Obstructions to directed embeddings of Eulerian digraphs in the plane

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January 23, 2003

Abstract

A 2-cell embedding of an Eulerian digraph in a closed surface is said to be directed if the boundary of each face is a directed closed walk in G. We prove Kuratowski-type theorems about obstructions to directed embeddings of Eulerian digraphs in the plane.

1 Introduction

Unless stated otherwise, all digraphs considered will be connected but may have loops as well as parallel (that is, multiple) arcs. For any two vertices u and v of a digraph G, the symbol $\overrightarrow{uv}$ will denote the set of all arcs in G that originate from u and terminate at v (shortly, $u \to v$ arcs, or arcs from u to v); $\overrightarrow{uu}$ is simply the set of all loops at u. We sometimes write uv for an arc belonging to $\overrightarrow{uv}$. For an arc $a \in \overrightarrow{uv}$, the *contraction* of a results in the digraph, denoted by G/a, that is obtained from G by identifying the vertices u and v, discarding a

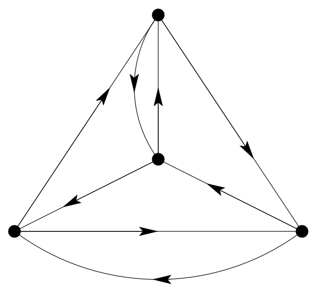

Figure 1: A directed planar embedding of an Eulerian digraph.

from the arc set, and forming loops out of all arcs in $\overrightarrow{uv}$ and $\overrightarrow{vu}$. If u and v are distinct vertices, and a and b are arcs of G such that $a \in \overrightarrow{uv}$ and $b \in \overrightarrow{vu}$ then the set $\{a, b\}$ is called a *digon* (*between u and v*). If there is a third arc $c \neq a, b$ between u and v, we say that the digon $\{a, b\}$ is *braced* (by c). A pair of parallel arcs are said to form a *bad digon.* The justification for the use of the adjective "bad" will become evident.

A digraph is *Eulerian* if at each vertex, the indegree and outdegree are the same. (Eulerian digraphs have a directed closed walk that uses every arc exactly once.) We say that an Eulerian digraph G is *directed planar* if G can be embedded (that is, "drawn" without crossings) in the plane in such a way that the boundary walk of each face is a *directed* closed walk in G. (See [1] for a discussion of directed embeddings of Eulerian digraphs in other surfaces.) Such an embedding is then called a *directed planar embedding* of G. For example, Figure 1 gives the essentially unique directed planar embedding of an Eulerian digraph with four vertices and eight arcs.

Observe that in a directed planar embedding of an Eulerian digraph, at each vertex the arcs pointing into the vertex have to alternate with those pointing out. Further, faces of a directed planar embedding fall into two classes according to the orientation of their boundary walks (clockwise and counterclockwise). Equivalently, the faces of a directed embedding can be properly two-coloured – say, white and black – such that the directed boundary walks of all black (white) faces are oriented clockwise (counterclockwise).

In the context of directed embeddings it is natural to introduce a partial order on the set of all Eulerian digraphs in such a way that the order "respects" the embeddings in some sense. We shall therefore say that an Eulerian digraph H is a *weak minor* of an Eulerian digraph G if H can be obtained from G by a non-empty sequence of the following operations:

- Contraction of an arc.
- Deletion of a loop.
- Discarding a digon.

It is obvious that, in a directed planar embedding of an Eulerian digraph G, a contraction of any arc a results again in a directed planar embedding of G/a.

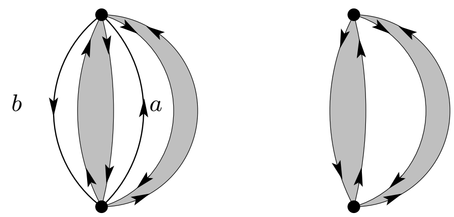

Figure 2: $G - \{a, b\}$ is directed planar if G is directed planar.

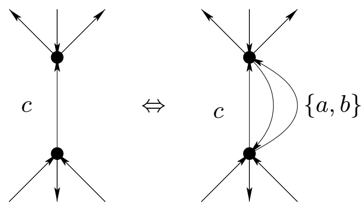

Figure 3: If $\{a, b\}$ is a braced digon, then G is directed planar if and only if $G - \{a, b\}$ is directed planar.

If a directed planar embedding of G contains a loop a then its deletion leads to a directed planar embedding of $G - a$ (we note that this is true even if the loop a did not bound a face in the original embedding). In fact, G is directed planar if and only if $G - a$ is directed planar. For digons, we have a similar situation:

Lemma 1 *Let $\{a, b\}$ denote a digon in an Eulerian digraph G. If G is planar, then $G - \{a, b\}$ is planar. Additionally, if $\{a, b\}$ is braced by an arc c, then $G - \{a, b\}$ is directed planar if and only if G is directed planar.*

Proof. If G is directed planar, then a directed planar embedding of $G - \{a, b\}$ is obtained by reversing (if necessary) the ordering of a subsequence of arcs at both u and v (see Figure 2.) (This type of operation is sometimes refered to as a "Whitney 2-flip".)

Suppose $G - \{a, b\}$ is directed planar and $\{a, b\}$ is braced by an arc c. Then we may introduce the digon $\{a, b\}$ into the planar embedding by placing it alongside the arc c while preserving directed planarity (see Figure 3.) $\square$

By the above, we see that directed planarity is preserved under the weak minor ordering. An Eulerian digraph G is said to be an *obstruction* to directed planarity (*under the weak minor order*) if G does not have a directed planar embedding yet each of its weak minors does.

It is immediate from Lemma 1 and the discussion preceeding that an obstruction G has no loops or braced digons. However G may have parallel arcs (bad digons) as we will discover.

Before we proceed to the presentation of the obstructions, we rule out another type of substructure from all directed planar digraphs:

Lemma 2 *Suppose G is a Eulerian digraph with a pair of bad digons $\{uv, uv\}$ and $\{wu, wu\}$ meeting a vertex u, where v and w are distinct vertices, and suppose further that u has no other incident arcs (that is, $indeg(u) = outdeg(u) = 2$). Then G is directed non-planar.*

Proof. The arcs incident with u must alternate cyclically (uv, wu, uv, wu). If G were planar, then the bad digon $\{uv, uv\}$ would form a closed curve in the plane. By the Jordan curve theorem, the pair of arcs $\{wu, wu\}$ must both lie on the same side on the bad digon $\{uv, uv\}$, forcing the four arcs at u to violate the directed embedding requirement. □

2 An infinite family of obstructions under the weak minor order

In this section we identify a particular infinite family of obstructions to directed planarity under the weak minor order. We begin with a number of preliminary results.

For any digraph G we denote by $\hat{G}$ the underlying simple undirected graph obtained from G by ignoring edge directions and deleting multiple edges and loops. We say a digraph G is *k-connected* if $\hat{G}$ is k-connected. This definition is motivated by the observation that a directed planar digraph G has essentially a unique directed embedding (up to the placement of the loops) if and only if $\hat{G}$ is 3-connected. (Recall that an undirected planar 3-connected graph has a unique embedding in the plane.)

Lemma 3 *Every obstruction under the weak minor order is 3-connected.*

Proof. The arguments are routine (in essence the same as when reducing the classical Kuratowski Theorem to 3-connected graphs, see [4]) and we leave them to the reader. □

For any $s \geq 2$, let K_3^s denote the digraph on the three vertices u, v, w with exactly s arcs from u to v, from v to w, and from w to u. (See Figure 4.) Clearly, K_3^s is an obstruction under the weak minor order for each $s \geq 2$. Indeed, contraction of any arc yields a directed planar graph with two vertices, and no digons or loops exist that may be deleted. Lemma 2 implies that K_3^2 is directed non-planar. Figure 5 gives the essentially unique directed embedding of K_3^2 (which is in the torus).

The following lemma implies that the only obstruction with parallel arcs (bad digons) is K_3^s, $s \geq 2$.

Lemma 4 *Let G be an obstruction to directed planarity under the weak minor order and suppose that G is not K_3^s, $s \geq 2$. Then G is loopless, and for any pair of adjacent vertices u and v, $|\overrightarrow{uv}| \leq 1$ and $|\overrightarrow{vu}| \leq 1$.*

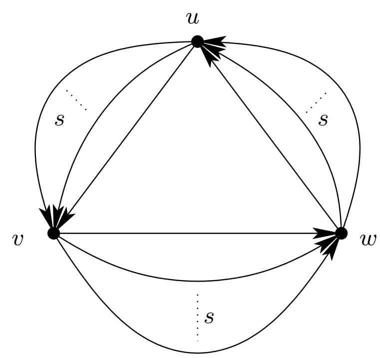

Figure 4: The graph K_3^s.

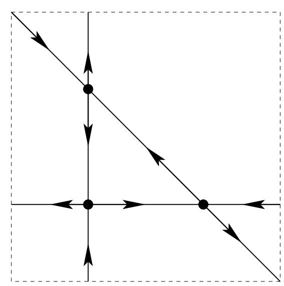

Figure 5: K_3^2 is directed non-planar

Proof. If there is a braced digon between u and v, then Lemma 1 implies that G is not an obstruction (that is, not minimal). Hence there is either a single digon between u and v, or (without loss of generality) $|\overrightarrow{uv}| = 0$. Suppose that $|\overrightarrow{vu}| = s \geq 2$. According to Lemma 3 the digraph $G - \{u, v\}$ is connected, and hence it contains a spanning tree T. Contraction of the arcs of T yields a digraph G' on 3 vertices. After removal of loops and digons from G' we obtain the digraph K_3^s with $s \geq 2$, a contradiction.

It follows that $|\overrightarrow{uv}| \leq 1$ and $|\overrightarrow{vu}| \leq 1$ for any pair of vertices $u, v \in G$, as required. □

We now present our first main result.

Theorem 1 *Let G be an obstruction under the weak minor order and suppose that $\hat{G}$ is planar. Then G is K_3^s for some $s \geq 2$.*

Proof. By Lemma 3, G and $\hat{G}$ are 3-connected and hence G has at least three vertices. If G has exactly three vertices, then it must contain some parallel arcs. By Lemma 4, G must be K_3^s for some $s \geq 2$. In the following we assume that G has at least four vertices and derive a contradiction.

Following Thomassen's proof of Kuratowski's Theorem from [4], the 3-connected planar graph $\hat{G}$ contains an edge uv whose contraction results in a 3-connected graph $\hat{G}'$. By Lemma 4, G either has a single arc between u and v, or the digon $\{uv, vu\}$. Let G' denote the Eulerian digraph obtained by contracting an arc between u and v, and removing any resulting loop (when the digon $\{uv, vu\}$ exists.) By minimality, G' is a directed planar Eulerian digraph. It has an essentially unique directed embedding in the plane.

Let w be the vertex of G' obtained by the identifying of the vertices u and v. Our strategy is to expand w back to our arc or digon between u and v, and show that this yields a directed planar embedding of G, or that G contains a weak minor that is non-planar. Both situations lead to a contradiction.

Now, since $\hat{G}'$ is 3-connected, the planar embedding of $\hat{G}' - w$ induced by the unique embedding of $\hat{G}'$ has a face boundary cycle C such that w is incident with only vertices of C in $\hat{G}'$. Let P_u be a minimal subpath of C that contains all the neighbours of u. Likewise, let P_v be a minimal subpath of C that contains all the neighbours of v. By Thomassen's proof of Kuratowski's Theorem [4], we can assume that P_v and P_u are internally disjoint (for otherwise $\hat{G}$ would contain a homeomorph of $K_{3,3}$.) The paths P_v and P_u may meet at one or both end-vertices, but no others.

Suppose that P_u and P_v have no vertices in common. We claim that G has a directed planar embedding. The rotation of the arcs for this embedding at each of the vertices of G other than u and v is identical to the rotation in the directed planar embedding of G'. In G' the arcs that were incident with u occur consecutively in the rotation at w (since $V(P_u) \cap V(P_v) = \emptyset$. Likewise, the arcs that were incident with v occur consecutively in the rotation at w. We order the arcs at u and v according to this order induced by w in G'. At this point we have a planar embedding of G with the arc or digon between u and v removed, with the property that every face is a directed walk, except possibly the face f with u and v on the boundary (see Figure 6.) If the boundary of f is a directed walk, then u and v must have a digon between them, which can be easily inserted to give a directed planar embedding of G. Otherwise, the boundary of f must consist of the union of two directed paths, with only the vertices u and v in common. It remains to insert the arc between u and v: since G is Eulerian, the introduction of this arc across face f creates a directed planar embedding of G.

Now, suppose that P_u and P_v have an end vertex x in common. The method of proof is similar to that of the above case. However, it is now conceivable that the rotation of the arcs at x induced by the planar directed embedding of G' cannot be applied to the arcs at x in G. This situation can only occur when a single arc a between x and u, and a single arc b between x and v create a digon in G' which is "flipped" in the planar embedding of G'. Figure 7 illustrates an example of this situation. It is worth noting that if either of u or v are joined to x by a digon, then the rotation at x in G can be made to correspond to the rotation in G'. Without loss of generality, we assume $a = xu$ (i.e. directed x to u) and $b = vx$.

We claim that G is not minimal directed non-planar. Firstly, if all arcs other than a, b, uv or vu are part of a digon, then C is a directed cycle. In fact, since G' is directed planar, C must be directed from x so that subpath P_u of

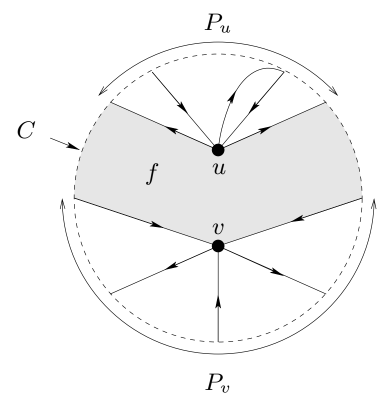

Figure 6: G with the arc or digon between u and v removed

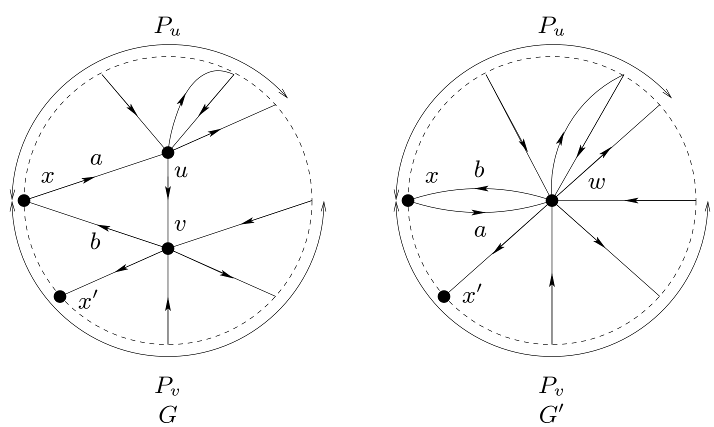

Figure 7: The arcs a and b are flipped

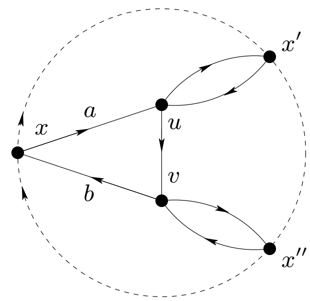

Figure 8:

C is followed by P_v. Now delete all digons except for one at u (to a vertex $x' \in V(P_u)$) and one at v (to a vertex $x'' \in V(P_v)$); we may assume the ends of these digons do not coincide (see Figure 8).

By 3-connectivity, there is a tree in $\hat{G}$ containing all vertices other than x, v and x''. Contract all edges of this tree to a vertex u', and contact the arc vx'' to a vertex x'''. The result is an Eulerian digraph such that $|\overrightarrow{vx'''}| - |\overrightarrow{x'''v}| = 2$, and $|\overrightarrow{u'v}| - |\overrightarrow{vu'}| = 2$. Removing redundant digons at x we obtain a graph satisfying the hypotheses in Lemma 2, and therefore is directed non-planar. Hence G is not mimimal, a contradiction.

Hence we now assume (without loss of generality) that there exists a single arc from v to a vertex $x' \neq x$ in P_v such that all other neighbours of v between x and x' in P_v are joined by a digon to v (see Figure 7). Let Q denote the subpath of P_v from x to x'. By 3-connectivity, there is a tree in $\hat{G}$ containing u and all neighbours of v other than those in Q. Contract all edges of this tree to a vertex u', and all edges in Q to a vertex x''. The result is an Eulerian digraph such that $|\overrightarrow{vx''}| - |\overrightarrow{x''v}| = 2$, and $|\overrightarrow{u'v}| - |\overrightarrow{vu'}| = 2$. Removing redundant digons at x we obtain a graph satisfying the hypotheses in Lemma 2, and therefore is directed non-planar. Hence G is not mimimal, a contradiction.

Therefore, we conclude that G contains 3 vertices and is K_3^s for some $s \geq 2$. $\square$

Lemma 5 *Let G be an obstruction under the weak minor order and suppose that $\hat{G}$ is not planar. Then $\hat{G}$ is either K_5 or a supergraph of $K_{3,3}$.*

Proof. By Kuratowski's theorem [3], $\hat{G}$ contains a subgraph homeomorphic to K_5 or $K_{3,3}$. If $\hat{G}$ contains K_5 and has exactly five vertices, then $\hat{G} = K_5$. If $\hat{G}$ has more than five vertices, then there exists an arc e in G such that, if $H = G/e$, then $\hat{H}$ still contains a subdivision of K_5 and hence G is not minimal.

If $\hat{G}$ contains no K_5, then $\hat{G}$ contains a subdivision of $K_{3,3}$. By an argument similar to that for K_5 we may assume that G has exactly six vertices. Since arc-deletion is not a weak minor operation, $\hat{G}$ may be a proper supergraph of $K_{3,3}$. $\square$

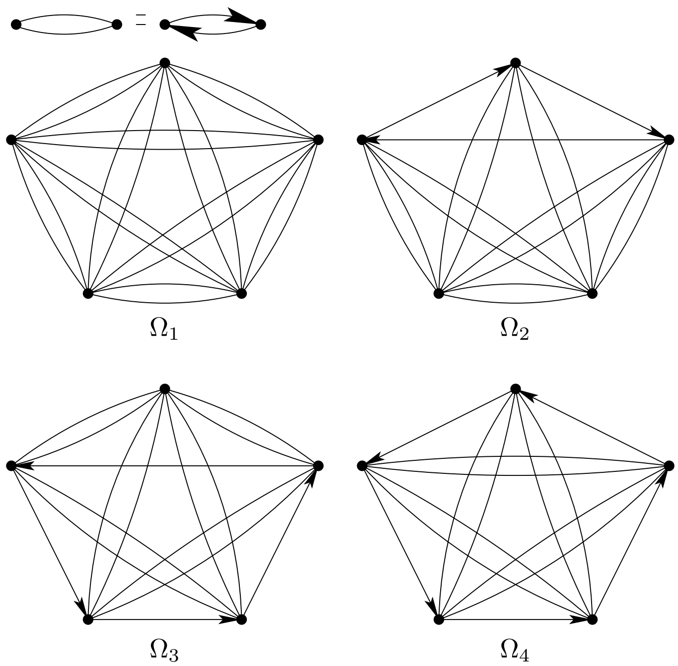

Figure 9: Obstructions based on K_5

3 The complete set of obstructions under the weak minor order

Here we present, in the form of figures, the complete set of obstructions under the weak minor order.

The main result of this section is the following.

Theorem 2 *An Eulerian directed graph G has a directed planar embedding if and only if none of the graphs K_3^s, $s \geq 2$ (Figure 4), $\Omega_1, \ldots, \Omega_4$ (Figure 9) and $\Theta_1, \ldots, \Theta_6$ (Figure 10) is a weak minor of G.*

Proof. Clearly, if any of the digraphs shown in Figures 4, 9 and 10 is a weak minor of G, then G has no directed planar embedding.

Hence assume that G has no directed planar embedding. Of all of the weak minors of G, choose a weak minor M that is minimal. If $\hat{M}$ is planar, then by Lemma 1 $\hat{M} = K_3^s$ for some $s \geq 2$. If $\hat{M}$ is not planar, then by Lemma 5 $\hat{M}$ is either K_5 or a supergraph of $K_{3,3}$. We first consider the case when $\hat{M} = K_5$. We show that M is one of $\Omega_1, \ldots, \Omega_4$ in Figure 9.

Since M is Eulerian, indegree equals outdegree at each vertex. The simplest case is when each pair of vertices is joined by a directed digon. Since $\hat{M}$ is not planar, M is not directed planar, but every weak minor of M is directed planar. This is obstruction Ω_1 in Figure 9.

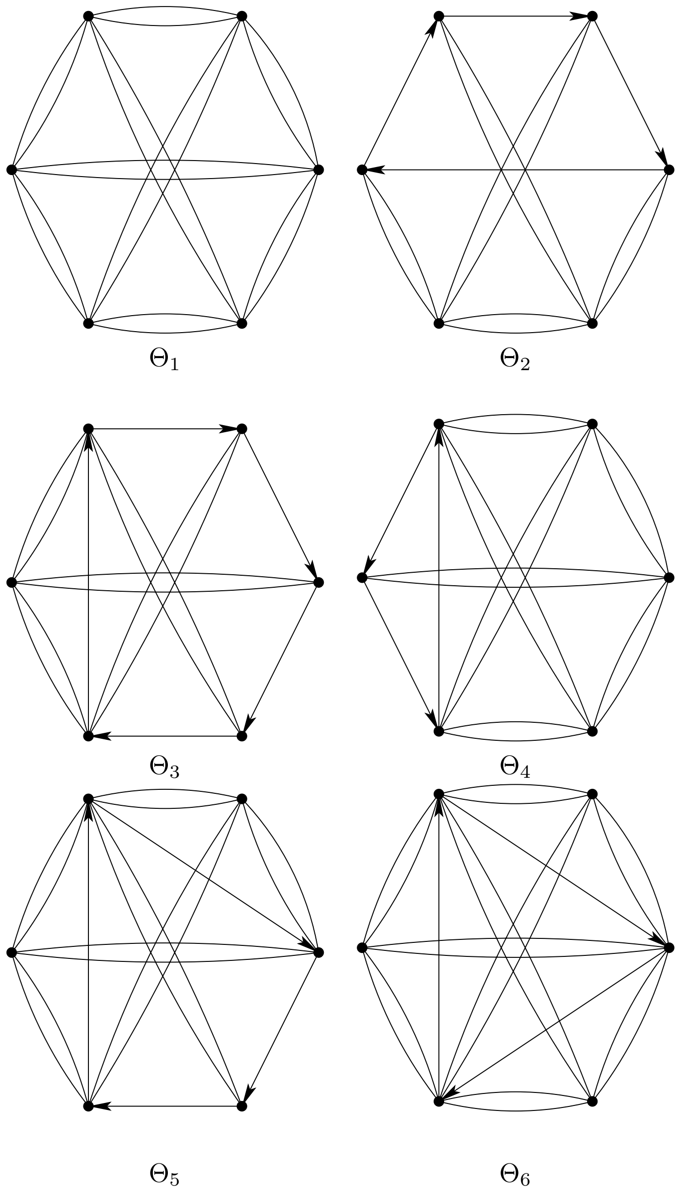

Figure 10: Obstructions based on $K_{3,3}$

Next we consider the case where a pair u, v of vertices is joined by a simple arc uv. Since M is Eulerian, there must be another vertex w that is adjacent from v by the single arc vw. Continuing in this fashion we see that the arc uv lies on a directed cycle of single arcs.

If this directed cycle is a triangle and if all other adjacencies are by means of directed digons, then contracting any arc or deleting any directed digon leads to a directed planar graph. This is obstruction Ω_2 in Figure 9.

There are two other possibilities for a directed cycle containing the arc uv. If uv lies on a directed quadrangle and all other adjacencies are directed digons, then we have obstruction Ω_3 in Figure 9. The final K_5 obstruction is where uv lies on a directed pentagon and all other adjacencies are directed digons. This is obstruction Ω_4 in Figure 9.

If there are two directed triangles and all other adjacencies are directed digons, then one of the five vertices lies on both directed triangles. If the directed triangles are xyz and xuv, contracting the digon between y and u or the digon between z and v produces two parallel arcs between two vertices (i.e. a bad digon) and so the resulting digraph is not directed planar. Thus M is not an obstruction.

The next possibility having the correct degrees is to have a directed triangle and a directed quadrangle and three digons. If we assume the triangle is directed xuz and the quadrangle is directed $xyzv$, then contracting the arc xv produces a bad digon. And finally if all arcs are simple, then we have two directed pentagons and they are $xyzuv$ and $xuyvz$. Again, contracting xz we obtain a bad digon.

Now suppose that $\hat{M}$ is a supergraph of $K_{3,3}$. If $\hat{M} = K_{3,3}$ and each adjacency is a digon, then every weak minor of M is directed planar and M is obstruction Θ_1 in Figure 10.

Consider a vertex x of M. If there is a simple arc from x to another vertex, that arc must lie on a directed cycle consisting of simple arcs since M is Eulerian. If the cycle is a quadrangle and all other adjacencies are digons, then M is obstruction Θ_2 in Figure 10. If the cycle is a hexagon and the remaining three adjacencies are directed digons, assume that the hexagon is $xaybzc$. Then contracting the digons xb and yc produces a bad digon. These are all the possibilities for M Eulerian and $\hat{M} = K_{3,3}$.

If $\hat{M}$ is $K_{3,3}$ plus one edge, then in M that edge must be a simple arc between two vertices, since if it were a digon we could delete it and still have a non-planar digraph. Suppose that the arc is xy and that the other vertex in the partite set is z. If a, b, c are vertices in the other partite set, what are the possibilities for M to be Eulerian?

It cannot happen that ax, bx, cx, ya, yb, and yc are all simple arcs. One of ax, bx, cx must be simple, and one of ya, yb, yc must be simple. Without loss of generality, assume that ax is simple. We now consider the case when ya is simple, so that xya is a directed triangle. This forces az to be a digon. If all adjacencies at b and c are digons, we obtain obstruction Θ_3 shown in Figure 10. If bz or zb is a simple arc, then contracting az and bx or by and az produces a

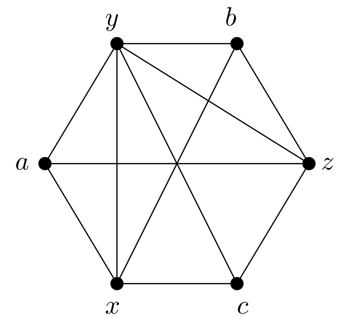

Figure 11: $\hat{M}$ is $K_{3,3}$ with two additional edges

bad digon. If bx or xb is a simple arc, then contracting by or contracting bz and az produces a bad digon. If ya is a digon, then za must be a simple arc. Then one of bz, cz must be a simple arc. Assume that bz is simple. Then yb must be simple and all other adjacencies are directed digons. This is obstruction Θ_4 in Figure 10.

The next possibility is that $\hat{M}$ is $K_{3,3}$ with two additional edges. We first observe that both must be in the same partite set because if they are in different sets, there will be a K_4 and contracting the arc between the two vertices of $\hat{M}$ that are not in this K_4 results in a K_5. We also observe that the additional adjacencies must be simple arcs or we could delete them and obtain a nonplanar digraph. The situation then is as depicted in Figure 11. Note also that it cannot occur that all arcs in Figure 11 are simple, since M is Eulerian.

If ay or ya is simple, contracting az or ax produces a bad digon. The analysis is analogous if the vertex a is replaced with b or c. Hence all adjacencies between y and a, b, c must be digons.

If zc is simple, then cx must be simple, and then $xyzc$ is a directed quadrangle. Suppose that bz and za, or az and zb, are simple arcs. Then either contracting by or ay produces a bad digon. Thus az and bz are digons. The analysis is similar if z is replaced with x, and we conclude that all adjacencies other than those in the directed quadrangle are digons. This is obstruction Θ_5 in Figure 10. A similar argument shows that if zb is a simple arc, then so is bx and obstruction Θ_5 results. The remaining possibility is that za and ax are simple arcs and the obstruction obtained is still Θ_5.

Finally, we consider the case where $\hat{M}$ is $K_{3,3}$ plus three edges. In M, the three additional adjacencies must be simple arcs and they must form a directed triangle, since if two of them are adjacent from (or to) the same vertex, contracting the third leads to a bad digon. (See Figure 12.)

If cz or zc is a simple arc, then contracting cy or cx produces a bad digon. If az or za is simple, then contracting ay or ax yields a bad digon. An analogous argument shows that all adjacencies other than the directed triangle xyz must be digons. This is obstruction Θ_6 in Figure 10. $\square$

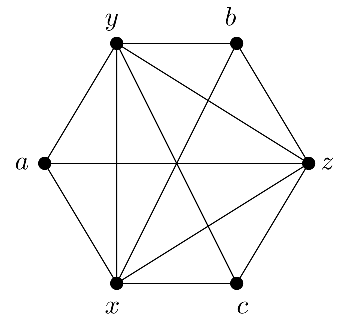

Figure 12: $\hat{M}$ is $K_{3,3}$ with three additional edges

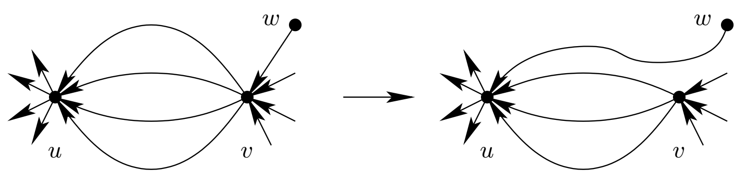

Figure 13: Slicing G at v

4 The strong minor order and associated obstruction

We now present a further set of minor operations for Eulerian digraphs which preserve directed planarity.

Slice

Suppose u and v are vertices in an Eulerian digraph G where all out-arcs from v terminate at u (and no uv arc exists). Let w denote the origin of an arc wv terminating at v. Let G' denote the Eulerian digraph obtained from G by removing one vu arc, wv, and inserting a wu arc. Then we say G' is obtain from G by a *slice* (at v). (See Figure 13.)

H-bowtie

This operation is analogous to to the well-known H-bowtie operation for undirected graphs. Suppose there exists six distinct vertices u_1, u_2, u, v, v_1 and v_2 and five digons $\{u_1u, uu_1\}$, $\{u_2u, uu_2\}$, $\{uv, vu\}$, $\{v_1v, vv_1\}$, and $\{v_2v, vv_2\}$ in an Eulerian digraph G, such that $indeg(u) = indeg(v) = 3$. Let G' denote the Eulerian digraph obtained from G by removing the digon $\{uv, vu\}$, identifying the vertices u and v and inserting new digons $\{u_1u_2, u_2u_1\}$ and $\{v_1v_2, v_2v_1\}$. (See Figure 14.)

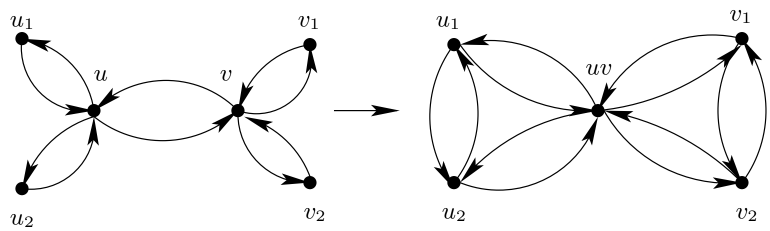

Figure 14: The H-bowtie operation at u and v

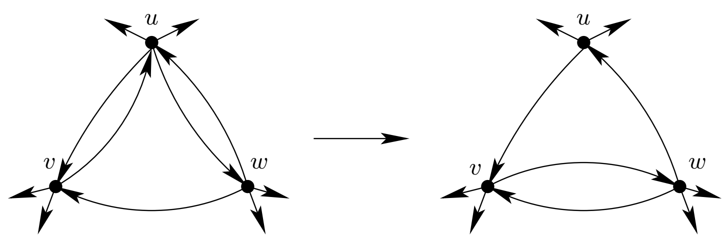

Figure 15: Spliting G at u

The remaining two operations pertain to non-separating sets of three vertices.

Split

Suppose $\{u, v, w\}$ is a non-separating set of three vertices in an Eulerian digraph G, and that the two digons $\{uv, vu\}$, $\{uw, wu\}$, and the arc wv exist (but not vw). Let G' be the digraph obtained by removing the arcs vu and uw, introducing a new arcs vw. Then we say that G' was obtain from G by a *split* (at u.) (See Figure 15.)

Triangle deletion

Suppose $\{u, v, w\}$ is a non-separating set of three vertices in an Eulerian digraph G, and that the digons $\{uv, vu\}$, $\{uw, wu\}$, and $\{wv, vw\}$ exist. Let G' be the digraph obtained by removing the arcs vu, uw and wv. Then we say that G' was obtain from G by *removing a triangle.* (See Figure 16.)

Refining the weak order by introducing these four additional operations yields an alternate Kuratowski-type characterisation for directed planarity:

Theorem 3 *An Eulerian digraph is directed planar if and only if it does not contain K_3^2 is a strong minor.*

Proof. We have already established that any Eulerian digraph containing K_3^2 as a weak minor (and hence as a strong minor) is directed non-planar. Furthermore, K_3^2 is minimal directed non-planar under the strong minor order since

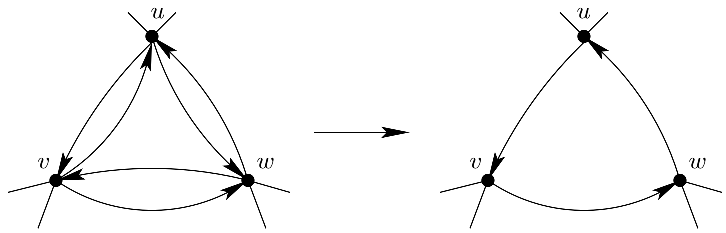

Figure 16: Removing a triangle from G

only the arc-contraction, slice, split and triangle deletion operations can be applied to K_3^2, all resulting in a directed planar graph.

By Theorem 2, a directed non-planar graph contains one of the graphs in Figures 4, 9 and 10 as a weak minor. It remains to show that all of these graphs (other than K_3^2) reduce to K_3^2 under the strong minor order.

Firstly, we note that performing a slice on the graph K_3^s, $s \geq 3$, and deleting the resulting digon, produces the graph K_3^{s-1}. Proceeding inductively, we have that all graphs K_3^s, $s \geq 3$ reduce to K_3^2 under the strong minor order.

Next, applying an H-bowtie operation on any of the obstructions Θ_1, Θ_2 and Θ_4 produces one of the obstructions based on K_5. Performing selective splits on the obstructions based on K_5 other than Ω_4 can reduce them all to obstruction Ω_4. Executing two more splits on obstruction Ω_4 results in a graph M, such that $\hat{M} = K_5$ and there are just three digons forming a triangle. Now, applying the triangle deletion operation on M yields a graph M' with single arcs only and $\hat{M}' = K_5$. Contracting an arc in M', deleting a resulting digon, and contracting out the degree two vertex yields the strong obstruction K_3^2.

Finally, we see that spliting obstruction Θ_5 (Θ_6 respectively) at a vertex that is the common neighbour of the ends of a single arc produces the obstruction Θ_3 (Θ_5). Performing on Θ_3 the only possible split results in a digraph based on $K_{3,3}$ with precisely three non-adjacent digons and a directed 6-cycle. Contracting an arc in each of the three digons yields K_3^2.

Hence we have shown that all weak minor obstructions $\Omega_1, \ldots, \Omega_4$ and $\Theta_1, \ldots, \Theta_6$ reduce to the single stong minor obstruction K_3^2, as required. $\square$

We conclude by mentioning that an alternative structural method for embedding digraphs is to force all in-arcs to appear consecutively in the cyclic rotation around every vertex. This type of "clustered" embedding of (not necessarily Eulerian) digraphs is the central subject in [5]. There are analogous characterisations of (clustered) planarity (see [2]) to the ones presented in this paper.

Acknowledgements

The authors worked pairwise hosted by one or another of the three home institutions. Each author gratefully thanks the other hosts. The first author gratefully acknowledges support of the Marsden Fund (Grant Number UOA-825) administered by the Royal Society of New Zealand, and for useful conversations with Dan Archdeacon and Jamie Sneddon.

References

[1] C.P. Bonnington, M. Conder, M. Morton, and P. McKenna, Embedding digraphs on orientable surfaces. J. Combin. Theory Ser. B 85 (2002), no. 1, 1–20.

[2] C.P. Bonnington, M. Conder, and J. Sneddon, A Kuratowski theorem for clustered planarity, in preparation.

[3] K. Kuratowski, Sur les problèmes des courbes gauches en topologie, Fundamenta Math. 15 (1933)

[4] C. Thomassen, Planarity and duality of finite and infinite graphs, J. Combin. Theory Ser. B 29 (1980), no. 2, 244–271.

[5] J. Sneddon, Some structural results for directed graphs, Ph.D Dissertation, University of Auckland, 2003.