COUNTABLE COMPACTNESS OF HYPERSPACES AND GINSBURG'S QUESTIONS

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Abstract. In this paper, we study the countable compactness and pseudocompactness of the hyperspace 2^X of a Hausdorff space X consisting of all nonempty closed subsets of X equipped with the Vietoris topology. Some open questions posed by Ginsburg in 1975 are considered. In particular, we give partial solutions to one of them.

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1. Introduction

All topological spaces are assumed to be Hausdorff. Given a space X, 2^X denotes the collection of all nonempty closed subsets of X. One of the most important and well-studied topologies on 2^X is the Vietoris topology τ_v, which is also known as the finite topology. To describe this topology, we need some notation. For a subset E of X, let $E^- = \{A \in 2^X : A \cap E \neq \emptyset\}$, and $E^+ = \{A \in 2^X : A \subseteq E\}$. Then τ_v has as a subbase all subsets of 2^X of the forms U^- and V^+, where U and V are open subsets of X. For any finite family $\mathcal{V}$ of subsets of X, let

$$\langle \mathcal{V} \rangle = \{F \in 2^X : F \subseteq \bigcup \mathcal{V},\ F \cap V \neq \emptyset \text{ for any } V \in \mathcal{V}\}.$$

It is a well-known fact that the collection of all subsets of 2^X of the form $\langle \mathcal{V} \rangle$, where $\mathcal{V}$ is a finite family of open subsets of X, is a base for τ_v. From now on, the hyperspace 2^X of X will always carry the topology τ_v.

One of the fundamental problems in the theory of hyperspaces is to decide how a topological property of X can be transfered to 2^X and vice versa. For instance, the famous Vietoris-Michael theorem [10, Theorem 4.2] asserts that a space X is compact if and only if 2^X is compact (See [2], [8], [9] and [10] for more results of this type). So, it is quite natural to ask the following question: *What can we say about the hyperspace 2^X of a countably compact or pseudocompact space X? Are there any analogs to the Vietoris-Michael theorem for countable compactness or pseudocompactness?* A space X is *countably compact* if every infinite subset has an accumulation point; and X is *pseudocompact* if every continuous real-valued function on it is bounded.

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When it is Tychonoff, a space is pseudocompact if and only if every sequence of nonempty open subsets has an accumulation point. In 1975, Ginsburg [6] considered the above question and discovered that the countable compactness (pseudocompactness) of 2^X has some nice connections to the countable compactness (pseudocompactness) of powers of X. Note that neither countable compactness nor pseudocompactness is (finitely) multiplicative in the realm of Tychonoff spaces as Novák [11] and Terasaka [13] showed independently. What Ginsburg proved are in fact the following:

(i) If all powers of X are countably compact, 2^X is countably compact;

(ii) If 2^X is countably compact, all finite powers of X are countably compact.

A result similar to (ii) also holds for pseudocompact Tychonoff spaces. In addition to these mentioned results, Ginsburg also discovered that one of Frolik's examples in [5] gives us a Tychonoff space X all of whose finite powers are countably compact but whose hyperspace 2^X is not pseudocompact. However, he was unable to solve the following question.

Question 1.1. [6, Remark 3.2] *Is there any relation between the pseudocompactness (countable compactness) of X^ω and that of 2^X?*

The main purpose of this paper is to consider Question 1.1 and problems related to it. Some partial solutions to this long-standing open question shall be given. More precisely, we give some counterexample to this question in one direction in Section 2, and show some positive results in the other direction in Section 3. Furthermore, we also show that under **MA**, $2^{\mathfrak{c}}$ is the best possible cardinal for powers of a countably compact space X to guarantee the countable compactness of 2^X.

2. Some Counterexamples

In this section, we give a general approach to produce examples of countably compact spaces whose hyperspaces with τ_v are not countably compact. Given a family $\{X_i : i \in A\}$ of countably compact spaces, let $\bigoplus\{X_i : i \in A\}$ be the disjoint union of $X_i's$. We define a countably compact space X as follows:

(i) If $|A| < \omega$, let $X = \bigoplus\{X_i : i \in A\}$;

(ii) If $|A| \geq \omega$, we pick up an arbitrary point $\infty \notin \bigoplus\{X_i : i \in A\}$, and then endow $X = \bigoplus\{X_i : i \in A\} \cup \{\infty\}$ with a topology such that each X_i with its original topology is clopen in X, and such that every neighbourhood of ∞ contains all but finitely many of the X_i. In this case, the space X is usually called the *one-point countable compactification* of $\bigoplus\{X_i : i \in A\}$. It can be checked easily that X is regular (Tychonoff) if and only if all $X_i's$ are regular (Tychonoff).

Theorem 2.1. *Let $\{X_i : i \in A\}$ be a family of countably compact spaces, and let X be the space defined above. If 2^X is countably compact, then the product space $\prod\{X_i : i \in A\}$ is countably compact.*

To prove Theorem 2.1, we need the following lemma which shall be used in the next section too.

Lemma 2.2. *Let X be a space, and let $\alpha > 0$ be a cardinal. If there are two α-sequences $(U_\xi : \xi < \alpha)$ and $(V_\xi : \xi < \alpha)$ of nonempty open sets in X; and a closed discrete sequence $(x_n : n < \omega)$ of points in the α-th power X^α of X, where $x_n = \langle x_n(\xi)\rangle_{\xi<\alpha}$ for each $n < \omega$, such that*
(i) $(U_\xi : \xi < \alpha)$ is pairwise disjoint,
(ii) $\overline{V}_\xi \subseteq U_\xi$ for all $\xi < \alpha$, and
(iii) $x_n(\xi) \in V_\xi$ for all $\xi < \alpha$ and all $n < \omega$,
then 2^X is not countably compact.

Proof. For each $n < \omega$, let $\overline{x}_n$ denote the closure of $\{x_n(\xi) : \xi < \alpha\}$ in X. We shall show that $(\overline{x}_n : n < \omega)$ has no accumulation points in 2^X, and thus 2^X is not countably compact. Suppose that it has an accumulation point $F \in 2^X$. We first establish that $|F \cap U_\xi| = 1$ for every $\xi < \alpha$. If this is not true, then there are two possible cases for us to consider.

Case (1). There exists some $\xi' < \alpha$ such that $F \cap U_{\xi'} = \emptyset$. In this case, $F \in (X \setminus \overline{V}_{\xi'})^+$. Thus, $\overline{x}_n \in (X \setminus \overline{V}_{\xi'})^+$ for infinitely many $n < \omega$. This is impossible, as $x_n(\xi') \in V_{\xi'}$ for all $n < \omega$.

Case (2). There exists some $\xi'' < \alpha$ such that $|F \cap U_{\xi''}| > 1$. In this case, F must meet two disjoint open sets $G_0, G_1 \subseteq U_{\xi''}$. It follows that $(G_0)^- \cap (G_1)^-$ is a τ_v-open neighborhood of F, and thus $\overline{x}_n \in (G_0)^- \cap (G_1)^-$ for infinitely many $n < \omega$. This implies that for infinitely many $n < \omega$, we have both $\{x_n(\xi) : \xi < \alpha\} \cap G_0 \neq \emptyset$ and $\{x_n(\xi) : \xi < \alpha\} \cap G_1 \neq \emptyset$ simultaneously. Again, it is impossible, since every x_n has exactly one coordinate contained in $U_{\xi''}$.

Now, let $F \cap U_\xi = \{x(\xi)\}$ for each $\xi < \alpha$. Then, we define a point $x \in X^\alpha$ such that $x = \langle x(\xi)\rangle_{\xi<\alpha}$. Next, we take an arbitrary basic open neighborhood

$$O = \prod_{i=0}^{k} O_{\xi_i} \times \prod\{X_\xi : \xi \in \alpha \setminus \{\xi_0, \xi_1 ..., \xi_k\}\}$$

of the point x in X^α, where $k < \min\{\alpha, \omega\}$, each O_{ξ_i} $(i \leq k)$ is an open neighborhood of $x(\xi_i)$ in X and $X_\xi = X$ for all $\xi \in \alpha \setminus \{\xi_0, \xi_1, ..., \xi_k\}$. Define

$$\mathcal{W} = \bigcap_{i=0}^{k} (U_{\xi_i} \cap O_{\xi_i})^-.$$

Then $\mathcal{W}$ is a τ_v-open neighborhood of F. Since F is an accumulation point of $(\overline{x}_n : n < \omega)$, for any $n < \omega$, there is a $j_n \geq n$ such that $\overline{x}_{j_n} \in \mathcal{W}$. Consequently,

$$\{x_{j_n}(\xi) : \xi < \alpha\} \cap (U_{\xi_i} \cap O_{\xi_i}) \neq \emptyset$$

for all $i \leq k$. It follows that $x_{j_n}(\xi_i) \in O_{\xi_i}$ for all $i \leq k$. Therefore, $x_{j_n} \in O$, and thus x is an accumulation point of $(x_n : n < \omega)$ in X^α. However, this is a contradiction, since $(x_n : n < \omega)$ is a closed discrete sequence in X^α. $\square$

Proof of Theorem 2.1. Suppose that $\prod\{X_i : i \in A\}$ is not countably compact. Then there exists a closed discrete sequence $(x_n : n < \omega)$ in $X^{|A|}$, where $x_n = \langle x_n(i)\rangle_{i\in A}$ for each $n < \omega$, such that $x_n(i) \in X_i$ for all $n < \omega$ and all $i \in A$. Note that each X_i is clopen in X. If we take $U_i = V_i = X_i$ for all $i \in A$, and apply Lemma 2.2, then we can conclude that 2^X is not countably compact. $\square$.

The next example gives a negative answer to Question 1.1 for countable compactness in one direction even when X^{ω_1} is countably compact.

Example 2.3. *There exists a countably compact Tychonoff space X such that $X^{\mathfrak{t}}$ is countably compact but 2^X is not countably compact.* For each $\mathcal{D} \in \omega^* = \beta\omega \setminus \omega$, let $X_{\mathcal{D}} = \beta\omega \setminus \{\mathcal{D}\}$ be endowed with the relative topology of $\beta\omega$. Let X be the one-point countable compactification of the disjoint union $\bigoplus\{X_{\mathcal{D}} : \mathcal{D} \in \omega^*\}$. Then X is totally countably compact (A space X is *totally countably compact* if every sequence of points in X has a subsequence which is contained in a compact subset of X). By [14, Theorem 3.3], $X^{\mathfrak{t}}$ is countably compact (Recall that $\mathfrak{t}$ is the minimal cardinality of towers and $\mathfrak{t} \geq \omega_1$). Let $Z = \prod\{X_{\mathcal{D}} : \mathcal{D} \in \omega^*\}$. For each $n < \omega$, let z_n denote the point in Z defined by $z_n(\mathcal{D}) = n$ for all $\mathcal{D} \in \omega^*$. Since Z contains the infinite closed discrete sequence $(z_n : n < \omega)$, it could not be countably compact. By Theorem 2.1, we conclude that 2^X is not countably compact. $\square$

Let $\mathcal{D} \in \omega^*$. We say a point $x \in X$ a $\mathcal{D}$-*limit* of a sequence $(A_n : n < \omega)$ of subsets of X if $\{n < \omega : A_n \cap U \neq \emptyset\} \in \mathcal{D}$ for each open neighborhood U of x. A space X is said to be $\mathcal{D}$-*compact* ($\mathcal{D}$-*pseudocompact*) [1], [6] provided that every sequence of points (nonempty open subsets) in X has a $\mathcal{D}$-limit. Every $\mathcal{D}$-compact ($\mathcal{D}$-pseudocompact) space is countably compact (pseudocompact).

Theorem 2.4. [7] *A space X is $\mathcal{D}$-compact for some $\mathcal{D} \in \omega^*$ if and only if $X^{2^{\mathfrak{c}}}$ is countably compact.*

It is shown in [6] that for a space X, 2^X is $\mathcal{D}$-compact ($\mathcal{D}$-pseudocompact) if and only if X is $\mathcal{D}$-compact ($\mathcal{D}$-pseudocompact). It follows immediately that 2^X is countably compact provided that $X^{2^{\mathfrak{c}}}$ is countably compact. Now, a natural question which arises from this fact is:

Question 2.5. *Is $2^{\mathfrak{c}}$ the best possible cardinal for the power of a countably compact space X to guarantee the countable compactness of 2^X?*

The answer to Question 2.5 is "yes" by Example 2.3, if we assume the first two steps of **GCH** (i.e., $\mathfrak{c} = 2^{\omega} = \omega_1$, and $2^{\omega_1} = \omega_2$). To get a better consistency answer, we need an extra concept. Recall that an ultrafilter $\mathcal{D} \in \omega^*$ is said to be a *selective ultrafilter*, or a *Ramsey ultrafilter* [3] if whenever $(A_n : n < \omega)$ is a countable decomposition of ω and $A_n \notin \mathcal{D}$ for all $n < \omega$, there is $A \in \mathcal{D}$ with $|A \cap A_n| \leq 1$ for all $n < \omega$.

Example 2.6. (**MA**) *There exists a Tychonoff space X such that for every $\alpha < 2^{\mathfrak{c}}$, X^α is countably compact but 2^X is not countably compact.* Assuming **MA**, Saks [12, Theorem 2.5] showed that there exists a $2^{\mathfrak{c}}$-sequence $(X_\varsigma : \varsigma < 2^{\mathfrak{c}})$ of subspaces of $\beta\omega$ such that for every proper $I \subset 2^{\mathfrak{c}}$, there is some selective ultrafilter $\mathcal{D} \in \omega^*$($\mathcal{D}$ depends on I) such that the partial product $\prod\{X_\varsigma : \varsigma \in I\}$ is $\mathcal{D}$-compact, but the full product $\prod\{X_\varsigma : \varsigma < 2^{\mathfrak{c}}\}$ is not countably compact. Let X be the one-point countable compactification of $\bigoplus\{X_\varsigma : \varsigma < 2^{\mathfrak{c}}\}$.

Since $\prod\{X_\varsigma : \varsigma < 2^{\mathfrak{c}}\}$ is not countably compact, by Theorem 2.1, 2^X is not countably compact. Hence, we are left to show that for any cardinal $\alpha < 2^{\mathfrak{c}}$, the α-th power X^α of X is countably compact. To this end, fix an $\alpha < 2^{\mathfrak{c}}$, and pick up an arbitrary sequence $(x_n : n < \omega)$ in X^α, where $x_n = \langle x_n(\xi)\rangle_{\xi<\alpha}$ for all $n < \omega$. We shall deduce that $(x_n : n < \omega)$ has an accumulation point in X^α. Note that a sequence in a product space has a $\mathcal{D}$-limit for some $\mathcal{D} \in \omega^*$ if and only if its projection on each factor has a $\mathcal{D}$-limit (with respect to the same $\mathcal{D}$). Thus, it suffices to prove that there is some $\mathcal{D} \in \omega^*$ such that for any $\xi < \alpha$, $(x_n(\xi) : n < \omega)$ has a $\mathcal{D}$-limit in X. To do so, we first choose a proper subset $I \subset 2^{\mathfrak{c}}$ such that

$$\{x_n(\xi) : \xi < \alpha, n < \omega\} \subseteq \bigcup\{X_\zeta : \zeta \in I\} \cup \{\infty\}.$$

Let $\mathcal{D} \in \omega^*$ be a selective ultrafilter such that $\prod\{X_\zeta : \zeta \in I\}$ is $\mathcal{D}$-compact. For a fixed $\xi < \alpha$, define $A_\xi = \{n < \omega : x_n(\xi) \neq \infty\}$. Since $\mathcal{D}$ is an ultrafilter, either

(i) $\omega \setminus A_\xi \in \mathcal{D}$, or

(ii) $A_\xi \in \mathcal{D}$.

If (i) holds, then ∞ is the $\mathcal{D}$-limit of $(x_n(\xi) : n < \omega)$ in X. If (ii) holds, we first define a countable subset

$$I(\xi) = \{\varsigma \in I : \{x_n(\xi) : n \in A_\xi\} \cap X_\varsigma \neq \emptyset\}$$

of I and then decompose ω into a countable disjoint union as

$$\omega = (\omega \setminus A_\xi) \cup \bigcup\{\{n \in A_\xi : x_n(\xi) \in X_\varsigma\} : \varsigma \in I(\xi)\}.$$

Now, we have the following two subcases:

(ii_a) There exists some $\varsigma \in I(\xi)$ with $\{n \in A_\xi : x_n(\xi) \in X_\varsigma\} \in \mathcal{D}$. In this case, X_ς is $\mathcal{D}$-compact, thus $(x_n(\xi) : n < \omega)$ has a $\mathcal{D}$-limit in $X_\varsigma \subset X$.

(ii_b) For every $\varsigma \in I(\xi)$, $\{n \in A_\xi : x_n(\xi) \in X_\varsigma\} \notin \mathcal{D}$. In this case, $|I(\xi)| = \omega$. Since $\mathcal{D}$ is a selective ultrafilter, there exists some $B_\xi \in \mathcal{D}$ and an injective mapping $f : B_\xi \to I(\xi)$ with $x_n(\xi) \in X_{f(n)}$ for all $n \in B_\xi$. It is easy to see that ∞ is the $\mathcal{D}$-limit of $(x_n(\xi) : n < \omega)$ in X.

It follows that in any case, $(x_n : n < \omega)$ has a $\mathcal{D}$-limit in X^α, which implies that it has an accumulation point in X^α. Therefore, X^α is countably compact. $\square$

It remains an open question whether the answer to Question 2.5 is affirmative in **ZFC**.

3. Positive Results for Homogeneous Spaces

In this section, we shall provide some positive answers to Question 1.1 in some special cases. Given a space X, let id_X be the identity mapping on X. The family of all homeomorphisms of X onto itself will be denoted by $\mathrm{Aut}(X)$. Recall that X is said to be *homogeneous* if for any two distinct points $x, y \in X$ there exists an $f \in \mathrm{Aut}(X)$ such that $f(x) = y$.

Theorem 3.1. *Let X be a regular homogeneous space. If 2^X is countably compact, then X^ω is countably compact.*

Proof. If X is finite, we have nothing to prove. So, we assume that X is infinite. Then, by the homogeneity of X and the countable compactness of 2^X, X must have no isolated points. By [6, Corollary 2.3], X is countably compact.

To prove that X^ω is countably compact, let $(x_n : n < \omega)$ be an arbitrary sequence in X^ω, where $x_n = \langle x_n(k) \rangle_{k<\omega}$ for each $n < \omega$. We shall deduce that $(x_n : n < \omega)$ has an accumulation point in X^ω. Let $z(0)$ be an accumulation point of the sequence $(x_n(0) : n < \omega)$. Then choose two open subsets $V_0, U_0 \subseteq X$ such that $z(0) \in V_0 \subseteq \overline{V}_0 \subseteq U_0$ and $\overline{U}_0 \neq X$. Next, we define

$$A_0 = \{n < \omega : f_0(x_n(0)) \in V_0\},$$

where $f_0 = \mathrm{id}_X$ (for the sake of unification). Let $z(1)$ be an accumulation point of the sequence $(x_n(1) : n \in A_0)$. Select an $f_1 \in \mathrm{Aut}(X)$ and two open subsets $V_1, U_1 \subseteq X$ such that $f_1(z(1)) \in V_1 \subseteq \overline{V}_1 \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and $\overline{U}_0 \cup \overline{U}_1 \neq X$ (In case $z(1) \notin \overline{U}_0$, then just take $f_1 = \mathrm{id}_X$). Then we define

$$A_1 = \{n \in A_0 : f_1(x_n(1)) \in V_1\}.$$

Since X has no isolated points, we can continue this process inductively, and thus obtain infinite sequences $(z(j) : j < \omega)$, $(U_j : j < \omega)$, $(V_j : j < \omega)$, $(A_j : j < \omega)$ and $(f_j : j < \omega)$ such that

(i) $f_j \in \mathrm{Aut}(X)$ for all $j < \omega$,

(ii) $(U_j : j < \omega)$ is a sequence of pairwise disjoint nonempty open sets in X with $\bigcup_{j \leq n} \overline{U}_j \neq X$ for all $n < \omega$,

(iii) $(V_j : j < \omega)$ is a sequence of nonempty open sets in X with $\overline{V}_j \subseteq U_j$ for all $j < \omega$,

(iv) for each $j < \omega$, A_j is an infinite subset of ω and $A_{j+1} \subseteq A_j$,

(v) $A_{j+1} = \{n \in A_j : f_{j+1}(x_n(j+1)) \in V_{j+1}\}$ for all $j < \omega$,

(vi) for each $j < \omega$, $z(j)$ is an accumulation point of $(x_n(j) : n \in A_j)$, and

(vii) $f_j(z(j)) \in U_j$ for all $j < \omega$.

Now, one can pick an infinite set $A = \{a_j : j < \omega\} \subseteq \omega$ such that $a_j \in A_j$ for all $j < \omega$. Then, by (iv), $|A \setminus A_j| < \omega$ for all $j < \omega$. For each $n < \omega$ and $j < \omega$, let $y_{a_n}(j) = f_j(x_{a_n}(j))$ and $y_{a_n} = \langle y_{a_n}(j)\rangle_{j<\omega}$.

Next, we shall show that the sequence $(y_{a_n} : n < \omega)$ has an accumulation point in X^ω by using Lemma 2.2. To this end, we choose an infinite sequence $(t_{a_n} : n < \omega)$ in X^ω such that for each $j < \omega$, $|\{n \in \omega : t_{a_n}(j) \neq y_{a_n}(j)\}| < \omega$, and $t_{a_n}(j) \in V_j$ for all $n, j < \omega$. This is possible, since for each $j < \omega$, V_j is infinite, and $y_{a_n}(j) \in V_j$ for all but finitely many $n < \omega$. It is easy to see that $(t_{a_n} : n < \omega)$ has an accumulation point in X^ω if and only if $(y_{a_n} : n < \omega)$ does. Since 2^X is countably compact, the sequence $(t_{a_n} : n < \omega)$ has an accumulation point in X^ω, which is also an accumulation point of $(y_{a_n} : n < \omega)$.

Let $y = \langle y(j)\rangle_{j<\omega}$ be any accumulation point of $(y_{a_n} : n < \omega)$ in X^ω. It can be easily checked that $x = \langle f_j^{-1}(y(j))\rangle_{j<\omega}$ is an accumulation point of $(x_{a_n} : n < \omega)$ in X^ω, and thus is also an accumulation point of $(x_n : n < \omega)$ in X^ω. Therefore, X^ω is countably compact. □

By replacing the closed discrete sequence of points in Lemma 2.2 with a suitable sequence of open sets, we obtain the following lemma.

Lemma 3.2. *Let X be a space without isolated points. For all $j, n < \omega$, let V_j, U_j and O_{nj} be nonempty open subsets of X satisfying*

(i) $(U_j : j < \omega)$ is pairwise disjoint,

(ii) $\overline{V}_j \subseteq U_j$ for all $j < \omega$, and

(iii) for each $j < \omega$, $O_{nj} \subseteq V_j$ for all but finitely many $n < \omega$.

Let $(k_n : n < \omega)$ be a strictly increasing sequence in ω. For each $n < \omega$, define

$$O_n = \textstyle\prod_{j=0}^{k_n} O_{nj} \times \prod\{X_j : j > k_n\},$$

where $X_j = X$ for every $j > k_n$. If 2^X is pseudocompact, then $(O_n : n < \omega)$ has an accumulation point in X^ω.

Proof. After making some adjustment to the matrix (O_{nj}) of open sets in a manner similar to what we have done in Theorem 3.1, we may require that $O_{nj} \subseteq V_j$ for all $n, j < \omega$. Let $\mathcal{G}_n = \{O_{n0}, ..., O_{nk_n}\}$ for each $n < \omega$. Since 2^X is pseudocompact, $(\langle \mathcal{G}_n\rangle : n < \omega)$ has an accumulation point $F \in 2^X$. We claim that $F \cap \overline{V}_j \neq \emptyset$ for all $j < \omega$. Suppose the contrary. Then $F \cap \overline{V}_{j_0} = \emptyset$ for some $j_0 < \omega$. It follows that $F \in (X \setminus \overline{V}_{j_0})^+$. Hence, we obtain an infinite subset

$$I = \{n > j_0 : \langle \mathcal{G}_n\rangle \cap (X \setminus \overline{V}_{j_0})^+ \neq \emptyset\}$$

of ω. Pick any $n \in I$ and an arbitrary $F_n \in \langle \mathcal{G}_n\rangle \cap (X \setminus \overline{V}_{j_0})^+$. Then, we have $\emptyset \neq F_n \cap O_{nj_0} \subseteq V_{j_0}$. But this is impossible, since $F_n \in (X \setminus \overline{V}_{j_0})^+$.

Next, select a point $x(j) \in F \cap \overline{V}_j$ for each $j < \omega$, and let $x = \langle x(j) \rangle_{j<\omega}$. We show that x is an accumulation point of $(O_n : n < \omega)$ in X^ω. To do this, let

$$W = \prod_{j=0}^{k} W_j \times \prod\{X_j : j > k\}$$

be an arbitrary basic open neighborhood of x in X^ω, where W_j $(j \leq k)$ is an open subset in X with $x(j) \in W_j$ and $X_j = X$ for all $j > k$. Let $\mathcal{W} = \bigcap_{j=0}^{k} (U_j \cap W_j)^-$. Then $\mathcal{W}$ is a τ_v-open neighborhood of F. Since F is an accumulation point of the sequence $(\langle \mathcal{G}_n \rangle : n < \omega)$, we have an infinite subset $J = \{n > k : \langle \mathcal{G}_n \rangle \cap \mathcal{W} \neq \emptyset\}$ of ω. For any $n \in J$, we pick up an arbitrary $H_n \in \langle \mathcal{G}_n \rangle \cap \mathcal{W}$, and decompose H_n into a finite disjoint union as $H_n = \bigcup_{j=0}^{k_n} (H_n \cap O_{nj})$. Since

$$(U_j \cap W_j) \cap (\bigcup\{H_n \cap O_{ni} : i \neq j, 0 \leq i \leq k_n\}) = \emptyset$$

and $(U_j \cap W_j) \cap H_n \neq \emptyset$, then $(H_n \cap O_{nj}) \cap (U_j \cap W_j) \neq \emptyset$ for every $j \leq k$. Thus $O_n \cap W \neq \emptyset$ for all $n \in J$, and x is an accumulation point of $(O_n : n < \omega)$. $\square$

The proof our next theorem is similar to that of Theorem 3.1.

Theorem 3.3. *Let X be a Tychonoff homogeneous space. If 2^X is pseudocompact, then X^α is pseudocompact for any cardinal α.*

Proof. As we have done in Theorem 3.1, to avoid triviality, we assume that X is infinite, and thus it has no isolated points. By [6, Corollary 2.7], all finite powers of X are peudocompact. It suffices to show that X^ω is pseudocompact, since an infinite power X^α of a Tychonoff space X is pseudocompact if and only if X^ω is pseudocompact. To do so, let $(G_n : n < \omega)$ be a sequence of nonempty basic open sets of X^ω. For each $n < \omega$, we may assume that

$$G_n = \prod_{i=0}^{k_n} G_{ni} \times \prod\{X_i : i > k_n\},$$

where each $k_n < \omega$, each G_{ni} $(i \leq k_n)$ is a nonempty open subset of X and $X_i = X$ for every $i > k_n$. Without loss of generality, we may require that $(k_n : n < \omega)$ is strictly increasing (In fact, for each $n < \omega$, we can always add some more open sets G_{ni} with $G_{ni} = X$ if necessary). Let $z(0)$ be an accumulation point of the sequence $(G_{n0} : n < \omega)$. By the regularity of X, we can choose two open sets V_0, U_0 of X such that $z(0) \in V_0 \subseteq \overline{V}_0 \subseteq U_0$, and $\overline{U}_0 \neq X$. Then, we define

$$A_0 = \{n < \omega : f_0(G_{n0}) \cap V_0 \neq \emptyset\},$$

where $f_0 = \mathrm{id}_X$. For each $n \in A_0$, let $O_{n0} = f_0(G_{n0}) \cap V_0$. Let $z(1)$ be an accumulation point of the sequence $(G_{n1} : n \in A_0)$. Select an $f_1 \in \mathrm{Aut}(X)$ and two open subsets V_1, U_1 of X such that $f_1(z(1)) \in V_1 \subseteq \overline{V}_1 \subseteq U_1$,

$U_0 \cap U_1 = \emptyset$ and $\overline{U}_0 \cup \overline{U}_1 \neq X$. (In case that $z(1) \notin \overline{U}_0$, then just take $f_1 = \mathrm{id}_X$). Let

$$A_1 = \{n \in A_0 : f_1(G_{n1}) \cap V_1 \neq \emptyset\}.$$

For each $n \in A_1$, let $O_{n1} = f_1(G_{n1}) \cap V_1$. Since X has no isolated points, one can continue this process inductively, which yields infinite sequences $(z(j) : j < \omega)$, $(V_j : j < \omega)$, $(U_j : j < \omega)$, $(A_j : j < \omega)$, $(f_j : j < \omega)$ and $(O_{nj} : n \in A_j)$ (for each $j < \omega$) such that

(i) $f_j \in \mathrm{Aut}(X)$ for all $j < \omega$,

(ii) $(U_j : j < \omega)$ is a sequence of pairwise disjoint nonempty open sets in X with $\bigcup_{j \leq n} \overline{U}_j \neq X$ for all $n < \omega$,

(iii) $(V_j : j < \omega)$ is a sequence of nonempty open sets in X with $\overline{V}_j \subseteq U_j$ for all $j < \omega$,

(iv) for each $j < \omega$, A_j is an infinite subset of ω and $A_{j+1} \subseteq A_j$,

(v) $A_{j+1} = \{n \in A_j : f_{j+1}(G_{n(j+1)}) \cap V_{j+1} \neq \emptyset\}$ for all $j < \omega$,

(vi) for each $j < \omega$, $z(j)$ is an accumulation point of $(G_{nj} : n \in A_j)$,

(vii) $f_j(z(j)) \in V_j$ for all $j < \omega$, and

(viii) for each $j < \omega$, $O_{nj} = f_j(G_{nj}) \cap V_j$ for all $n \in A_j$.

Now, we can pick an infinite set $A = \{a_j : j < \omega\} \subseteq \omega$ such that $a_j \in A_j$ for every $j < \omega$. For each $n < \omega$, let $O_{a_n} \subseteq X^\omega$ be defined by

$$O_{a_n} = \textstyle\prod_{j=0}^{n} O_{a_n j} \times \prod_{j=n+1}^{k_{a_n}} f_j(G_{a_n j}) \times \prod\{X_j : j > k_{a_n}\},$$

where $X_j = X$ for every $j > k_{a_n}$. Moreover, define $Q_{a_n} \subseteq X^\omega$ as

$$Q_{a_n} = \textstyle\prod_{j=0}^{k_{a_n}} f_j(G_{nj}) \times \prod\{X_j : j > k_{a_n}\}$$

for each $n < \omega$, where $X_j = X$ for all $j > k_{a_n}$.

It is clear that $(U_j : j < \omega)$, $(V_j : j < \omega)$, $(O_{a_n j} : n, j < \omega)$ and $(O_{a_n} : n < \omega)$ satisfy conditions in Lemma 3.2. By the pseudocompactness of 2^X, $(O_{a_n} : n < \omega)$ has an accumulation point in X^ω, say $y = \langle y(j) \rangle_{j<\omega}$. Since $O_{a_n} \subseteq Q_{a_n}$ for all $n < \omega$, y is also an accumulation point of $(Q_{a_n} : n < \omega)$. It follows that the point $x = \langle f_j^{-1}(y(j)) \rangle_{j<\omega}$ is an accumulation point of $(G_{a_n} : n < \omega)$ in X^ω, thus is an accumulation point of $(G_n : n < \omega)$. Therefore, X^ω is pseudocompact. □

In contrast to counterexamples in Section 2, Theorem 3.1 and Theorem 3.3 give positive answers to Question 1.1 for countable compactness and pseudocompactness respectively in the class of homogeneous Tychonoff spaces in the other direction.

4. Additional Remarks

In [6], Ginsburg also posed the following interesting question.

Question 4.1. *Characterize those spaces X whose hyperspaces 2^X are countable compact (pseudocompact).*

In 1998, Natsheh considered Question 4.1 and claimed to provide a sufficient condition for 2^X to be pseudocompact (see *Questions Answers Gen. Topology*, **16** (1998), 213–217 for details). According to Natsheh, a subset C in a topological space X is called a *C_δ-set* if there exists a sequence $(V_n : n < \omega)$ of nonempty open sets in X such that $C = \bigcap_{n<\omega} \overline{V}_n$. What he claimed to prove is the following: *If X is a pseudocompact normal space and for each sequence $(C_n : n < \omega)$ of C_δ-sets in X there exists an $F \in 2^X$ such that $F \cap C_n \neq \emptyset$ for all $n < \omega$ and $F \subseteq \bigcup_{n<\omega} C_n$, then 2^X is pseudocompact.*

Remark 4.2. *There is a gap in Natsheh's proof.* In fact, he used the following false statement in his argument: A Tychonoff space X is pseudocompact if and only if $\bigcap_{n<\omega} \overline{G}_n \neq \emptyset$ for every non-increasing sequence $(G_n : n < \omega)$ of nonempty "basic" open sets of X. Therefore, Question 4.1 remains open.

Remark 4.3. *Theorem 3.3 is still valid when X is regular and pseudocompactness is replaced by feeble compactness.* Recall that a space X is *feebly compact* if every locally finite family of nonempty open sets is finite (Note that feebly compact spaces are called *$\mathcal{G}$-pseudocompact* in [6]). Every feebly compact space is pseudocompact. For Tychonoff spaces, these two notions are equivalent. When X is Tychonoff, 2^X is pseudocompact if and only if 2^X is feebly compact [6, Proposition 2.6].

Remark 4.4. *There are no analogs to Theorem 2.4 for pseudocompactness.* Consider the space Z in Example 2.3. Given any cardinal α, we can re-write Z^α as

$$Z^\alpha = \prod\{X_{\langle \mathcal{D},\xi\rangle} : \langle \mathcal{D},\xi\rangle \in \omega^* \times \alpha\}.$$

Pick an arbitrary point $x_* \in Z^\alpha$ such that $x_*(\langle \mathcal{D},\xi\rangle) \in \omega$ for all $\langle \mathcal{D},\xi\rangle \in \omega^* \times \alpha$. Then $\Sigma(x_*)$, the Σ-product with the base point x_*, is dense in Z^α. Let $(x_n : n < \omega)$ be any sequence in $\Sigma(x_*)$, and $A = \bigcup_{n<\omega} \operatorname{supp}(x_n)$, where

$$\operatorname{supp}(x_n) = \{\langle \mathcal{D},\xi\rangle \in \omega^* \times \alpha : x_n(\langle \mathcal{D},\xi\rangle) \neq x_*(\langle \mathcal{D},\xi\rangle)\}.$$

Since $\prod\{X_{\langle \mathcal{D},\xi\rangle} : \langle \mathcal{D},\xi\rangle \in A\}$ is countably compact, $(\pi_A(x_n) : n < \omega)$ has an accumulation point, where $\pi_A : Z^\alpha \to \prod\{X_{\langle \mathcal{D},\xi\rangle} : \langle \mathcal{D},\xi\rangle \in A\}$ is the projection. So, $(x_n : n < \omega)$ has an accumulation point in Z^α. This implies that Z^α is pseudocompact. But, Z is not $\mathcal{D}$-pseudocompact for any $\mathcal{D} \in \omega^*$. Unfortunately, we don't know whether 2^Z is pseudocompact or not. In fact, it is still an unsolved problem that whether 2^X is pseudocompact for a Tychonoff space X whenever all powers of X are pseudocompact.

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