On Monotonically Orthocompact Spaces *

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Abstract: In this paper we study some properties of transitive neighbornets, and make use of them to discuss the properties of ortho-refinable spaces. We prove that ortho-refinable spaces are irreducible and iso-compact, that every closed subspace of an ortho-refinable space is ortho-refinable, and other related results.
Key words: neighbornets, ortho-refinable, irreducible, iso-compact, open mapping

1.4INTRODUCTION

A base $\mathcal{B}$ for a space X is an ortho-base if for each subcollection $\mathcal{A}$ of $\mathcal{B}$, either $(i) \cap \mathcal{A}$ is open, or $(ii) \cap \mathcal{A}$ is a nonisolated singleton $\{x\}$ and $\mathcal{A}$ is a base for the neighborhoods of x [6]. A proto-metrizable space is a paracompact space with an ortho-base. In [2], Gartside and Moody show that the spaces which admit a monotonic uniformity can be characterized as proto-metrizable spaces, i,e., as the monotonically normal spaces having an ortho-base. Furthermore, in [3] they show that proto-metrizable spaces can also be characterized by a certain monotonic covering property which they call monotonic paracompactness: To each open cover $\mathcal{C}$ of a proto-metrizable space X there is an open star-refinement $m(\mathcal{C})$ of $\mathcal{C}$ so that $m(\mathcal{C}_\infty)$ refines $m(\mathcal{C}_\in)$ whenever $\mathcal{C}_\infty$ and $\mathcal{C}_\in$ are open covers of X and $\mathcal{C}_\infty$ refines $\mathcal{C}_\in$. In [5], Junnila and Kunzi introduce another monotonic covering property which they call monotonic orthocompactness. They show that each space with an ortho-base is a monotonically orthocompact space, and that each monotonically normal monotonically orthocompact space is proto-metrizable. An obvious problem has arisen, whether this result is more general than the following characterization of proto-metrizability given by Nyikos in [8]: A space is proto-metrizable iff it is a monotonically normal space having an ortho-base.

Obviously, if X is monotonically normal, the following are equivalent: (i) X is monotonically orthocompact, (ii) X has an ortho-base. In [6] Lindgren and Nyikos prove that, if X is developable, the following are equivalent: (i) X is orthocompact, (ii) X has an ortho-base. Since monotonically orthocompact spaces are orthocompact, when X is developable, the following are also equivalent: (i) X is monotonically orthocompact, (ii) X has an ortho-base. So, in [5] Junnila and Kunzi ask the question whether each monotonically orthocompact space has an ortho-base. In this paper, we study some properties of neighbornets and discuss the monotonically orthocompact spaces. Furthermore, we prove that each space with a Noetherian base of sub-infinite $rank$ is monotonically orthocompact, and we give an example of a monotonically orthocompact space without an ortho-base, so the above question is solved.

In this paper, all the spaces are T_1-spaces, the ordinal numbers are denoted by α, β etc., and the cardinal numbers are denoted by κ, λ etc..

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2. SOME PROPERTIES OF NEIGHBORNETS

Let X be a set, a subset R of $X \times X$ is called a relation on X. We usually denote the set $\{y : (x, y) \in R\}$ by $R\{x\}$. It follows that $R = \cup\{\{x\} \times R\{x\} : x \in X\}$, and thus R is determined by $\{R\{x\} : x \in X\}$. It is obvious that if R_1 and R_2 are two relations on X, then $R_1 \subset R_2$ iff for each $x \in X$ we have $R_1\{x\} \subset R_2\{x\}$. For each $A \subset X$, we denote $\cup\{R\{x\} : x \in X\}$ by $R(A)$. The inverse of a relation R, denoted by R^{-1}, is defined by that for all $x, y \in X$, $(x, y) \in R$ iff $(y, x) \in R^{-1}$. Suppose R and S are two relations on X, we define the relation $R \circ S = \{(x, z) : \exists y \in X$ such that $(x, y) \in R$ and $(y, z) \in S\}$. In particular, for each natural number n, let $R^1 = R$, $R^{n+1} = R \circ R^n$. It is easy to see that, for each $x \in X$, we have $(R \circ S)\{x\} = R(S\{x\})$, $(R \cap S)\{x\} = R\{x\} \cap S\{x\}$, $(R \cup S)\{x\} = R\{x\} \cup S\{x\}$. About the inverse relation, we have $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$, $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$, and $S^{-1} \subset R^{-1}$ whenever $S \subset R$. A relation R on X is transitive if $R^2 \subset R$, i.e., $(x, z) \in R$ whenever $(x, y) \in R$ and $(y, z) \in R$, or $z \in R\{x\}$ whenever $y \in R\{x\}$ and $z \in R\{y\}$.

In [4], Junnila has studied the properties on neighbornets in detail. Here, we also give some other properties of neighbornets. For convenience, if $\mathcal{C}$ is a family of open subsets of a space X, we denote $\{C \in \mathcal{C} : x \in C\}$ by $(\mathcal{C})_{\S}$.

Definition 2.1 A relation R on a space X is called a neighbornet of X if $R\{x\}$ is a neighborhood of x for each $x \in X$ [8]. R is called a partial neighbornet if for each $x \in X$, $R\{x\} \cup R^{-1}\{x\} = \emptyset$ or $R\{x\}$ is a neighborhood of x in X [2].

Proposition 2.2 Let T be a transitive partial neighbornet of a space X, Y a subspace of X. Then $R = T \cap (Y \times Y)$ is also a transitive partial neighbornet of Y.

Proof For each $y \in Y$, we have that $R\{y\} = (T \cap (Y \times Y))\{y\} = T\{y\} \cap (Y \times Y)\{y\} = T\{y\} \cap Y$, $R^{-1}\{y\} = (T^{-1} \cap (Y \times Y)^{-1})\{y\} = T^{-1}\{y\} \cap (Y \times Y)^{-1}\{y\} = T^{-1}\{y\} \cap Y$, and thus $R\{y\} \cup R^{-1}\{y\} = (T\{y\} \cap Y) \cup (T^{-1}\{y\} \cap Y) = (T\{y\} \cup T^{-1}\{y\}) \cap Y$. If $R\{y\} \cup R^{-1}\{y\} \neq \emptyset$, then we have $T\{y\} \cup T^{-1}\{y\} \neq \emptyset$. It follows that $T\{y\}$ is a neighborhood of y in X, and thus $R\{y\}$ is a neighborhood of y in Y. Therefore R is a partial neighbornet of Y.

Let $(x, y), (y, z) \in R = T \cap (Y \times Y)$, then we have $(x, y), (y, z) \in T$ and $(x, y), (y, z) \in Y \times Y$. Since T is a transitive partial neighbornet, we have $(x, z) \in T$. Note that $(x, z) \in Y \times Y$, then $(x, z) \in T \cap (Y \times Y)$. Therefore, R is a transitive partial neighbornet of Y.

Corollary 2.3 Let T be a transitive neighbornet of a space X, Y a subspace of X. Then $R = T \cap (Y \times Y)$ is also a transitive neighbornet of Y.

Definition 2.4 A neighbornet R of a space X is said to be open if for each $x \in X$, the set $R\{x\}$ is an open subset of X[4].

Proposition 2.5 A transitive neighbornet is open [4].

Definition 2.6 A collection $\mathcal{U}$ of open subsets of a space X is said to be interior-preserving if for each $\mathcal{U}' \subset \mathcal{U}$, the intersection $\cap\mathcal{U}'$ is open in X [1].

Proposition 2.7 Let T be a transitive partial neighbornet of a space X, then $\{T\{x\} : x \in X\}$ is an interior-preserving open family of X.

Proof Since T is a transitive partial neighbornet of X, for each $x \in X$, we have that $T\{x\}$ is an open subset of X.

Suppose $B \subset X$, and $\cap\{T\{x\} : x \in B\} \neq \emptyset$. Let $y \in \cap\{T\{x\} : x \in B\}$. Then, for each $x \in B$, we have $y \in T\{x\}$, and thus $T\{y\} \subset T(T\{x\}) = T^2\{x\} \subset T\{x\}$. It follows that $T\{y\} \subset \cap\{T\{x\} : x \in B\}$. Clearly $B \neq \emptyset$. Choose $x \in B$, then $y \in T\{x\}$, we have $(x, y) \in T$, and thus $(y, x) \in T^{-1}$, $x \in T^{-1}\{y\}$. It is to say that $T^{-1}\{y\} \neq \emptyset$. By Definition 2.1, we know that $T\{y\}$ is an open neighborhood of y. Hence, the intersection $\cap\{T\{x\} : x \in B\}$ is open in X.

We prove that $\{T\{x\} : x \in X\}$ is an interior-preserving open family of X.

Corollary 2.8 Let T be a transitive neighbornet of a space X, then $\{T\{x\} : x \in X\}$ is an interior-preserving open cover of X.

Proposition 2.9 Let $\mathcal{C}$ be an interior-preserving open family of a space X, $R\{x\} = \cap(\mathcal{C})_{\S}$ for each

$x \in X$. Then R is a transitive partial neighbornet of X, and $\{R\{x\} : x \in \cup\mathcal{C}\}$ is an open refinement of $\mathcal{C}$.

Proof Since $\mathcal{C}$ is an interior-preserving open family of X, then for each $x \in X$, $R\{x\} = \cap(\mathcal{C})_{\S}$ is an open subset of X. If $x \in \cup\mathcal{C}$, we have $x \in R\{x\}$, and thus $R\{x\} \neq \emptyset$. It follows that $R\{x\}$ is an open neighborhood of x. If $x \notin \cup\mathcal{C}$, then $R\{x\} = \emptyset$. Suppose $R^{-1}\{x\} \neq \emptyset$, then there exists some $y \in X$ such that $y \in R^{-1}\{x\}$, and thus $y \in R\{x\}$, a contradiction. So, we have that $R^{-1}\{x\} = \emptyset$, and thus $R\{x\} \cup R^{-1}\{x\} = \emptyset$. In light of Definition 2.1, R is a partial neighbornet of X.

Let $(x, y), (y, z) \in R$, then $y \in R\{x\}$, $z \in R\{y\}$. Since $y \in R\{x\} = \cap(\mathcal{C})_{\S}$, for each $C \in (\mathcal{C})_{\S}$ we have $y \in C$. It follows that $\cap(\mathcal{C})_{\dagger} \subset \cap(\mathcal{C})_{\S}$, and thus $R\{y\} \subset R\{x\}$. So, we have $z \in R\{y\} \subset R\{x\}$, and thus $(x, z) \in R$. Therefore, R is a transitive partial neighbornet of X.

By the definition of $R\{x\}$, it is easy to see that $\{R\{x\} : x \in X\}$ is an open refinement of $\mathcal{C}$.

Corollary 2.10 Let $\mathcal{C}$ be an interior-preserving open cover of a space X, $R\{x\} = \cap(\mathcal{C})_{\S}$ for each $x \in X$. Then R is a transitive neighbornet of X, and $\{R\{x\} : x \in \cup\mathcal{C}\}$ is an open refinement of $\mathcal{C}$.

3. 4 ON MONOTONICALLY ORTHOCOMPACT SPACES

In [6], Lindgren and Nyikos firstly introduce the concept of ortho-base, study the spaces which have an ortho-base, and give some basic results for them. In [5], Junnila and Kunzi make use of neighbornets to give another new characterization of the spaces having an ortho-base, as a generalization they also introduce the concept of monotonically orthocompact spaces.

Definition 3.1 A base $\mathcal{B}$ for a space X is called an ortho-base if whenever $\mathcal{B}'$ is a subset of $\mathcal{B}$, $x \in \cap\mathcal{B}'$, and $x \notin int(\cap\mathcal{B}')$, then $\mathcal{B}'$ is a local base at x [6, 5].

Theorem 3.2 Let $\mathcal{B}$ be an ortho-base of a space X, then every point x in X has a local base which is linearly ordered by reverse inclusion [6].

Theorem 3.3 Let $\mathcal{B}$ be an ortho-base for a space X, then

(1) $\{\cap\mathcal{B}' : \mathcal{B}' \subset \mathcal{B}, \cap\mathcal{B}'$ is open $\}$ is an ortho-base of the space X, i.e., the collection of all open intersections of subsets of $\mathcal{B}$ is an ortho-base.

(2) Let Y be a subspace of X, then the collection $\{Y \cap B : B \subset \mathcal{B}\}$ is an ortho-base of Y[6].

Theorem 3.4 Let X be a topological space with an ortho-base $\mathcal{B}$, assume that $\mathcal{B}$ is closed under open intersections. Then there is an operator $T : \Gamma \longrightarrow \mathcal{E}$ from the set Γ of all collections of open subsets of X to the set $\mathcal{E}$ of all transitive partial neighbornets of X, such that

(1) $T(\mathcal{C}_\infty) \subset \mathcal{T}(\mathcal{C}_\in)$ whenever $\mathcal{C}_\infty, \mathcal{C}_\in \in -$ and $\mathcal{C}_\infty$ is a partial refinement of $\mathcal{C}_\in$.

(2) For each $\mathcal{C} \in -$, the collection $\{T(\mathcal{C})\{\S\} : \S \in \mathcal{T}(\mathcal{C})\{\S\}\}$ is a subcollection of $\mathcal{B}$ and a refinement of $\mathcal{C}[\bigtriangledown]$.

Definition 3.5 A topological space X is called monotonically orthocompact provided that there is an operator $T : \Xi \longrightarrow \varphi$ from the set Ξ of all open covers of X to the set φ of all transitive neighbornets of X, such that

(1) $T(\mathcal{C}_\infty) \subset \mathcal{T}(\mathcal{C}_\in)$ whenever $\mathcal{C}_\infty, \mathcal{C}_\in \in \div$ and $\mathcal{C}_\infty$ is a refinement of $\mathcal{C}_\in$.

(2) $\{T(\mathcal{C})\{\S\} : \S \in \mathcal{X}\}$ refines $\mathcal{C}$ whenever $\mathcal{C} \in \div$.

Here T is called a monotonically orthocompact operator on X[5].

Definition 3.6 A space X is said to be orthocompact if for each open cover $\mathcal{C}$ of X, there is an interior-preserving open refinement $\mathcal{R}$ of $\mathcal{C}$.

By Theorem 3.4 and Definition 3.5, we know that the spaces having an ortho-base are monotonically orthocompact. From Proposition 2.5 and Definition 3.5, it is also easy to see that the monotonically orthocompact spaces are orthocompact. Next, we shall discuss some properties of monotonically orthocompact spaces.

Theorem 3.7 A closed subspace Y of a monotonically orthocompact space X is also a monotonically orthocompact space.

Proof Suppose $T : \Xi \longrightarrow \varphi$ is a monotonically orthocompact operator on X. Let $\mathcal{U}$ be an open cover of Y, then each element U of $\mathcal{U}$ has the form $U = V \cap Y$, where V is an open subset of X. For each $U \subset \mathcal{U}$, let $V_U = V \cup (X \setminus Y)$, then V_U is an open subset of X and $U = V_U \cap Y$. Set $\mathcal{V}_\mathcal{U} = \{\mathcal{V}_\mathcal{U} : \mathcal{U} \in \mathcal{U}\}$, then $\mathcal{V}_\mathcal{U}$ is an open cover of X.

Since $T(\mathcal{V}_\mathcal{U})$ is a transitive neighbornet of X, by Corollary 2.3 we conclude that $T(\mathcal{V}_\mathcal{U}) \cap (\mathcal{Y} \times \mathcal{Y})$ is a transitive neighbornet of Y. Set $P(\mathcal{U}) = \mathcal{T}(\mathcal{V}_\mathcal{U}) \cap (\mathcal{Y} \times \mathcal{Y})$, it suffices to prove that P is a monotonically orthocompact operator on Y.

Let $\mathcal{U}_\infty$,$\mathcal{U}_\in$ be two open covers of Y, such that $\mathcal{U}_\infty$ refines $\mathcal{U}_\in$. As in the above assignment, we have $\mathcal{V}_{\mathcal{U}_\infty} = \{\mathcal{V}_{\mathcal{U}_\infty} : \mathcal{U}_\infty = \mathcal{V}_{\mathcal{U}_\infty} \cap \mathcal{Y}, \mathcal{U}_\infty \in \mathcal{U}_\infty\}$, and $\mathcal{V}_{\mathcal{U}_\in} = \{\mathcal{V}_{\mathcal{U}_\in} : \mathcal{U}_\in = \mathcal{V}_{\mathcal{U}_\in} \cap \mathcal{Y}, \mathcal{U}_\in \in \mathcal{U}_\in\}$. For each $V_{U_1} \in \mathcal{V}_{\mathcal{U}_\infty}$, there is $U_2 \in \mathcal{U}_\in$ such that $U_1 \subset U_2$, so we have $V_{U_1} \cap Y \subset V_{U_2} \cap Y$. It follows that $V_{U_1} = (V_{U_1} \cap Y) \cup (X \setminus Y) \subset (V_{U_2} \cap Y) \cup (X \setminus Y) = V_{U_2} \in \mathcal{V}_{\mathcal{U}_\in}$. Consequently $\mathcal{V}_{\mathcal{U}_\infty}$ refines $\mathcal{V}_{\mathcal{U}_\in}$, so we have $T(\mathcal{V}_{\mathcal{U}_\infty}) \subset \mathcal{T}(\mathcal{V}_{\mathcal{U}_\in})$, and thus $P(\mathcal{U}_\infty) \subset \mathcal{P}(\mathcal{U}_\in)$.

Let $\mathcal{U}$ be an arbitrary open cover of Y, then $\mathcal{V}_\mathcal{U}$ is an open cover of X. By Definition 3.5, we have that $\{T(\mathcal{V}_\mathcal{U})\{\S\} : \S \in \mathcal{X}\}$ refines $\mathcal{V}_\mathcal{U}$. For each $y \in Y$, by the fact that T is a transitive neighbornet of X, we know $T(\mathcal{V}_\mathcal{U})\{\dagger\}$ is an open neighborhood of y in X. Hence $T(\mathcal{V}_\mathcal{U})\{\dagger\} \not\subset \mathcal{X} \setminus \mathcal{Y}$, and it follows that there is $V_U \in \mathcal{V}_\mathcal{U}$ such that $T(\mathcal{V}_\mathcal{U})\{\dagger\} \subset \mathcal{V}_\mathcal{U}$, where $U = V_U \cap Y \in \mathcal{U}$. Since $P(\mathcal{U}) = \mathcal{T}(\mathcal{V}_\mathcal{U}) \cap (\mathcal{Y} \times \mathcal{Y})$ is a transitive neighbornet of Y and $P(\mathcal{U})\{\dagger\} = (\mathcal{T}(\mathcal{V}_\mathcal{U}) \cap (\mathcal{Y} \times \mathcal{Y}))\{\dagger\} = \mathcal{T}(\mathcal{V}_\mathcal{U})\{\dagger\} \cap \mathcal{Y} \subset \mathcal{V}_\mathcal{U} \cap \mathcal{Y}$, we have that $P(\mathcal{U})\{\dagger\}$ is an open subset of Y and $P(\mathcal{U})\{\dagger\} \in \mathcal{U}$, so $\{P(\mathcal{U})\{\dagger\} : \dagger \in \mathcal{Y}\}$ is a refinement of $\mathcal{U}$.

Theorem 3.8 Let X be a monotonically orthocompact space, and $f : X \longrightarrow Y$ be a finite to one open mapping. Then $f(X)$ is a monotonically orthocompact space.

Proof Let T be a monotonically orthocompact operator on X, $\mathcal{C}$ an arbitrary open cover of $f(X)$, then $f^{-1}(\mathcal{C}) = \{\{^{-\infty}(\mathcal{C}) : \mathcal{C} \in \mathcal{C}\}$ is an open cover of X. Thus, $T(f^{-1}(\mathcal{C}))$ is a transitive neighbornet of X. By Corollary 2.8, we know that $\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{X}\}$ is an interior-preserving open refinement of $f^{-1}(\mathcal{C})$.

Let $\mathcal{D} = \{(\mathcal{T}(\{^{-\infty}(\mathcal{C}))\{\S\}) : \S \in \mathcal{X}$, then $\mathcal{D}$ is a family of open subsets of $f(X)$. For each $x \in X$, there exists $C \in \mathcal{C}$ such that $T(f^{-1}(\mathcal{C}))\{\S\} \subset \{^{-\infty}(\mathcal{C})$, and thus $f(T(f^{-1}(\mathcal{C}))\{\S\}) \subset \{(\{^{-\infty}(\mathcal{C})) = \mathcal{C}$. So, $\mathcal{D}$ is an open refinement of $\mathcal{C}$. Let $A \subset X$, and $y \in \cap\{f(T(f^{-1}(\mathcal{C}))\{\S\}) : \S \in \mathcal{A}\}$. By the fact that f is a finite to one mapping , we can assume $f^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$. Let $A_i = \{x \in A : x_i \in T(f^{-1}(\mathcal{C}))\{\S\}\}(\infty \leq \rangle \leq \backslash)$, then $A = \cup\{A_i : 1 \leq i \leq n\}$. Without loss of generality, for $1 \leq i \leq n$, we can assume that A_i is not empty. Since $\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{X}\}$ is an interior-preserving family of open subsets of X, we know that $\cap\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{A}_\rangle\}$ is an open subset of X, and thus $f(\cap\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{A}_\rangle\})$ is an open subset of $f(X)$. Clearly, for $1 \leq i \leq n$ we have $y = f(x_i) \in f(\cap\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{A}_\rangle\}) \subset \cap\{\{(\mathcal{T}(\{^{-\infty}(\mathcal{C}))\{\S\}) : \S \in \mathcal{A}_\rangle\}$. It follows that $y \in \cap\{f(\cap\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{A}_\rangle\}) : \infty \leq \rangle \leq \backslash\} \subset \cap\{\cap\{\{(\mathcal{T}(\{^{-\infty}(\mathcal{C}))\{\S\}) : \S \in \mathcal{A}_\rangle\} : \infty \leq \rangle \leq \backslash\} = \cap\{\{(\mathcal{T}(\{^{-\infty}(\mathcal{C}))\{\S\}) : \S \in \mathcal{A}\}$. But $\cap\{f(\cap\{T(f^{-1}(\mathcal{C}))\{\S\} : \S \in \mathcal{A}_\rangle\}) : \infty \leq \rangle \leq \backslash\}$ is the intersection of finitely many open subsets of X, so is open. Hence we have that y is an interior point of the set $\cap\{f(T(f^{-1}(\mathcal{C}))(\S)) : \S \in \mathcal{A}\}$. Therefore, the set $\cap\{f(T(f^{-1}(\mathcal{C}))\{\S\}) : \S \in \mathcal{A}\}$ is an open subset of $f(X)$. This indicates that $\mathcal{D}$ is an interior-preserving open refinement of $\mathcal{C}$.

For each $y \in f(X)$, let $R\{y\} = \cap(\mathcal{D})_\dagger$. By Corollary 2.10 we know that R is a transitive neighbornet of $f(X)$, and $\{R\{y\} : y \in f(X)\}$ is an open refinement of $\mathcal{D}$.

Define an operator $P : \Xi \longrightarrow \varphi$ from the set of all open covers of $f(X)$ to the set of all transitive neighbornets of $f(X)$ such that, for each open cover $\mathcal{C}$ of $f(X)$, set $P(\mathcal{C}) = \mathcal{R}$, which is the transitive neighbornet defined in the above. It remains to prove that P is a monotonically orthocompact operator on $f(X)$.

In fact, let $\mathcal{C}_\infty, \mathcal{C}_\in$ be two open covers of $f(X)$ such that $\mathcal{C}_\infty$ refines $\mathcal{C}_\in$. Then $f^{-1}(\mathcal{C}_\infty)$, $f^{-1}(\mathcal{C}_\in)$ are open covers of X, and $f^{-1}(\mathcal{C}_\infty)$ refines $f^{-1}(\mathcal{C}_\in)$. So, we have $T(f^{-1}(\mathcal{C}_\infty)) \subset \mathcal{T}(\{^{-\infty}(\mathcal{C}_\in))$, and thus $\{T(f^{-1}(\mathcal{C}_\infty))\{\S\} : \S \in \mathcal{X}\}$ is an open refinement of $T(f^{-1}(\mathcal{C}_\in))\{\S\} : \S \in \mathcal{X})$. Hence, $\mathcal{D}_\infty = \{\{(\mathcal{T}(\{^{-\infty}(\mathcal{C}_\infty))\{\S\}) : \S \in \mathcal{X}\}$ is an open refinement of $\mathcal{D}_\in = \{\{(\mathcal{T}(\{^{-\infty}(\mathcal{C}_\in))\{\S\}) : \S \in \mathcal{X}\}$ in $f(X)$. It follows that we have $R_1\{y\} \subset R_2\{y\}$ for each $y \in f(X)$, and thus $R_1 \subset R_2$. Therefore,

$P(\mathcal{C}_\infty) \subset \mathcal{P}(\mathcal{C}_\in)$. Let $\mathcal{C}$ be an open cover of $f(X)$, then $\mathcal{D} = \{\{(\mathcal{T}(\{^{-\infty}(\mathcal{C}))\{\S\}) : \S \in \mathcal{X}\}$ is an interior-preserving open refinement of $\mathcal{C}$. Since $\{R\{y\} : y \in f(X)\}$ is an open refinement of $\mathcal{D}$, and $P(\mathcal{C}) = \mathcal{R}$, we have that $\{P(\mathcal{C})\{\dagger\} : \dagger \in \{(\mathcal{X})\}$ is an open refinement of $\mathcal{C}$. By Definition 3.5, we know that P is a monotonically orthocompact operator on $f(X)$.

4. 4 AN EXAMPLE OF MONOTONICALLY ORTHOCOMPACT SPACE WITHOUT ORTHO-BASE

For the concept of *rank* of a collection of sets defined by Nagata [7], Lindgren and Nyikos give further general studies in [6]. They introduce the concept of Noetherian collection of subsets of a set X, and discuss the properties of Noetherian bases.

Definition 4.1 A collection $\mathcal{A}$ of subsets of a set X is incomparable if, given any two members A_1 and A_2 of $\mathcal{A}$ neither $A_1 \subset A_2$ nor $A_2 \subset A_1$ obtains [6].

Definition 4.2 Let κ be a cardinal number, X be a set, $\mathcal{A}$ be a collection of subsets of X, and $x \in X$. The collection $\mathcal{A}$ is of $rank \leq \kappa$ at x in X if for every incomparable subcollection $\mathcal{A}' = \{\mathcal{A} \in \mathcal{A} : \S \in \mathcal{A}\}$, its cardinality is $\leq \kappa$. It is of $rank$ κ at x if it is of $rank \leq \kappa$ at x and if there exists an incomparable subcollection $\mathcal{A}_\prime = \{\mathcal{A} \in \mathcal{A} : \S \in \mathcal{A}\}$ of $\mathcal{A}$, and the cardinality of $\mathcal{A}_0$ is κ. The collection $\mathcal{A}$ is of $rank \leq \kappa$ if $\mathcal{A}$ is of $rank \leq \kappa$ at every point of X. It is of $rank$ κ if $\mathcal{A}$ is of $rank \leq \kappa$ and $\mathcal{A}$ is of $rank$ κ at some point of X.

If κ is a limit cardinal, the collection $\mathcal{A}$ is of $rank$ $(\kappa-)$ at x if it is of $rank \leq \kappa$ at x, is not of $rank$ λ at x for each $\lambda < \kappa$, and there exists an incomparable subcollection $\mathcal{A}' = \{\mathcal{A} \in \mathcal{A} : \S \in \mathcal{A}\}$, with cardinality $\geq \lambda$. It is of $rank$ $(\kappa-)$ if $\mathcal{A}$ is of $rank \leq \kappa$, is not of $rank$ κ, and is of $rank$ $(\kappa-)$ at some point of X. It is of $rank$ $(\kappa - -)$ if $\mathcal{A}$ is of $rank \leq \kappa$, is not of $rank$ κ, nor of $rank$ $(\kappa-)$, and for every $\lambda < \kappa$ there is a point of X at which $\mathcal{A}$ is of $rank \geq \lambda$.

A collection of some finite $rank$ n or of $rank$ $(\aleph_0-)$ or of $rank$ $(\aleph_0 - -)$ is said to be of sub-infinite $rank$ [6].

Definition 4.3 A collection $\mathcal{A}$ of subsets of a set X is Notherian, if every ascending sequence $A_1 \subset A_2 \subset \ldots$ of members of $\mathcal{A}$ is finite [6].

In [6], Lindgren and Nyikos prove that each space with a Notherian base of sub-infinite $rank$ is (hereditarily) metacompact. Here, we also have the next result.

Theorem 4.4 Every space with a Notherian base of sub-infinite $rank$ is monotonically orthocompact.

Proof Suppose $\mathcal{B}$ is a Notherian base of sub-infinite $rank$ for the space X, and $\mathcal{U}$ is an open cover of X. For each $x \in X$, let $\mathcal{B}_\mathcal{U}(\S)$ be the collection of maximal members of $\mathcal{B}$ that contain x and are contained in some member of $\mathcal{U}$, then $\mathcal{B}_\mathcal{U}(\S)$ is finite. Set $\mathcal{V} = \cup\{\mathcal{B}_\mathcal{U}(\S) : \S \in \mathcal{X}\}$, then $\mathcal{V}$ is a point-finite (certainly an interior-preserving) open refinement of $\mathcal{U}$. In fact, for each $x \in X$, if $x \in V \in \mathcal{V}$, then $V \in \mathcal{B}_\mathcal{U}(\S)$. Otherwise, there exists $y \in X$ such that $V \in \mathcal{B}_\mathcal{U}(\dagger)$, $x \neq y$. By the definition of $\mathcal{B}_\mathcal{U}(\S)$, there exists $B \in \mathcal{B}_\mathcal{U}(\S)$ such that $x \in V \subset B$ with $V \neq B$, this is a contradiction with $V \in \mathcal{B}_\mathcal{U}(\dagger)$.

By Corollary 2.10, we conclude that there exists a transitive neighbornet $R_\mathcal{U} = \{\{x\} \times \cap(\mathcal{V})_\S : \S \in \mathcal{X}\}$ of X such that $\{R_\mathcal{U}\{x\} : x \in X\}$ is an open refinement of $\mathcal{V}$. Clearly $\{R_\mathcal{U}\{x\} : x \in X\}$ is also an open refinement of $\mathcal{U}$.

Define an operator $T : \Xi \longrightarrow \varphi$ from the set Ξ of all open covers of X to the set φ of all transitive neighbornets of X such that, for each $\mathcal{U} \in \div$ we assign the transitive neighbornet $R_\mathcal{U}$ defined as the above. Then T is a monotonically orthocompact operator. In fact, let $U_1, U_2 \in \Xi$ such that U_1 refines U_2. Then $\mathcal{B}_{\mathcal{U}_\infty}(\S)$ is a partial refinement of $\mathcal{B}_{\mathcal{U}_\in}(\S)$, and thus $\mathcal{V}_\infty$ refines $\mathcal{V}_\in$. It follows that $R_{\mathcal{U}_\infty} \subset R_{\mathcal{U}_\in}$, and $T(\mathcal{U}_\infty) \subset \mathcal{T}(\mathcal{U}_\in)$. On the other hand, for each $\mathcal{U} \subset \div$ we have that $\{T(\mathcal{U})\{\S\} : \S \in \mathcal{X}\} = \{\mathcal{R}_\mathcal{U}\{\S\} : \S \in \mathcal{X}\}$ refines $\mathcal{U}$. Therefore, the space X is monotonically orthocompact.

Theorem 4.5 The finite product of spaces, each of which has a Noetherian base of sub-infinite *rank*, has a Noetherian base of sub-infinite *rank* [6].

Definition 4.6 Let $(X,\mathcal{T})$ be a topological space and let M be a subset of X. Set $\mathcal{T}' = \{\mathcal{U} \cup \mathcal{B} : \mathcal{U} \in \mathcal{T}, \mathcal{B} \subset \mathcal{M}\}$, then $(X,\mathcal{T})$ is called the discretization of X by M [1].

In [6], Lindgren and Nyikos give an example of hereditarily metacompact space which has no base of finite rank. Now, we also indicate that it is a monotonically orthocompact space having no ortho-base. Therefore the question asked by Junnila and Kunzi in [5] is solved.

Example. Let D^* be the discretization of the ordinal space $\omega_1 + 1$ by ω_1, then D^* has a Noetherian *rank* 1 base $\{\{x\} : x < \omega_1\} \cup \{\{x : x > \alpha\} : \alpha < \omega_1\}$. Its product with $\omega_0 + 1$ denoted by $D = D^* \times (\omega_0 + 1)$ is known as Dieudonné plank. Since $\omega_0 + 1$ has a Noetherian *rank* 1 base $\{\{x\} : x < \omega_0\} \cup \{\{x : x > n\} : n \in \omega_0\}$, by Theorem 4.5 we know that D has a Noetherian base of sub-infinite *rank*. Thus, in light of Theorem 4.4, D is a monotonically orthocompact space. On the other hand, the point $(\omega_1 + 1, \omega_0 + 1)$ obviously has no local base which is linearly ordered by inverse inclusion. From the Theorem 3.2 we know that D has no ortho-base.

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