, then X itself is weakly Volterra.*

Proof. Suppose that U and V are any two dense G_δ-sets in X. Then $U \cap Y$ and $V \cap Y$ are two dense G_δ-sets in the subspace Y. Since Y is weakly Volterra, then $U \cap V \supset (U \cap V) \cap Y \neq \emptyset$. Hence, X is weakly Volterra. □

Remark 3.3. In Lemma 3.2, "Y is open" can be replaced with a weaker condition " there exists a G_δ-set H in Y such that int$H$ is dense in Y".

Lemma 3.4. [5] *A space is Volterra if and only if every nonempty open subspace is weakly Volterra.*

Lemma 3.5. *If a space X contains a dense open subspace that is not weakly Volterra, then X itself is not weakly Volterra.*

Proof. Let $Y \subset X$ be a dense open subspace that is not weakly Volterra. Then there are two disjoint dense G_δ-sets U and V in Y. Pick two dense G_δ-sets $\hat{U}$ and $\hat{V}$ in X with $U = \hat{U} \cap Y$ and $V = \hat{V} \cap Y$. Suppose that X is weakly Volterra. Then, by Theorem 2.4, int$\overline{\hat{U} \cap \hat{V}} \neq \emptyset$. It follows that $Y \cap (\hat{U} \cap \hat{V}) \neq \emptyset$. This is a contradiction, since $U \cap V = \emptyset$. □

Since every non-weakly Volterra subspace in a topological space must be a set of first category, our next lemma can be treated as an analogy of the famous Banach category theorem in topology and analysis.

Lemma 3.6. *In any space X, the union of any family of nonempty open non-weakly Volterra subspaces is not weakly Volterra.*

Proof. Let $\mathcal{U}$ be a family of nonempty open subspaces of X such that each member of $\mathcal{U}$ is not weakly Volterra in X. Let $\Im_{NV}$ be the set of all collections of nonempty open subsets of X with the following two properties:

(a) each collection $\mathcal{V} \in \Im_{NV}$ is pairwise disjoint; and

(b) for each collection $\mathcal{V} \in \Im_{NV}$ and each member $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ such that $V \subset U$.

Then, by Zorn's lemma, $\Im_{NV}$ has a maximal element $\mathcal{V} = \{V_\alpha : \alpha \in A\}$. Let $V = \bigcup\{V_\alpha : \alpha \in A\}$. By the maximality of $\mathcal{V}$, we have $\bigcup\{U : U \in \mathcal{U}\} \subset \overline{V}$. Moreover, it follows from (b) and Lemma 3.2 that for each $\alpha \in A$, V_α is not weakly Volterra as an open subspace of X. Thus, there are two families $\{F_\alpha : \alpha \in A\}$ and $\{H_\alpha : \alpha \in A\}$ of G_δ-sets of X such that

(c) $F_\alpha \cap H_\alpha = \emptyset$ for all $\alpha \in A$; and

(d) $F_\alpha \subset V_\alpha \subset \overline{F_\alpha}$ and $H_\alpha \subset V_\alpha \subset \overline{H_\alpha}$ for all $\alpha \in A$.

Let $F = \bigcup\{F_\alpha : \alpha \in A\}$ and $H = \bigcup\{H_\alpha : \alpha \in A\}$. By ($a$) and ($c$), $F \cap H = \emptyset$. For each $\alpha \in A$, let $F_\alpha = \bigcap\{F_\alpha^n : n \geq 1\}$ and $H_\alpha = \bigcap\{H_\alpha^n : n \geq 1\}$, where F_α^n and H_α^n are nonempty open subsets of X contained in V_α such that $F_\alpha^{n+1} \subset F_\alpha^n$ and $H_\alpha^{n+1} \subset H_\alpha^n$ for all $\alpha \in A$ and all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, put $F_n = \bigcup\{F_\alpha^n : \alpha \in A\}$ and $H_n = \bigcup\{H_\alpha^n : \alpha \in A\}$. Simple computation yields $F = \bigcap\{F_n : n \geq 1\}$ and $H = \bigcap\{H_n : n \geq 1\}$. By ($d$), we have $V_\alpha \subset \overline{F_\alpha}^V$ and $V_\alpha \subset \overline{H_\alpha}^V$ for each $\alpha \in A$. Since $\{F_\alpha : \alpha \in A\}$ and $\{H_\alpha : \alpha \in A\}$ are two discrete families in the subspace V of X, then $V \subset \bigcup\{\overline{F_\alpha}^V : \alpha \in A\} = \overline{F}^V$ and $V \subset \bigcup\{\overline{H_\alpha}^V : \alpha \in A\} = \overline{H}^V$. Thus, F and H are two disjoint dense G_δ-sets in the subspace V of X. Consequently, V is not a weakly Volterra subspace of X. It follows from Lemma 3.5 that $\overline{V}$ is not a weakly Volterra subspace of X. Since $\bigcup\{U : U \in \mathcal{U}\} \subset \overline{V}$, by Lemma 3.2 again, $\bigcup\{U : U \in \mathcal{U}\}$ is not a weakly Volterra subspace of $\overline{V}$. Therefore, $\bigcup\{U : U \in \mathcal{U}\}$ is not a weakly Volterra subspace of X. $\square$

As an immediate application of Lemma 3.6, we obtain the following decomposition lemma for an arbitrary topological space.

Lemma 3.7. *Let X be an arbitrary topological space. Then there are two open (possibly empty) subspaces X_{NV} and X_V of X such that*

(a) *$X = \overline{X_{NV}} \cup \overline{X_V}$ and $X_{NV} \cap X_V = \emptyset$;*

(b) *every nonempty open subspace of X_{NV} is not weakly Volterra in X; and*

(c) *every nonempty open subspace of X_V is Volterra in X.*

Furthermore, X is a Volterra space if and only $X_{NV} = \emptyset$, and X is a weakly Volterra space if and only if $X_V \neq \emptyset$.

Proof. Let X_{NV} be the union of all nonempty open non-weakly Volterra subspaces of X, and let $X_V = X \setminus \overline{X_{NV}}$. By Lemma 3.6, X_{NV} is not weakly Volterra as an open subspace of X. It is obvious that every nonempty open subspace of X_V is weakly Volterra. Thus, following from Lemma 3.4, every nonempty open subspace of X_V is Volterra. So, we have shown that X_{NV} and X_V fufil (a), (b) and (c). By Lemma 3.4 again, X is Volterra if and only if $X_{NV} = \emptyset$. If $X_V \neq \emptyset$, then X_V is a weakly Volterra subspace of X. By Lemma 3.2, the space X itself is weakly Volterra. Conversely, suppose that X is weakly Volterra, and $X_V = \emptyset$. Then $X = \overline{X_{NV}}$. Since X_{NV} is not weakly Volterra, then by Lemma 3.5, the space X itself is not weakly Volterra either. This is a contradiction. $\square$

Now we are able to show Theorem 3.1 by applying the previous lemmas.

Proof of Theorem 3.1. The necessity is trivial. To prove the sufficiency, suppose that X is a weakly Volterra space. Then, by Lemma 3.7, X_V is a nonempty open Volterra subspace of X. Now, let U be any nonempty open subspace of X. Then there exists a point $x \in X_V$ and a homeomorphism $f : X \to X$ such that $f(x) \in U$. The space $U \cap f(X_V)$, being a nonempty open subspace of the Volterra space $f(X_V)$, is also Volterra. Thus, it follows from Lemma 3.2 that U is a weakly Volterra subspace of X. Finally, by Lemma 3.4, the space X itself is Volterra. $\square$

It is well-known that a homogeneous space is Baire if and only if it is of second category. In the next example, we give a simple T_1 homogeneous space which is Volterra, but is not Baire.

Example 3.8. *There is a T_1 homogeneous Volterra space that is not Baire.* Let $X = \mathbb{R}$ be the set of all reals. Let $\mathcal{T}_1$ be the lower topology on X, that is, $\mathcal{T}_1 = \{\emptyset, X\} \cup \{(a, +\infty) : a \in X\}$. Let $\mathcal{T}_2$ be the co-countable topology on X. Equipe X with the topology $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$. Then X is a T_1 homogeneous space. Every dense G_δ-set A of X can be expressed by either $A = X \setminus S$, or $A = (a, +\infty) \setminus S$, or $A = [a, +\infty) \setminus S$, where $a \in X$ and $S \subset X$ is countable. Hence, the intersection of any finitely many dense G_δ-sets of X meets every nonempty member of $\mathcal{T}$. It follows that X is Volterra. On the other hand, X is not Baire, because the subsets $U_n = (n, +\infty)$ of X are all open and dense but their intersection over $\mathbb{N}$ is empty. $\square$

By Lemma 3.7, a nonempty space is not weakly Volterra if and only if no nonempty open subspace is weakly Volterra. Our next result, which says *a semi-open subspace of a given space is not weakly Volterra if and only if it is nowhere weakly Volterra*, is a slight extension of this fact. Recall that a set A of a space X is *semi-open* if $\mathrm{int}A$ is dense in A. It is clear that in any topological space, all open subspaces are semi-open.

Theorem 3.9. *Let A be a nonempty semi-open subspace of a space X. Then A is not weakly Volterra in X if and only if for every open subset U of X*

with $U \cap A \neq \emptyset$, there exists a nonempty open subset V of X contained in U such that $V \cap A$ is not weakly Volterra in X.

Proof. The necessity follows from Lemma 3.2 directly. So, we shall consider the sufficiency. First, suppose that A is a nowhere dense subset of X. Let U and V be any two dense G_δ-sets in A. If A is weakly Volterra, then by Theorem 2.4, $U \cap V$ is a somewhere dense set in the subspace A. We shall derive a contradiction. Let G be any nonempty open subset of A, and let H be an open subset of X with $G = H \cap A$. Then $H \cap \mathrm{int}A \neq \emptyset$, as $\mathrm{int}A$ is dense in A. Since A is a nowhere dense set of X, then $U \cap V$ is a nowhere dense subset of X as well. Thus, there exists a nonempty open subset O of X contained in $H \cap \mathrm{int}A$ such that $O \cap (U \cap V) = \emptyset$. This shows that $U \cap V$ is a nowhere dense set in the subspace A, which is a contradiction. Hence, A is not weakly Volterra in this case.

Next, we shall consider the case that A is a somewhere dense subset of X. Let $U = \mathrm{int}\overline{A}$. Then U is a nonempty open subset of X. Let $\mathcal{U} = \{U_\beta : \beta \in B\}$ be the family of all nonempty open subsets of X such that for each $\beta \in B$, $U_\beta \subset U$ and $U_\beta \cap A$ is not a weakly Volterra subspace of X. Note that for each $\beta \in B$, $U_\beta \cap A$ is not weakly Volterra as an open subspace of the subspace A. It follows from Lemma 3.6 that $\bigcup\{U_\beta \cap A : \beta \in B\}$ is not weakly Volterra in the subspace A. By hypothesis, $\bigcup\{U_\beta \cap A : \beta \in B\}$ is a dense open subspace of $A \cap U$. Hence, it follows from Lemma 3.5 that $A \cap U$ cannot be weakly Volterra. Furthermore, since $\mathrm{int}A \subset A \cap U$, by Lemma 3.2, $\mathrm{int}A$ is not weakly Volterra in X. Finally, as $\mathrm{int}A$ is dense and open in the subspace A, by Lemma 3.5 again, we conclude that A is not a weakly Volterra subspace of X. $\square$

Note that the condition "A is semi-open" in Theorem 3.9 is not needed in the proof of necessity. However, the authors don't know whether this condition can be dropped from the proof of the sufficiency.

4. The hyperspace of nonempty compact subsets of a space

Given a topological space X, let $\mathcal{K}(X)$ denote the collection of all nonempty compact subsets of X. For any finite family $\mathcal{U} = \{U_1, U_2, \cdots, U_n\}$ of subsets of X, we define $\langle \mathcal{U} \rangle \subseteq \mathcal{K}(X)$ by

$$\langle \mathcal{U} \rangle = \{F \in \mathcal{K}(X) : F \subset \bigcup\{U_i : 1 \leq i \leq n\}, \text{ and}$$
$$F \cap U_i \neq \emptyset \text{ for all } i \leq n\}.$$

In what follows, $\mathcal{K}(X)$ shall be equipped with the so-called *Vietoris topology* τ_v, which has the family of all subsets of $\mathcal{K}(X)$ of the form $\langle \mathcal{U} \rangle$ as a base, where $\mathcal{U}$ runs through all finite families of open subsets of X.

Proposition 4.1. *For any topological space X, if $\mathcal{K}(X)$ is Volterra (resp. weakly Volterra) then X is Volterra (resp. weakly Volterra).*

Proof. First, it is routine to check that the following hold for any family $\{B_\alpha : \alpha \in A\}$ of subsets of X:

(a) $\langle \bigcap\{B_\alpha : \alpha \in B\}\rangle = \bigcap\{\langle B_\alpha\rangle : \alpha \in A\}$; and

(b) for any $\alpha \in A$, B_α is dense (resp. nonempty) in X if and only if $\langle B_\alpha\rangle$ is dense (resp. nonempty) in $\mathcal{K}(X)$.

Now suppose that $\mathcal{K}(X)$ is Volterra (resp. weakly Volterra). Let U and V be two dense G_δ-sets in X. Then $\langle U\rangle$ and $\langle V\rangle$ are dense G_δ-sets in $\mathcal{K}(X)$. Since $\mathcal{K}(X)$ is Volterra (resp. weakly Volterra), then $\langle U\rangle \cap \langle V\rangle$ is dense (resp. nonempty) in $\mathcal{K}(X)$. By (a) and (b) above, $U \cap V$ is dense (resp. nonempty) in X. Therefore, X is Volterra (resp. weakly Volterra). □

Next, we shall show that the converse of Proposition 4.1 does not hold in general by using a space constructed by Fleissner and Kunen in [1].

Example 4.2. *There exists a metric Baire space X such that $\mathcal{K}(X)$ is not even weakly Volterra.*

Proof. For any cardinal $\kappa > \omega$, let $C_\omega\kappa = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$. For any $f \in {}^\omega\kappa$, let $f^* = \sup\{f(n) : n \in \omega\}$. Next, define a metric ρ on ${}^\omega\kappa$ by

$$\rho(f,g) = \begin{cases} 0, & \text{if } f = g; \\ \frac{1}{2^n}, & \text{if } f \neq g, \text{ where } n = \min\{m \in \omega : f \restriction m \neq g \restriction m\}. \end{cases}$$

Then the metric space $({}^\omega\kappa, \rho)$ is simply denoted by J_κ. Let $M = J_2 \times J_{\mathfrak{c}^+}$ be given the product metric d, i.e., $d(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \rho_1(x_1, x_2) + \rho_2(y_1, y_2)$. Now, let $\{A_y : y \in J_2\}$ be a family of pairwise disjoint stationary subsets of $C_\omega\mathfrak{c}^+$. Consider the subspace $X = \{\langle y, f\rangle \in M : f^* \in A_y\}$ of M. It is shown in Example 4 of [1] that X is a Baire space.

Claim 1. X^2 is not a weakly Volterra space.

For each $n \in \omega$, let

$$U_n = \{\langle\langle y_1, f_1\rangle, \langle y_2, f_2\rangle\rangle \in X^2 : y_1 \neq y_2, \text{ and}$$
$$f_1(n) < f_2^*, f_2(n) < f_1^*\}.$$

Each U_n is open in X^2, so the set $U_E = \bigcap\{U_{2n} : n \in \omega\}$ is a G_δ-set in X^2. In fact, U_E is also dense in X^2. To see this, suppose given any point $\langle\langle y_1, f_1\rangle, \langle y_2, f_2\rangle\rangle \in X^2$ and any two positive integers m_1 and m_2. We shall show $V \cap U_E \neq \emptyset$, where

$$V = \{\langle\langle z_1, g_1\rangle, \langle z_2, g_2\rangle\rangle \in X^2 : z_i \restriction m_i = y_i \restriction m_i, g_i \restriction m_i = f_i \restriction m_i\}$$

is an arbitrary basic open set in X^2. Without loss of generality, we may assume $m_1 \leq m_2$. Choose $\alpha_i \in A_{y_i}$ $(i = 1, 2)$ such that for any $i = 1, 2$,

$$a_i > \sup(\{f_1(n) : n \leq m_1\} \cup \{f_2(n) : n \leq m_2\}).$$

For each $i = 1$, or 2, let us define $g_i : \omega \to \mathfrak{c}^+$ as follows:

$$g_i(n) = \begin{cases} f_i(n), & \text{if } n \leq m_i; \\ f_i(1), & \text{if } n > m_i \text{ and } n \text{ is even}; \\ \alpha_i, & \text{if } n > m_i \text{ and } n \text{ is odd}. \end{cases}$$

Then $\langle\langle y_1, g_1\rangle, \langle y_2, g_2\rangle\rangle \in V \cap U_E$. In a similar way, it can be shown that $U_O = \bigcap\{U_{2n+1} : n \in \omega\}$ is also a dense G_δ-set of X^2. However $U_E \cap U_O = \emptyset$, because if we had $\langle\langle y_1, f_1\rangle, \langle y_2, f_2\rangle\rangle \in \bigcap\{U_n : n \in \omega\}$ then $y_1 \neq y_2$ and for all $n \in \omega$ we would have $f_1(n) < f_2^*$ so that $f_1^* \leq f_2^*$. By symmetry, $f_1^* = f_2^*$, which is contrary to $A_{y_1} \cap A_{y_2} = \emptyset$.

Claim 2. $\mathcal{K}(X)$ *is not weakly Volterra.*

To show this, we need to introduce some auxiliary tools. For any finite family $\mathcal{U} = \{U_1, U_2, \cdots, U_n\}$ of subsets of X, let $\mathcal{U}^* \subset X^\omega$ be defined by

$$\mathcal{U}^* = \prod\{U_i : 1 \leq i \leq n\} \times \prod\{\bigcup\{U_j : 1 \leq j \leq n\} : i > n\}.$$

Let X^ω be equipped with a topology τ^* by taking

$$\mathfrak{S} = \{\mathcal{U}^* : \mathcal{U} \text{ is a finite family of open subsets of } X\}$$

as a base. We denote the space X^ω with this topology by X^ω_*. Suppose that $\mathcal{K}(X)$ is weakly Volterra. Let $\mathcal{F}(X)$ be the family of all finite subsets of X. Then $\mathcal{F}(X)$ is a subset of $\mathcal{K}(X)$. Moreover, $\mathcal{F}(X)$ with the relative topology is a dense metrisable subspace of $\mathcal{K}(X)$. By an argument similar to that in Proposition 2.6 of [6], we conclude that $\mathcal{K}(X)$ is of second category. Then it follows from Theorem 3.10 of [9] that X^ω_* is also a space of second category. Since the canonical projection mapping $\pi : X^\omega_* \to X^2$, defined by $\pi(\langle x_i\rangle) = \langle x_1, x_2\rangle$ for all $\langle x_i\rangle \in X^\omega_*$, is an open and continuous surjection, then X^2 is of second category. But, Claim 1 shows that X^2 is not even weakly Volterra. This is a contradiction. $\square$

Remark 4.3. For any given space X, the associated space X^ω_*, or a more general space X^κ_* (where $\kappa \geq \omega$), has been studied in [9] and [10] respectively. In particular, τ^* is called *the pinched-cube topology* in [10].

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