ASYMPTOTIC BALAYAGE IN HARDY AND BERGMAN SPACES

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Abstract. For a range of Hardy and Bergman spaces X and sets of uniqueness K we show that for any functional $\phi \in X^*$ there exists a sequence of measures (μ_n) supported on K converging weak* to ϕ. In particular, we consider H^2 of the right half plane and obtain a Carleman-type formula for the continuous wavelet transform.

Keywords: Measures, Balayage, Hardy spaces, Bergman spaces, Sets of uniqueness.
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1. Introduction

1.1. **Notation and conventions.** Let $\mathbb{D}$ denote the unit disc in the complex plane:

$$\mathbb{D} = \{z = x + iy : |z| < 1\},$$

and $\mathbb{T}$ the unit circle:

$$\mathbb{T} = \{z = x + iy : |z| = 1\}.$$

For $1 \leq p < \infty$, $H^p(\mathbb{D})$ is the Hardy class of functions $f(z)$ analytic in $\mathbb{D}$ with

$$\|f\|_p^p = \sup_{r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

For $1 \leq p < \infty$, $L_a^p(\mathbb{D})$ is the Bergman space of functions $f(z)$ analytic on $\mathbb{D}$ with

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where dA denotes Lebesgue area measure on $\mathbb{D}$.

The disc algebra $A(\mathbb{D})$ is algebra of functions analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

Let Ω be a measurable subset of the plane. A *Banach function space*, (*BFS*) $X = X(\Omega)$ is a Banach space of analytic functions defined on Ω.

Let K be a compact subset of the complex plane. $M(K)$ is the Banach space of regular Borel measures on K with finite total variation.

For K a measurable subset of $\mathbb{T}$ or $\mathbb{R}$, $|K|$ denotes the linear measure of K, and χ_K the characteristic function of K.

Theorem A (etc) will denote a known result in the literature. Numbered results will be proved.

1.2. **Background.** This paper is motivated by a problem in systems theory: to what extent can a transfer function be reconstructed from measurements restricted to an interval K? Transfer functions are normally assumed to belong to a Hardy space H^p, in which case it follows from the F&M Riesz theorem that reconstruction is possible; moreover, the Carleman formula can be used to perform this reconstruction.

In practise, however, the measurements are both corrupted and finite in number and we cannot directly apply the Carleman formula. In this situation we can formulate a well-posed and applicable problem by asking for the best analytic approximant to an L^p function on the interval, subject to a norm constraint outside the interval where the measurements are taken. It turns out in this more general context that the Carleman formula is still very useful; a full discussion can be found in [2] and [3].

Carleman's formula states that if K is a subset of the unit circle $\mathbb{T}$ which has positive linear measure and $\{\Phi_n : n = 1, 2, \dots\}$ is the system of outer functions defined by

$$\Phi_n(z) = \exp\left[n\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\chi_K\frac{dt}{2\pi}\right]$$

then ([1], page 2) for $f \in H^p$ the sequence f_n

$$f_n(z) = \frac{1}{2\pi}\int_K f(e^{i\theta})\frac{\Phi_n(e^{i\theta})}{\Phi_n(z)}\frac{1}{1-ze^{-i\theta}}d\theta$$

converges locally uniformly in the disc to $f(z)$. Using Toeplitz operators and the boundedness of the Riesz projection Patil [12] showed that $f_n \to f$ in H^p for $1 < p < \infty$.

This formula is an example of a sweeping process: the map $f \mapsto f(z)$ (for $z \in \mathbb{D}$ fixed) defines a bounded linear functional on H^p and the measures $d\mu_n = \dfrac{\Phi_n(e^{i\theta})}{\Phi_n(z)}\dfrac{1}{1-ze^{-i\theta}}\dfrac{d\theta}{2\pi}$ functionals on $L^p(K)$.

In applications to systems theory we are often interested in reconstructing coefficient functionals in a given basis expansion of a transfer function from measurements of the function given on a set of uniqueness (See §3 for an example).

For a given BFS X and set of uniqueness K, we ask whether for each functional $\phi \in X^*$ there exists a sequence of measures $d\mu_n$ supported on K converging weak* to ϕ.

The breakdown of this paper is as follows. In §2 we define *asymptotic balayage* (AB) and we show, Theorem 2 that AB is possible for a range of Hardy and Bergman spaces. In §3 we consider the particular case of the wavelet transform and the Hardy space H^2 of a half plane and obtain a Carleman-type formula, Theorem 5, for the continuous wavelet transform. In particular we obtain, Theorem 5 (ii), a wavelet version of the Patil result, using distinct methods.

1.3. **Sets of uniqueness.** Suppose that $X = X(\Omega)$ is a BFS. A subset K of $\overline{\Omega}$ is called a set of uniqueness for X if $f \in X$ and $f|_K = 0$ implies $f \equiv 0$.

We consider $X = H^p(\mathbb{D})$, $1 \leq p < \infty$ or $X = A(\mathbb{D})$ and subsets K of the unit circle. The F&M Riesz theorem ([9], p.51) says that such K of positive measure are sets of uniqueness for X.

We also consider the Bergman spaces $L^p_a(\mathbb{D})$. A classification of sets of uniqueness for L^p_a is not known but an important subclass are the *sets of sampling*: a discrete subset (z_j) of $\mathbb{D}$ is called a set of sampling for L^p_a if there exist positive constants A and B such that for all $f \in L^p_a$ the following inequalities hold:

$$A\|f\|_p^p \leq \sum_j |f(z_j)|^p (1-|z_j|)^2 \leq B\|f\|_p^p.$$

The existence of sets of sampling is proved in the paper of Coifman and Rochberg [6] and an exact classification given in Seip [14].

2. Asymptotic balayage

Let X be a BFS on $\mathbb{D}$ and $K \subset \overline{\mathbb{D}}$ a set of uniqueness for X. Then we say that *asymptotic balayage onto* K is possible for X if for any functional ϕ in X^* there exists a sequence $(\mu_n) \in M(K)$ such that

$$\int f d\mu_n \to \phi(f), \quad n \to \infty, \tag{1}$$

for every f belonging to X. That is, ϕ is a weak* limit of measures supported on K. If instead, for each functional ϕ in X^* there exists a measure $\mu \in M(K)$ such that

$$\int f d\mu = \phi(f),$$

we say that *exact* (or *classical*) *balayage* onto K is possible.

Remarks .

(i) The Carleman formula shows that the point evaluation functionals $\phi(f) = f(z)$, for $z \in \mathbb{D}$ are weak* limits of measures supported on K.
(ii) The classical case is $X = h(\Omega)$, the space of harmonic functions on Ω with continuous extension to $K = \partial\Omega$. Here AB reduces to exact balayage [10].

In this paper we are interested in the case that $X = H^p$, L^p_a or A.

Lemma 1. *Suppose that K is a set of positive measure on $\mathbb{T}$ and that ϕ_k, $k = 0, 1, \ldots$ are the functionals on H^1 given by*

$$\phi_k(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi}, \quad f \in H^1.$$

Then for each k there exists a sequence of measures μ_n supported on K such that

$$\int_K f d\mu_n \to \phi_k(f), \quad f \in X.$$

Proof. Fix $r < 1$. For $k = 0, 1, 2, \ldots$ we may write

$$\phi_k(f) = \frac{1}{2\pi r} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-ik\theta} d\theta.$$

We approximate $\phi_k(f)$ by Riemann sums: writing $\theta_j = 2\pi j/N$ we have

$$\frac{1}{rN} \sum_{j=1}^{N} f(re^{in\theta_j}) e^{-ik\theta_j} \to \phi_k(f),$$

as $N \to \infty$. Next using the Carleman formula ([1], page 2) we have for each j, $j = 1, \ldots, N$

$$\left| f(re^{i\theta_j}) - \int_K f(e^{i\theta}) d\mu_{n,j} \right| \leq \frac{e^{-n|K|}}{1-r} \|f\|_1.$$

where $d\mu_{n,j} = \dfrac{\Phi_n(e^{i\theta})}{\Phi_n(re^{i\theta_j})} \dfrac{1}{1 - re^{i(\theta_j - \theta)}} \dfrac{d\theta}{2\pi}$.

Therefore given $\epsilon > 0$ we may choose N and n so that

$$\left| \phi_k(f) - \frac{1}{rN} \sum_{j=1}^{N} \int_K f(e^{i\theta}) d\mu_{n,j}(e^{i\theta}) \right| < \epsilon \|f\|_1.$$

□

Theorem 2. (i) *For $X = H^p$, $1 \leq p < \infty$, or $A(\mathbb{D})$ and K a subset of the unit circle of positive linear measure AB onto K is possible.*

(ii) *For $X = L^p_a$, $1 < p < \infty$, and $K = (z_j)$ a discrete subset of $\mathbb{D}$ which is a set of uniqueness for X and for which there exists $B > 0$ with*

$$\sum_j |f(z_j)|^p (1 - |z_j|)^2 \leq B \|f\|_p^p, \qquad f \in L^p_a,$$

AB onto K is possible.

Proof. We first consider the case $X = H^p$ or L^p_a where $1 < p < \infty$. We define the operator $T : X \to L^p(K)$ or $\ell^p((1 - |z_j|)^2)$ by

$$T : f \mapsto f|_K$$

Since K is a set of uniqueness for X it follows that the map T is injective. The Hahn-Banach theorem implies that T^* has weak* dense range. But

$$\begin{aligned} T^* &: L^q \to X^*, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ T^* &: \mu \mapsto \mu(Tf) = \mu(f|_K), \quad f \in X. \end{aligned}$$

Since X is reflexive, T^* has norm dense range. Thus if ϕ belongs to X^*, there exists $(\mu_n) \subset L^q(K)$ with $\|T^*\mu_n - \phi\| \to 0$ as $n \to \infty$. This implies that

$$\int f\,d\mu_n \to \int f\,d\mu, f \in X,$$

which completes the proof for $X = H^p$ or L^p_a, $1 < p < \infty$.

For $X = H^1$ or $X = A(\mathbb{D})$ the preceding argument does not work since one is only able to infer that the unit ball of X^* is metrisable ([13], Theorem 3.16). However it is possible to proceed by more concrete arguments.

We consider next $X = A(\mathbb{D})$. By Fejèr's theorem there exist trigonometric polynomials p_n such that for all $f \in C(\mathbb{T})$

$$\int_{-\pi}^{\pi} f p_n \frac{d\theta}{2\pi} \to \int_{-\pi}^{\pi} f\,d\mu, \quad n \to \infty. \tag{2}$$

Here the p_n are formed from the n'th Césaro means of the Fourier coefficients for μ. In particular (2) holds for $f \in A(\mathbb{D})$. Now if $p_n = \sum_{k=-n}^{n} a_k e^{-ik\theta}$,

$$\int_{-\pi}^{\pi} f p_n \frac{d\theta}{2\pi} = \sum_{k=-n}^{n} a_k \int_{-\pi}^{\pi} f e^{-ik\theta} \frac{d\theta}{2\pi}.$$

Now since $f \in A(\mathbb{D})$,

$$\int_{-\pi}^{\pi} f e^{-ik\theta} \frac{d\theta}{2\pi} = 0,$$

for $k = -1, -2, -3, \ldots$ and so

$$\int_{-\pi}^{\pi} f p_n \frac{d\theta}{2\pi} = \sum_{k=0}^{n} a_k \int_{-\pi}^{\pi} f e^{-ik\theta} \frac{d\theta}{2\pi}.$$

We then apply Lemma 1 and deduce that there exists a sequence of measures μ_n supported on K such that

$$\int_K f\,d\mu_n \to \int f\,d\mu,$$

for every f belonging to $A(\mathbb{D})$. The proof for H^1 is the same and will be omitted. $\square$

It is interesting to know when asymptotic balayage reduces to classical balayage.

Theorem 3. *Exact balayage obtains for $X = H^p$, L^p_a or $A(\mathbb{D})$ if, and only if, there exists a positive constant A such that*

$$\begin{aligned} \sup_{z \in K} |f(z)| &\geq A\|f\|_\infty, \ \textit{if} \quad X = A(\mathbb{D}) \\ \|f\|_{L^p(K)} &\geq A\|f\|_p, \ \textit{if} \quad X = H^p, 1 \leq p < \infty \end{aligned}$$

$$\left(\sum_j |f(z_j)|^p(1-|z_j|)^2\right)^{1/p} \quad \geq \quad A\|f\|_p, \quad f \in X = L^p_a.$$

Proof. We consider the operator

$$\begin{aligned} T &: X \to Y, \\ T &: f \mapsto f|_K, \end{aligned}$$

where $Y = C(K)$ or $L^p(K)$ when $X = A(\mathbb{D})$ or $X = H^p(\mathbb{D})$ or L^p_a respectively. This is an injection since K is a set of uniqueness for X.

Note that exact balayage is possible if T^* is a surjection. We know from the Hahn-Banach theorem that T^* has weak* dense range. We need to show that T^* has closed range. We prove:

(*) *T^* has weak* closed range if, and only if, there exists $A > 0$ such that $\|Tf\| \geq A\|f\|$, $f \in X$.*

Suppose first that T^* is weak* closed. Then by Banach's closed range theorem range(T) is closed in Y so is complete. Therefore by the open mapping theorem there exists a positive constant A such that for each y in Y there is an x in X with $y = Tx$ and $A\|x\| \leq \|y\|$. Thus $\|T^{-1}\| \leq A$ or $\|T\| \geq A$.

Next suppose that $\|Tf\| \geq A\|f\|$ for some positive constant A. Then T has closed range: let (y_n) be a Cauchy sequence in range$T$ and suppose $y_n \to y \in Y$. Then $y_n = Tx_n$ and the condition assumed on T implies that (x_n) is a Cauchy sequence in X and thus converges to $x \in X$. By the continuity of T, we note that $y = Tx$ and deduce that T has closed range. Thus by the closed range theorem again, T^* is weak* closed in X^* which establishes the other implication in (*) and hence the theorem.

□

Remark . The proof of Theorems 2 and 3 may be applied to an arbitrary set of uniqueness. The restriction operator $T : f \mapsto f|_K$ defines a map into a normed space Y with norm given by $\|y\| = \|f\|$ where $y = Tf$. By completing the space we have a mapping between Banach spaces.

3. A Carleman formula for the continuous wavelet transform

In [8] the continuous wavelet transform and its discrete analogue are used to obtain wavelet decompositions of a range of Hardy-Sobolev Spaces $H^{2,p}$ using as wavelets simple rational functions. In a systems theory context rational functions are desirable since they correspond to finite dimensional systems and a central problem is to approximate or reconstruct a transfer function using rational functions in the H^2 or H^∞ norms.

The main result in this section, Theorem 5, is a Carleman formula for the continuous wavelet transform. In particular part (ii) is a wavelet analogue of the Patil result. The principal ingredient in the proof of Theorem 5 is an atomic decomposition of H^2 using Cauchy kernels, Theorem 6.

We define the Hardy class H^2 for the right half plane. For g belonging to $L^2((0,\infty))$ we write $G = (\mathcal{L}g)(s)$ for the Laplace transform of g

$$G(s) = (\mathcal{L}g)(s) = \int_0^\infty g(t)e^{-st}dt.$$

$H^2(\mathbb{C}_+)$ denotes the Hardy space of functions $F(s)$ analytic in the right half plane and such that

$$\|F\|_2 = \sup_{x>0} \int_\infty^\infty |F(x+iy)|^2 dy < \infty.$$

By the Paley-Wiener theorem every $F(s)$ belonging to $H^2(\mathbb{C}_+)$ is the Laplace transform of some $f(t) \in L^2((0,\infty))$ such that $\|F\|_2 = (2\pi)^{1/2}\|f\|_2$. Upper case letters will be used to denote the Laplace transform of the corresponding lower case letter. *e.g.* $\Psi(s) = (\mathcal{L}\psi)(s)$. (For further details of Hardy spaces defined on a half plane see [9], Chapter 8.)

First we recall the foundations of the theory of the continuous wavelet transform applied to $H^2(\mathbb{C}_+)$. For further details the reader is referred to the recent paper [8].

A function $\Psi(s) = \mathcal{L}(\psi)(s)$, $s = x+iy$, $x > 0$ belonging to $H^2(\mathbb{C}_+)$ is called an *admissible wavelet* provided the following condition obtains:

$$C_\Psi = \int_0^\infty \frac{|\psi(t)|^2}{t} dt < \infty.$$

For a positive parameter a and real b define

$$\Psi_{a,b}(y) = a^{1/2}\Psi\left(\frac{y-b}{a}\right).$$

Daubechies [7] proves the following basic results.

Theorem A. *Suppose that $\Psi(s) \in H^2$ is admissible. Then the following two statements hold:*

(i) *For $F, G \in H^2$*

$$\langle F, G\rangle = \frac{1}{C_\Psi}\int_{-\infty}^\infty \int_0^\infty \frac{dadb}{a^2} \langle F, \Psi_{a,b}\rangle \langle \Psi_{a,b}, G\rangle.$$

(ii) *For $F \in H^2$ we have*

$$\left\| F(y) - \frac{1}{C_\Psi} \iint_{\substack{A_1 \le a \le A_2 \\ |b| \le B}} \frac{dadb}{a^2} \langle F, \Psi_{a,b}\rangle \Psi_{a,b} \right\|_2 \to 0,$$

as $A_1 \to 0$ and $A_2, B_1 \to \infty$.

Suppose that K is a subset of the imaginary axis of positive linear measure. We define a system $G_n(z)$, $n = 1, 2, \dots$ of bounded analytic functions by

$$G_n(z) = \exp\left[\frac{n}{\pi}\int_{-\infty}^{\infty} \chi_K \frac{tz+i}{t+iz}\frac{dt}{1+t^2}\right].$$

Note that $|G_n(iy)| = 1$ for almost all real y with $iy \notin K$. By transforming to the disc we see also that $|G_1(z)| > 1$ for $z \in \mathbb{C}_+$ and since $G_n(z) = (G_1(z))^n$ it follows that $|G_n(z)| \to \infty$ as $n \to \infty$.

The following identity will be used in the proof of Theorem 5.

Lemma 4. *For F, $G \in H^2$ we have*

$$\frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{dadb}{a^2}\left\langle FG_n, \Psi_{a,b}\right\rangle \left\langle \Psi_{a,b}, \frac{H}{\overline{G}_n}\right\rangle = \langle F, H\rangle$$

Proof. For F, $H \in H^2$, we have FG_n and $H/|G_n| \in L^2$ and so by Proposition 2.4.1 in [7], page 24,

$$\begin{aligned}\langle F, H\rangle &= \left\langle FG_n, \frac{H}{\overline{G}_n}\right\rangle \\ &= \frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dadb}{a^2}\left\langle FG_n, \Psi_{a,b}\right\rangle \left\langle \Psi_{a,b}, \frac{H}{\overline{G}_n}\right\rangle.\end{aligned}$$

For $a < 0$, $\Psi_{a,b}$ is antianalytic so that $\langle FG_n, \Psi_{a,b}\rangle = 0$. Hence

$$\langle F, H\rangle = \frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{dadb}{a^2}\left\langle FG_n, \Psi_{a,b}\right\rangle \left\langle \Psi_{a,b}, \frac{H}{\overline{G}_n}\right\rangle.$$

□

We now state the main result of this section.

Theorem 5. *Suppose that K is a subset of the imaginary axis of positive linear measure and χ_K the characteristic function of K. Then with the system G_n defined above we have*

(i) *For F, $G \in H^2$*

$$\frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{dadb}{a^2}\left\langle FG_n\chi_K, \Psi_{a,b}\right\rangle \left\langle \Psi_{a,b}, \frac{H}{\overline{G}_n}\right\rangle \to \langle F, H\rangle,$$

(ii) *For $F \in H^2$ we have*

$$\left\| F - \frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\ |b|\le B}} \frac{dadb}{a^2}\left\langle FG_n\chi_K, \Psi_{a,b}\right\rangle \frac{\Psi_{a,b}}{G_n}\right\|_2 \to 0,$$

as $A_1 \to 0$ and A_2, $B_1 \to \infty$.

In order to prove Theorem 5 we establish an atomic decomposition of H^2 using normalised Cauchy kernels following the techniques used by Bonsall [4], [5]. The theorem is essentially proved in Luecking [11] for the disc case although details for the norm inequalities equivalent to the first part of the next result are lacking. We establish the norm inequalities using the method in [7], Chapter 3, which has the merit of giving explicit bounds for the constants A and B given below.

We define mixed norm spaces $\ell^{2,1}$ and $\ell^{2,\infty}$ by

$$\ell^{2,1} = \{\boldsymbol{\lambda} = (\lambda_{j,k}) : \|\boldsymbol{\lambda}\|_{2,1} = \sum_j (\sum_k |\lambda_{j,k}|^2)^{1/2} < \infty\},$$

and

$$\ell^{2,\infty} = \{\boldsymbol{\lambda} = (\lambda_{j,k}) : \|\boldsymbol{\lambda}\|_{2,\infty} = \sup_j (\sum_k |\lambda_{j,k}|^2)^{1/2} < \infty\}.$$

It is readily seen that $\ell^{2,\infty}$ may be identified with the Banach space dual of $\ell^{2,1}$.

Theorem 6. *Let $C(y) = (1+iy)^{-1} = \mathcal{L}(e^{-t})$ be the Cauchy kernel for the right half plane. For a fixed positive parameter b_0 and for $j,\ k \in \mathbb{Z}$, define the system $C_{j,k}$ by setting $C_{j,k}(y) = 2^{j/2}C(2^j y - kb_0)$.*

(i) *Then for b_0 positive and sufficiently small there exist constants $A = A(b_0)$ and $B = B(b_0)$ such that for $F \in H^2$ the following pair of inequalities hold:*

$$A\|F\|_2^2 \le \sup_{j\in\mathbb{Z}} \frac{1}{2^j} \sum_{k\in\mathbb{Z}} |F(2^j + i2^j kb_0)|^2 \le B\|F\|_2^2$$

(ii) *Every F may be decomposed in the sense of the H^2 norm in the form*

$$F = \sum \lambda_{j,k} C_{j,k} \tag{3}$$

where $\boldsymbol{\lambda} = (\lambda_{j,k}) \in \ell^{2,1}$ and moreover

$$B^{-1/2}\|F\|_2 \le \inf \|\boldsymbol{\lambda}\|_{2,1} \le A^{-1/2}\|F\|_2,$$

where the infimum is taken over all decompositions (3) of F.

3.1. **Proof of Theorem 5 (i):** By Lemma 4,

$$\frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{dadb}{a^2} \langle FG_n, \Psi_{a,b}\rangle \left\langle \Psi_{a,b}, \frac{H}{\overline{G}_n}\right\rangle = \langle F, H\rangle .$$

Therefore

$$\begin{aligned}\langle F, H\rangle &- \frac{1}{C_\Psi}\int_{\infty}^{\infty}\int_0^{\infty} \frac{dadb}{a^2} \langle FG_n\chi_K, \Psi_{a,b}\rangle \left\langle \frac{\Psi_{a,b}}{G_n}, H\right\rangle \\ &= \frac{1}{C_\Psi}\int_{\infty}^{\infty}\int_0^{\infty} \frac{dadb}{a^2} \langle FG_n\chi_{\mathbb{R}\setminus K}, \Psi_{a,b}\rangle \left\langle \frac{\Psi_{a,b}}{G_n}, H\right\rangle\end{aligned}$$

We apply Theorem 6 to H and write $H = \sum \lambda_{j,k} C_{j,k}$ with $\sum_j (\sum_k |\lambda_{j,k}|^2)^{1/2} < \infty$. Then using the reproducing nature of the $C_{j,k}$'s we obtain

$$
\begin{aligned}
&\big|\langle F, H\rangle - \frac{1}{C_\Psi}\int_\infty^\infty \int_0^\infty \frac{dadb}{a^2}\langle FG_n\chi_K, \Psi_{a,b}\rangle \left\langle \frac{\Psi_{a,b}}{G_n}, H\right\rangle\big| \\
&\quad = \left|\frac{1}{C_\Psi}\int_\infty^\infty \int_0^\infty \frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K}, \Psi_{a,b}\right\rangle \left\langle \frac{\Psi_{a,b}}{G_n}, H\right\rangle\right| \\
&\quad = \left|\frac{1}{C_\Psi}\int_\infty^\infty \int_0^\infty \frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K}, \Psi_{a,b}\right\rangle \left\langle \frac{\Psi_{a,b}}{G_n}, \sum_{j,k}\lambda_{j,k}C_{j,k}\right\rangle\right| \\
&\quad = \left|\frac{1}{C_\Psi}\sum_{j,k}\overline{\lambda}_{j,k}\int_\infty^\infty \int_0^\infty \frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K}, \Psi_{a,b}\right\rangle \left\langle \frac{\Psi_{a,b}}{G_n}, C_{j,k}\right\rangle\right| \\
&\quad = \left|\frac{1}{C_\Psi}\sum_{j,k}\frac{\overline{\lambda}_{j,k}}{G_n(2^j + i2^j kb_0)}\int_\infty^\infty \int_0^\infty \frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K}, \Psi_{a,b}\right\rangle \langle \Psi_{a,b}, C_{j,k}\rangle\right| \\
&\quad \leq \sum_j \left(\sum_k \frac{|\lambda_{j,k}|^2}{|G_n(2^j + i2^j kb_0)|^2}\right)^{1/2} \times \left(\sum_k |\left\langle FG_n\chi_{\mathbb{R}\setminus K}, C_{j,k}\right\rangle|^2\right)^{1/2} \\
&\quad = \sum_1 + \sum_2
\end{aligned}
$$

Here $\sum_2$ is a sum over j,k with N taken so large that

$$
\sum_{|j|<N}\left(\sum_{|k|\geq N}\frac{|\lambda_{j,k}|^2}{|G_n(2^j + i2^j kb_0)|^2}\right)^{1/2} + \sum_{|j|\geq N}\left(\sum_k \frac{|\lambda_{j,k}|^2}{|G_n(2^j + i2^j kb_0)|^2}\right)^{1/2} < \epsilon
$$

This is possible since $(\lambda_{j,k}) \in \ell^{2,1}$ and $|G_n|^{-1} < 1$.

Next we choose n so large that

$$
\sum_1 = \sum_{|j|<N}\left(\sum_{|k|\leq N}\frac{|\lambda_{j,k}|^2}{|G_n(2^j + i2^j kb_0)|^2}\right)^{1/2} < \epsilon
$$

Now $\left\langle FG_n\chi_{\mathbb{R}\setminus K}, C_{j,k}\right\rangle$ is a constant multiple of the analytic projection of $FG_n\chi_{\mathbb{R}\setminus K}$ at $2^j + i2^j kb_0$, and so since $|G_n(iy) = 1|$ *a.e.* for $iy \notin K$ we have

$$
\sup_j \sum_k |\left\langle FG_n\chi_{\mathbb{R}\setminus K}, C_{j,k}\right\rangle|^2 \leq B\|F\|^2
$$

Thus we have

$$\left|\langle F,H\rangle - \frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dadb}{a^2}\langle FG_n\chi_K,\Psi_{a,b}\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right| < 2B^{1/2}\|F\|_2\epsilon$$

Since ϵ was arbitrary the limit in (i) obtains as $n\to\infty$.

3.2. **Proof of Theorem 5 (ii):** By the Hahn-Banach theorem and Lemma 4

$$\begin{aligned}
&\left\|F - \iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\langle FG_n\chi_K,\Psi_{a,b}\rangle\frac{\Psi_{a,b}}{G_n}\right\|_2\\
&\qquad= \sup_{\|H\|_2=1}\left|\left\langle F - \frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\langle FG_n\chi_K,\Psi_{a,b}\rangle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right|\\
&\qquad= \left|\frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dadb}{a^2}\langle FG_n,\Psi_{a,b}\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right.\\
&\qquad\quad - \frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\langle FG_n\chi_K,\Psi_{a,b}\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\\
&\qquad\quad - \frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K},\Psi_{a,b}\right\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\\
&\qquad\quad \left.+ \frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K},\Psi_{a,b}\right\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right|\\
&\qquad\le \left|\frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dadb}{a^2}\langle FG_n,\Psi_{a,b}\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right.\\
&\qquad\quad \left.- \frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\langle FG_n,\Psi_{a,b}\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right|\\
&\qquad\quad + \left|\frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K},\Psi_{a,b}\right\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right|\\
&\qquad= \left|\frac{1}{C_\Psi}\iint_D\frac{dadb}{a^2}\langle FG_n,\Psi_{a,b}\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right|\\
&\qquad\quad + \left|\frac{1}{C_\Psi}\iint_{\substack{A_1\le a\le A_2\\|b|\le B}}\frac{dadb}{a^2}\left\langle FG_n\chi_{\mathbb{R}\setminus K},\Psi_{a,b}\right\rangle\left\langle\frac{\Psi_{a,b}}{G_n},H\right\rangle\right|\\
&\qquad= S_1 + S_2
\end{aligned}$$

where D is the complementary domain to

$$\left.\begin{array}{r} A_1\le a\le A_2\\ |b|\le B\end{array}\right\}$$

We proceed to analyse S_1. By Lemma 4

$$\frac{1}{C_\Psi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dadb}{a^2}\left\langle FG_n,\Psi_{a,b}\right\rangle\left\langle \Psi_{a,b},\frac{H}{\overline{G}_n}\right\rangle=\left\langle F,H\right\rangle,$$

and so

$$\frac{1}{C_\Psi}\iint_D\frac{dadb}{a^2}\left\langle FG_n,\Psi_{a,b}\right\rangle\left\langle \frac{\Psi_{a,b}}{G_n},H\right\rangle\to 0$$

as $A_1\to 0$ and A_2, $B\to\infty$.

We now proceed as in (i) and apply Theorem 6 to H and write $H=\sum_{j,k}\lambda_{j,k}C_{j,k}$ with $\sum_j(\sum_k|\lambda_{j,k}|^2)^{1/2}<\infty$. We deduce that $|\left\langle\Psi_{a,b},H/G_n\right\rangle|$ can be made arbitrarily small by taking n sufficiently large. Hence as $n\to\infty$, $A_1\to 0$ and $A_2,B\to\infty$ the limit in (ii) obtains.

3.3. **Proof of Theorem 6.** To prove the first part of Theorem 6 it is sufficient to find a discrete analogue of

$$\int_{-\infty}^{\infty}|F(2^j+iy)|^2dy$$

for $j\in\mathbb{Z}$. We follow the method in [7]. Note first that $F(2^j+iy)=\mathcal{L}(e^{-2^jt}f(t))$. By applying the Poisson summation formula $\sum_{l\in\mathbb{Z}}\exp(ilax)=2\pi a^{-1}\sum_{k\in Z}\delta(x-2\pi ka^{-1})$ we obtain the following set of deductions:

$$\begin{aligned}
\sum_k|\left\langle F,C_{j,k}\right\rangle|^2&=\sum_k\int_{-\infty}^{\infty}F(iy)\overline{C_{j,k}}(y)dy\int_{-\infty}^{\infty}\overline{F}(iy')C_{j,k}(y')dy'\\
&=2\pi\sum_k\int_0^{\infty}f(t)2^{-j/2}\overline{c}(2^{-j}t)e^{i2^{-j}kb_0t}dt\\
&\qquad\times\int_0^{\infty}\overline{f}(t')2^{-j/2}c(2^{-j}t')e^{-i2^{-j}kb_0t'}dt'\\
&=\frac{(2\pi)^2}{b_0}\sum_k\int_0^{\infty}\int_0^{\infty}f(t)\overline{f}(t')\overline{c}(2^{-j}t)c(2^{-j}t')\\
&\qquad\times\delta(t'-t-2\pi k2^jb_0^{-1})\,dt\,dt'\\
&=\frac{(2\pi)^2}{b_0}\sum_k\int_0^{\infty}f(t)\overline{f}(t-2\pi 2^jk)\overline{c}(2^{-j}t)\\
&\qquad\times c(2^{-j}t-2\pi kb_0)dt\\
&=\frac{(2\pi)^2}{b_0}\int_0^{\infty}|f(t)|^2|c(2^{-j}t)|^2dt+\mathrm{rest}(f)\\
&=\frac{2\pi}{b_0}\int_{-\infty}^{\infty}|F(2^{-j}+iy)|^2dy+\mathrm{rest}(f)
\end{aligned}$$

$$\leq \frac{2\pi}{b_0}\|F\|_2^2 + |\operatorname{rest}(f)|,$$

where

$$\operatorname{rest}(f) = \frac{(2\pi)^2}{b_0}\sum_{k\neq 0}\int_0^\infty f(t)\overline{f}(t-2\pi k 2^j)\overline{c}(2^{-j}t)$$
$$\times\, c(2^{-j}t - 2\pi k b_0^{-1})dt.$$

By the Cauchy-Schwarz inequality for integrals we have

$$|\operatorname{rest}(f)| \leq \frac{(2\pi)^2}{b_0}\sum_{k\neq 0}\left\{\int_0^\infty |f(t)|^2|c(2^{-j}t)||c(2^{-j}t - 2\pi k b_0^{-1})|dt\right\}^{1/2}$$
$$\times\left\{\int_0^\infty |f(t-2\pi k 2^j)|^2|c(2^{-j}t)||c(2^{-j}t - 2\pi k b_0^{-1})|dt\right\}^{1/2}.$$

We make the substitution $t - 2\pi k 2^j b_0^{-1} \mapsto t$ in the second integral and noting that $c(t) = e^{-t}$, $t > 0$ and is zero otherwise deduce that

$$|\operatorname{rest}(f)| \leq \frac{(2\pi)^2}{b_0}\sum_{k\neq 0}\left\{\int_0^\infty |f(t)|^2|c(2^{-j}t)||c(2^{-j}t - 2\pi k b_0^{-1})|dt\right\}^{1/2}$$
$$\times\left\{\int_0^\infty |f(t)|^2|c(2^{-j}t)||c(2^{-j}t + 2\pi k b_0^{-1})|dt\right\}^{1/2}.$$

Now for k with $k \geq 0$,

$$|c(2^{-j}t)||c(2^{-j}t - 2\pi k b_0^{-1})| \leq 1,$$

while

$$|c(2^{-j}t)||c(2^{-j}t + 2\pi k b_0^{-1})| = |c(2^{-j}t)|^2 e^{-2\pi k b_0^{-1}},$$

and so

$$\sum_{k\geq 0} \leq \frac{2\Pi}{b_0}\|F\|_2\|F(2^{-j}+iy)\|_2\sum_{k\geq 0} e^{-2\pi k b_0^{-1}}.$$

Similarly for $k < 0$ we have $|c(2^{-j}t)c(2^{-j} + 2\pi k b_0^{-1})| \leq 1$, and

$$\begin{aligned} |c(2^{-j}t)||c(2^{-j}t - 2\pi k b_0^{-1})| &= |c(2^{-j}t)|^2 e^{-2\pi k b_0^{-1}} \\ &= |c(2^{-j}t)|^2 e^{-2\pi k b_0^{-1}}. \end{aligned}$$

Therefore in either case

$$\| \operatorname{rest}(f) \| \leq \ \|f\|_2^2 \sum_{k \neq 0} e^{-\pi |k| b_0^{-1}}.$$

By the integral test it is easily seen that the sum on the right decreases with b_0. Hence for sufficiently small b_0 we may take

$$A = \frac{2\pi}{b_0} \left(1 - \sum_{k \neq 0} e^{-2\pi k b_0^{-1}} \right),$$

and

$$B = \frac{2\pi}{b_0} \left(1 + \sum_{k \neq 0} e^{-2\pi k b_0^{-1}} \right).$$

Thus the inequalities in (i) follow.

To prove the second claim of the theorem we consider the operator $T : \ell^{2,1} \to H^2$ where $\ell^{2,1}$ is the Banach space of sequences defined above and T is defined by

$$T : \boldsymbol{\lambda} \mapsto \sum_{j,k} \lambda_{j,k} C_{j,k}.$$

Let E be the subset of $\ell^{2,1}$ defined by

$$\begin{aligned} \lambda_{j,k} &= \langle G, C_{j,k} \rangle, j = j_0 \quad \text{fixed} \\ &= 0, \quad \text{otherwise} \end{aligned}$$

By the first part of the theorem $\|\boldsymbol{\lambda}\|_{2,1} < \infty$. Also

$$\begin{aligned} \|T^* G\|_{2,\infty} &= \sup_{\boldsymbol{\lambda} \in \ell^{2,1}} \frac{|(T^* G)(\boldsymbol{\lambda})|}{\|\boldsymbol{\lambda}\|_{2,1}} \\ &\geq \sup_E \frac{\left| \sum_{j,k} \lambda_{j,k} \langle G, C_{j,k} \rangle \right|}{\|\boldsymbol{\lambda}\|_{2,1}} \\ &= \sup_E \frac{\sum_k |\langle G, C_{j,k} \rangle|^2}{(|\langle G, C_{j,k} \rangle|^2)^{1/2}} \\ &= \sup_E (| \langle G, C_{j,k} \rangle |^2)^{1/2} \\ &\geq A \|G\|_2. \end{aligned}$$

Thus T^* has zero kernel and closed range so by Banach's closed range theorem T is a surjection. The proof of the inequalities given in (ii) is elementary and identical to the proof of Theorem 1 equation (2) in [4] and will be omitted.

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