Deformations of Surfaces in 4–space

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Abstract

In this paper we describe geometric properties of embedded liftabilities of immersed 3–manifolds in 4–space into 5–space. It is known that a regular homotopy class of an immersed orientable surface in 3–space is constructed by a pair of an embedded surface and an immersed circle on it. We found an isotopy between embedded lifts of these constructed immersions in 4–space, which covers a regular homotopy of their projections in 3–space. Also we construct non-liftable immersed 3–spheres without quadruple points.

1 Introduction

We assume that all manifolds and maps are smooth. Let M^m and N^n be manifolds with the dimensions m and n respectively. We denote the interval $[0,1]$ by I.

A map $f\colon M^m \to N^n$ $(m < n)$ is called an *immersion* if $df_x\colon T_xM^m \to T_{f(x)}N^n$ has full rank for each $x \in M^m$. Two immersions $f, g\colon M^m \to N^n$ are said to be *regularly homotopic* if there is a homotopy

$$H\colon M^m \times I \to N^n \tag{1}$$

such that setting $H_t(x) = H(x,t)$ for $t \in I$ and $x \in M^m$, then $H_0(x) = f(x)$, $H_1(x) = g(x)$ and H_t is an immersion for all $t \in I$. The homotopy H is called a *regular homotopy* from f to g. Regular homotopy is an equivalence relation.

A *crossing set* of an immersion $f\colon M^m \to N^n$ is the set

$$C(f) = \left\{ x \in M^m | \#(f^{-1}(f(x))) > 1 \right\}. \tag{2}$$

Among intermediate maps of regular homotopy there is a finite number of immersions having non–transversal crossings (see Remark 1.1).

For a point $x \in C(f)$ with $\#(f^{-1}(f(x))) = k$, $f(x)$ is called a k*–tuple point*. If M^m is compact, then $C(f)$ is a union of

$$C_k(f) = \{x \in M^m | \#(f^{-1}(f(x))) = k\}, \quad (k = 2, 3, \ldots, \ell).$$

The crossing set $C(f)$ is a union of immersed n–manifolds in M^m $(n < m)$. We denote the collection of immersed n–manifolds $(n < m)$ in $C(f) \subset M^m$ by $\mathcal{C}(f)$:

$$C(f) = \bigcup_{C \in \mathcal{C}(f)} C. \tag{3}$$

Remark 1.1. *During a regular homotopy* $\{H_t\}, t \in I$ *there are the finite number of points in* I *such that* H_t *does not have transversal crossings.*

A regular homotopy $H\colon M^m \times I \to N^n$ induces a *track*

$$\widehat{H}\colon I \times M^m \to I \times N^n$$

defined by $\widehat{H}(t, x) = (t, H_t(x))$, which is an immersion.

For an immersion $f\colon M^m \to N^n$, a map $\widetilde{f}\colon M^m \to N^n \times \mathbf{R}^k$ is called a *lift over* f *via* p_1 if $p_1 \circ \widetilde{f} = f$, where $p_1\colon N^n \times \mathbf{R}^k \to N^n$ is the projection. If $\widetilde{f}$ is an embedding, then we will call $\widetilde{f}$ an *embedded lift*. We know that the term "lift" is often used as an embedded lift but we distinguish embedded lifts and lifts with crossing points.

Theorem 1.1. [Carter and Saito][CS] *Let* F^2 *be a closed surface and let* $f\colon F^2 \to \mathbf{R}^3$ *be an immersion. Then the crossing set* $C(f)$ *satisfies the following two conditions if and only if* f *has an embedded lift* $\widetilde{f}\colon M^2 \to \mathbf{R}^4$*; that is,* $p_1 \circ \widetilde{f} = f$.

> $(CS1)$ *each pair of immersed curves* $\{C_a, C_b\}$ *with* $f(C_a) = f(C_b)$ *has two colours* a *for* C_a *and* b *for* C_b*, and*

($CS2$) *for* $\{p_1, p_2, p_3\} \subset C_3(f)$ *with* $f(p_1) = f(p_2) = f(p_3)$, *there are neighbourhoods* $U(p_i)$ *in* F^2 *such that three pairs of smooth arcs in* $U(p_i) \cap C(f)$ $(i = 1, 2, 3)$, *have colours,* $\{(a, a), (a, b), (b, b)\}$.

We call these conditions CS–conditions and we call the second condition ($CS2$) the (3,2)–colouring condition.

We will introduce similar conditions for the case of immersions from orientable closed 3–manifolds to $\mathbf{R}^4$.

Let M^3 be an orientable closed 3–manifold and let $f : M^3 \to \mathbf{R}^4$ be an immersion. Then f may have quadruple points, triple arcs and double surfaces as the crossing sets. We will provide the following conditions:

(Y1) $\mathcal{C}(f)$ consists of pairs of two coloured immersed surfaces $\{F_a, F_b\}$ called a–surfaces and b–surfaces respectively and

(Y2) for every set $\{p_1, p_2, p_3\} \subset C_3(f)$ with $f(p_1) = f(p_2) = f(p_3)$, there are neighbourhoods $U(p_i)$ of p_i in M^3, such that three pairs of smooth discs in $U(p_i) \cap C(f)$ $(i = 1, 2, 3)$, have colours, $\{(a, a), (a, b), (b, b)\}$.

The main theorems of this paper are the following.

Theorem 1.2. *Let* M^3 *be an orientable closed 3–manifold and let* $f\colon M^3 \to \mathbf{R}^4$ *be an immersion. Then* f *has an embedded lift into* $\mathbf{R}^5$ *via* $p\colon \mathbf{R}^5 \to \mathbf{R}^4$ *if and only if* $C(f)$ *satisfies* $(Y1)$ *and* $(Y2)$.

We apply Theorem 1.2 to show some regular homotopy on a surface in $\mathbf{R}^3$ is covered by an isotopy in $\mathbf{R}^4$.

It is known that every regular homotopy class of an immersed surface in $\mathbf{R}^3$ is represented by a immersed surface constructed with an embedding $g : F^2 \to \mathbf{R}^3$ and an oriented immersed circle $\gamma : S^1 \to F^2$ [Hug] [Yas1]. We denote the constructed map by $g_\gamma : F^2 \to \mathbf{R}^3$. The immersed circle γ can be modified by smoothing operations at crossing points of γ so that we obtain a set of disjoint embedded circles on the surface F^2. We denote the set by δ. The embedding g and the set of circles δ define an immersion.

Then we have the following theorem.

Theorem 1.3. *Let γ be an oriented circle on a closed surface F^2 and let δ be a set of disjoint oriented simple circles on F^2. Then there is a regular homotopy from g_γ to g_δ and this regular homotopy is covered by an isotopy from the lift $\widetilde{g}_\gamma$ to the lift $\widetilde{g}_\delta$.*

Examples 1. *Let i: $S^2 \to \mathbf{R}^3$ be the inclusion. Let $\bigcirc_+$: $S^1 \to S^2$ be an oriented embedding on the sphere and let $\bigcirc_-$ be a reversed oriented embedding on the sphere. Let 8: $S^1 \to S^2$ be the eight–immersion. Then*

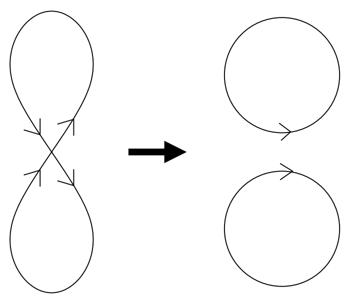


Figure 1: Smoothing 8–immersion to $\{\bigcirc_+, \bigcirc_-\}$–immersion.

we obtain two immersions i_8 and $i_{\{\bigcirc_+,\bigcirc_-\}}$ constructed with the inclusion $i : S^2 \to \mathbf{R}^3$, 8–immersion and its smoothing $\{\bigcirc_+, \bigcirc_-\}$ (see Section 5). The immersion $i_{\{\bigcirc_+,\bigcirc_-\}}$ is regularly homotopic to i_8 and the homotopy is covered by an isotopy from the embedding lift $\widetilde{i}_{\{\bigcirc_+,\bigcirc_-\}}$ to the embedding lift $\widetilde{i}_8$.

It is known that there exists a non–liftable immersed 3–sphere in $\mathbf{R}^4$ with quadruple points [Yas2]. Finally, we construct a non–liftable immersed 3–sphere in $\mathbf{R}^4$ without quadruple points.

Theorem 1.4. *There is a non–liftable immersed 3–sphere in $\mathbf{R}^4$ without quadruple points.*

It is also known that for every orientable closed 3–manifold M^3, there is an immersion from M^3 into $\mathbf{R}^4$ without quadruple points such that it has an embedded lift into $\mathbf{R}^5$ [Yas1]. Thus using connected sums with non–liftable immersed 3–spheres, we can construct an immersed orientable closed 3–manifolds in $\mathbf{R}^4$ without quadruple points so that it does not have an embedded lift so the following holds.

Corollary 1.1. *For every orientable closed 3–manifold M^3, there is a non-liftable immersion from M^3 into $\mathbf{R}^4$ without quadruple points.*

We will prove Theorem 1.2 in Section 3, Theorem 1.3 in Section 6.1 and Theorem 1.4 in Section 7.1.

2 Liftabilities of immersed 3–manifolds

Let M^m and N^n be orientable manifolds with dimensions m and n respectively. Let $f\colon M^m \to N^n$ $(m < n)$ be an immersion. Poenaru [Poe] introduced geometric invariants to detect whether an immersion $f\colon M^m \to N^n$ has an embedded lift $\widetilde{f}\colon M^m \to N^n \times I$ via p. For $m = 2$ and $N^n = \mathbf{R}^3$, Carter and Saito gave CS–conditions [CS] (Theorem 1.1).

We will discuss liftabilities of immersions from orientable closed 3–manifold M^3 in $\mathbf{R}^4$ with regular homotopy tracks and colouring conditions of their intermediate maps. A regular homotopy track over an orientable surface forms an immersed 3–manifold in $\mathbf{R}^4$ with immersed boundary. Let M^3 be an orientable closed 3–manifold. Then there exists an immersion $f\colon M^3 \to \mathbf{R}^4$ [Hir]. We will discuss about a liftability of the immersion into $\mathbf{R}^5$ and we will prove Theorem 1.2.

2.1 Models of k–tuple points

We provide some models of multiple points of surfaces which will be used later. Let

$$B^3(0;r) = \{\ x = (x_1, x_2, x_3) \in \mathbf{R}^3 \mid |x| < r\ \} \tag{4}$$

A model of a set of k–tuple points of planes in $\mathbf{R}^3$ denoted by P_k is given as followings.

Set

$$P'_2 = \begin{cases} x_2 = 0 & \text{or} \\ x_3 = 0 & \end{cases} \tag{5}$$

Let

$$P_2 = P'_2 \cap B^3(0;1) \tag{6}$$

P_2 consists of two discs intersecting at the segment $\{(x_1, 0, 0) \mid |x_1| < 1\}$. We call the segment a *double segment.*

Let $P' = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 = 0\ \}$ and let $P'_3 = P'_2 \cup P'$. Let

$$P_3 = P'_3 \cap B^3(0;1) \tag{7}$$

P_3 consists of three discs containing a *triple point* at the origin.

Let $P'' = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + x_2 - x_3 = 0\ \}$. and let $P'_4 = P'_3 \cup P''$. Set

$$P_4 = P_4\prime \cap B^3(0;1) \tag{8}$$

P_4 consists of four discs containing a *quadruple point* at the origin.

2.2 Crossing sets for immersed 3–manifolds

Let M^3 be an orientable closed 3–manifold and let $f : M^3 \to \mathbf{R}^4$ be an immersion. Then

$$C(f) = \bigcup_{i=2}^{4} C_i(f), \tag{9}$$

where $C_2(f)$ is a set of 2–dimensional open submanifolds, $C_3(f)$ is a set of 1–dimensional open submanifolds and $C_4(f)$ is a set of discrete points of M^3 respectively. We call $f(C_2(f))$ *double surfaces*, $f(C_3(f))$ *triple arcs* and $f(C_4(f))$ *quadruple points*. The set $\mathcal{C}(f)$ consists of a set of immersed surfaces in M^3.

Note 2.1. *The set $f(C(f))$ is a union of immersed surfaces in $f(M^3)$. These surfaces in $\mathcal{C}(f)$ are not necessary to be orientable (see an appendix of* [Ape] and [Yas2]*).*

2.2.1 Triple curves and quadruple points

Let f: $M^3 \to \mathbf{R}^4$ be an immersion. Then there may be a set of triple arcs in $f(M^3) \subset \mathbf{R}^4$. Let τ be a triple arc and let $y \in \tau$. Then there are three distinct points $p_1,\ p_2,\ p_3 \in C_3(f)$ such that $f(p_1) = f(p_2) = f(p_3) = y$.

Let $U(p_i)$ be neighbourhoods of p_i $(i = 1, 2, 3)$ in M^3 and let

$$D_i = U(p_i) \cap C(f) \tag{10}$$

Then D_i is diffeomorphic to P_2 (see (6)).

The immersion f may have some quadruple points. We denote the quadruple point by $\rho \in f(M^3)$. We may assume that there are four points $q_1, \ldots, q_4 \in M^3$ and there are small neighbourhoods $U(q_i)$ of q_i in M^3 $(i = 1, \ldots, 4)$ such that for each $i (= 1, \ldots, 4)$, $U(q_i) \cap C(f)$ is diffeomorphic to the model P_3 (see (7)) denoted by T_i. Each T_i consists of three discs forming a triple point. If we can colour these discs of $\{T_i | \ i = 1, \ldots, 4\}$ with two colours a and b such that their combinations are $\{(a,a,a), (a,a,b), (a,b,b), (b,b,b)\}$, then we will call the colouring $(4,3)$–*condition.* Then the following holds.

Lemma 2.1. *Let $f\colon M^3 \to \mathbf{R}^4$ be an immersion. with a quadruple point $\rho \in f(M^3)$. Let $U(q_i)$ $(i = 1, \ldots, 4)$ be neighbourhoods of $f^{-1}(\rho)$ in M^3. For $\{T_i \subset U(q_i) \mid i = 1, \ldots, 4\}$, there are four triples of pairs of two colours. These triples satisfy $(3,2)$–condition if and only if the set $\{T_i | i = 1, \ldots, 4\}$ satisfies $(4,3)$–condition.*

Proof. Each quadruple point ρ is formed as an intersection of four triple curves. If we view each triple curve as a vertex and view a disc contains the triple curves as an edge, then we obtain a complete graph with four vertices: every pair of vertices is joined with an edge. We may replace each edge with a pair of edges because each edge corresponds to a double disc. We put two colours a and b on each of the pair of edges. We denote the graph by G_Q. Similarly, we can obtain a graph for every T_i in which the graph with three vertices and three edges. We denote the graph by G_{T_i} $(i = 1, \ldots, 4)$. Each edge of G_{T_i} is coloured by a or b.

If all triple curves around the quadruple point ρ satisfy $(3,2)$–condition, then this is interpreted as a relation among $G_{T_i}, (i = 1, \ldots, 4)$ of G_Q. Every vertex of the graph G_Q represents a triple curve which is represented by three discs D_1, D_2 and D_3, where the colouring of these satisfies the $(3,2)$–condition. $T_1, \ldots, T_4$ contain pre–images of triple curves so that for every triple curve, the colourings of D_1, D_2 and D_3 determine a colouring of $T_1, \ldots, T_4$.

Therefore, we can find complete 3–graphs G_{T_i} $(i = 1, \ldots, 4)$ as subgraphs of G_Q such that every vertex of G_Q is contained in three of these subgraphs; that is, if we fix a vertex v of G_Q, then there are three subgraphs each of which contains v. These three pairs of colours of edges around v are $\{(a,a), (a,b), (b,b)\}$. Such subgraphs can be found as shown in Figure 2

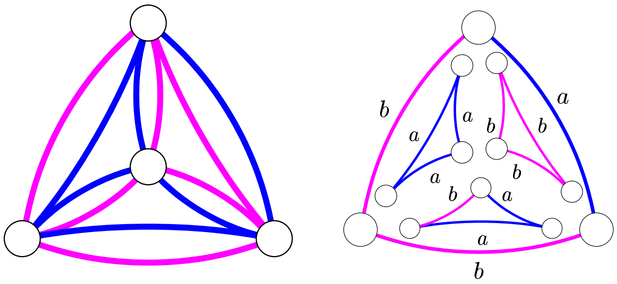


Figure 2: The graph G_Q and its four cycles

Furthermore, we can find such four subgraphs each of which has three vertices so that each pair of subgraphs has different colourings:

$$\{(a,a,a),(a,a,b),(a,b,b),(b,b,b)\}$$

This colouring is equivalent to the $(4,3)$–condition.

Conversely, if the set $\{T_1,\dots,T_4\}$ satisfies the $(4,3)$–condition, then each T_i induces a 3–complete graph where each vertex corresponds to the double curves of T_i and each edge corresponds to a disc of T_i. We denote these graphs by $G_{T_i},\dots,G_{T_4}$. They form a graph G_Q so that $\{T_i|\ i=1,\dots,4\}$ satisfies $(3,2)$–condition. This implies that $\{G_{T_i}|\ i=1,\dots,4\}$ gives the $(3,2)$–condition for each T_i. $\square$

The lemma implies that the colouring conditions around a quadruple point is determined by colouring conditions of all triple curves around it. Thus we have the following.

3 Proof of Theorem 1.2

Proof. We assume that there is an embedded lift $\widetilde{f}\colon M^3\to\mathbf{R}^5$ over f via p. Then $C(f)$ is a union of immersed surfaces in M^3. There are pairs of immersed surfaces $\{F,F'\}\subset\mathcal{C}(f)$ such that $f(F)=f(F')$. For each pair, lower surface will be coloured by b and the upper surface will be coloured by a with respect to the last coordinate of $\mathbf{R}^5$. This implies that f satisfies (Y1).

Choose a coordinate system $(x_1,\dots,x_4)$ of $\mathbf{R}^4$ so that the projection $p_1\colon\mathbf{R}^4\to\mathbf{R}$ defined by $p_1(x_1,\dots,x_4)=x_1$ gives a Morse function $h=p_1\circ f$.

Let $q_0, \ldots, q_k \in M^3$ be critical points with respect to the last coordinate of $\mathbf{R}^5$, $\mathbf{R}$. We may assume that $h(q_i) = c_i < c_{i+1} = h(q_{i+1})$ and we also assume that

$$C(f) \cap \{q_0, \ldots, q_k\} = \emptyset. \tag{11}$$

Take an interval $[c, d] \subset \mathbf{R}$ which does not contain critical values. Then it is not difficult to see that $f(h^{-1}([c, d]))$ is a regular homotopy track on some surfaces. This means that for $y \in [c, d]$, $h^{-1}(y)$ is a set of disjoint surfaces and each of them is immersed into $\mathbf{R}^3$ by f. We denote the collection of these surfaces by $\mathcal{F}_y$. Let $F_y \in \mathcal{F}_y$ be a closed surface in M^3. Since the condition (11) we may assume that y is a regular value for h. The restriction $f|_{F_y}$ is an immersion into $\{y\} \times \mathbf{R}^3$ and $f|_{F_y}$ has an embedded lift into $p^{-1}(p_1^{-1}(y)) \cong \mathbf{R}^4$, since f has an embedded lift. Thus the restricted immersion satisfies the CS–conditions. Let T_y be a triple point of $f|_{F_y}(F_y)$ in $p_1^{-1}(y) \cong \mathbf{R}^3$ and let $V(\tau_y)$ be a neighbourhood of τ_y in $p_1^{-1}(y) \cong \mathbf{R}^3$. This means that for each y, $V(\tau_y) \cap f(C(f))$ is homeomorphic to P_3.

For an interval $[c, d]$ of regular values y, the set $\{\tau_y : y \in [c, d]\}$ forms a triple arc τ in $f(M^3) \subset \mathbf{R}^4$. Each disc of P_3 has two proper arcs crossing at the middle point of each arc. Each pair of arcs on discs has a colour a or b and the colourings on discs are $\{a, a\}$, $\{a, b\}$ and $\{b, b\}$. These colourings are induced from the CS–conditions for $f_{F_y}\colon F_y \to p_1^{-1}(y) \cong \mathbf{R}^3$. Along a triple arc τ these coloured arcs form coloured discs. Then these discs satisfy the condition (Y2).

If y is a critical value, then $h^{-1}(y)$ is not a set of immersed surfaces. However, the colouring of the crossing set on $h^{-1}(y)$ is determined by the condition (Y1). Therefore, $C(f)$ satisfies conditions (Y1) and (Y2).

Conversely, assume that f satisfies conditions (Y1) and (Y2). Then for each regular value $y \in \mathbf{R}$, $f(M^3) \cap p_1^{-1}(y)$ is an immersed surface in $p_1^{-1}(y) \cong \mathbf{R}^3$ and above conditions induce the CS–conditions for this immersed surface. This implies that $p_1^{-1}(y) \cap f(M^3)$ has an embedded lift into $p^{-1}\left(p_1^{-1}(y)\right) \cong \mathbf{R}^4$. Therefore, for each $y \in [c_i + \varepsilon, c_{i+1} - \varepsilon]$, $f|F_y$ has an embedded lift for $i = 0, \ldots, k-1$.

On the other hand, $C(f)$ satisfies the condition (Y1). This implies that CS–conditions for $C(f|_{F_y})$ are continuously preserved for all $y \in [c_i + \varepsilon, c_{i+1} - \varepsilon]$.

Take an interval $[c_i + \varepsilon, c_{i+1} - \varepsilon]$ $(0 = 1, \ldots, k-1)$. Let $M_a = h^{-1}((-\infty, a])$. Then let $M_{i,\varepsilon} = Cl(M_{c_{i+1}-\varepsilon} \setminus M_{c_i+\varepsilon})$. The restricted immersion $f|_{M_{i,\varepsilon}}$ is expressed as a regular homotopy track.

Take a quadruple point $\rho \in f(M_{i,\varepsilon})$. Let $V(\rho)$ be a neighbourhood of ρ in $\mathbf{R}^4$. Then there are four points $x_1, \ldots, x_4 \in M_{i,\varepsilon}$ such that $f(x_i) = \rho$ $(i = 1, \ldots, x_4)$. From the condition (Y2) and Lemma 2.1, there are neighbourhoods
$U(x_1), \ldots, U(x_4)$ in M^3 which have the following properties:

(i) For each i, $U(x_i) \cap C(f)$ is diffeomorphic to P_3, and we denote this by T_i and

(ii) $\{T_1, \ldots, T_4\}$ satisfies the (4,3)–condition.

The $(4,3)$–condition implies that each T_i has an embedded lift. Therefore, the neighbourhood $U(\rho)$ has an embedded lift into $\mathbf{R}^5$. Thus $f|_{M_{c_i}}$ has an embedded lift into $p^{-1}(p_1^{-1}([c_i-\varepsilon, c_{i+1}-\varepsilon])) \subset \mathbf{R}^5$ for $i = 0, \ldots, k$.

Take an interval $[c_{i+1} - \varepsilon, c_{i+1} + \varepsilon]$, containing critical values $c_{i+1} = h(p_{i+1})$ $(i = 1, \ldots, k-1)$. For each $y \in [c_{i+1} - \varepsilon, c_{i+1} + \varepsilon]$, with $y \neq c_{i+1}$, $f|_{F_y}$ has an embedded lift in $p_1^{-1}(y)$ so it satisfies the CS–conditions. These colourings of crossing sets on F_y are preserved for $c_i + \varepsilon \leq y < c_{i+1}$ since the condition (11). Therefore, $f|_{M_{c_i}}$ has an embedded lift into $\mathbf{R}^5$.

We start to construct an embedded lift from $f|_{M_{c_1}}$, which has an embedded lift $\widetilde{f}|_{M_{c_1}}$ into $\mathbf{R}^5$. Inductively $f|_{M_{c_i}}$ has an embedded lift $\widetilde{f}|_{M_{c_i}}$ because colouring conditions are preserved on the interval $[c_0, c_i]$ and This can be extended to $\widetilde{f}|_{M_{c_{i+i}}}$ because of conditions (Y1) and (Y2).

This implies that f has an embedded lift into $\mathbf{R}^5$. $\square$

4 Elementary deformations

A homotopy on an immersed closed surface in a compact 3–manifold can be expressed by a sequence of six types of local deformations (see [HN1][HN2]) shown in Figure 3 and 4. The deformation from the left column of the arrow to the right column is denoted by type $*^+$ h–move otherwise denoted by type $*^-$ h–move, $(* = \mathrm{I}, \ldots, \mathrm{V})$. Note that type VI h–move is not distinguished by the directions of arrows. Each of h–moves is explained in the following.

Type $\mathrm{I}^{\pm}$ h–moves consist of a pair of discs and they create or eliminate a simple double circle.

Type II$^{\pm}$ h–moves consist of three discs and they create or eliminate a pair of double circles intersecting at a pair of triple points.

Type III$^{\pm}$ h–moves consist of a neighbourhood of a triple point and a disc so they have a triple of discs and a disc. They create or eliminate three double circles intersecting at six triple points around the triple point.

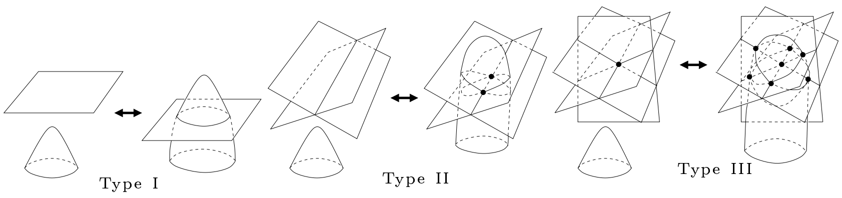


Figure 3: Type I–III h–moves

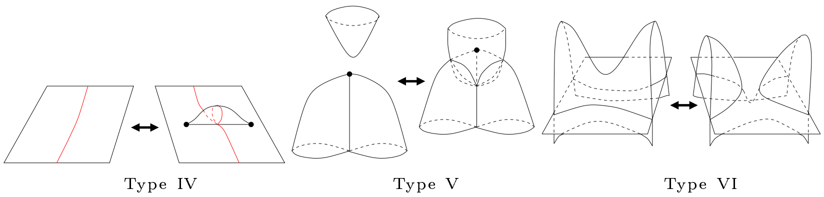


Figure 4: Type IV–VI h–moves

Type IV$^{\pm}$ h–moves consist of only one disc and they create or eliminate a pair of branch points and a double segment terminated by the pair of branch points.

Type V$^{\pm}$ h–moves consist of a pair of discs and they create or eliminate a triple point and two looped double circles based on the triple point.

Type VI h–moves consist of two discs, which create a pair of proper double segments, say α from a_1 to a_2 and β from b_1 to b_2, where a_i and b_j are boundary points. We may assume that these boundary points are ordered as $\{a_1, a_2, b_1, b_2\}$ with respect to the orientation of the boundary.

As one disc move, these two segments move closer and touch at the middle point of each, then new segments γ from a_1 to b_2 and δ from a_2 to b_1 will appear (see the right picture of Figure 4).

Assume that the Type VI$^{\pm}$ h–moves are applicable to the pair of discs and use the same notations as in the above. Let a be the middle point of the double segment α and let b be the middle point of the double segment β. then we can find a disc bounded by a pair of arcs bounded by the pair of points, a and b; one of them is on the first discs and the other is on the other disc. This disc is called a *descendent disc.* On the other hand, if we can find a descendent disc, whose interior is disjoint from other part of the immersed surface, then we can apply the Type $\mathrm{VI}^{\pm}$ moves.

The following theorems can be found in [HN1, HN2].

Theorem 4.1. *Let f, g: $F^2 \to N^3$ be two immersions from a compact surface into a 3–manifold. If $f \simeq g$, then there is a sequence of h–moves realising this homotopy.*

Theorem 4.2. *Let f, g: $F^2 \to N^3$ be two immersions from a compact surface into a 3–manifold N^3. Then there is a sequence of h–moves which consist of type* I, II, III *and* VI *that deform the immersed surface $f(F^2)$ to $g(F^2)$ if and only if $f \simeq_r g$.*

5 Constructing Immersed Surfaces

In this section we assume all surfaces are closed orientable. Let f: $F^2 \to \mathbf{R}^3$ be an immersion from a surface into $\mathbf{R}^3$. It is known that there is an embedding g: $F^2 \to \mathbf{R}^3$ and an immersion γ: $S^1 \to F^2$ such that f is regularly homotopic to an immersion obtained from γ and g [Hug][Yas1]. In this section we describe a construction of the immersed surfaces.

5.1 Bug Constructions

5.1.1 Bugs

Let F^2 be a surface and let g: $F^2 \to \mathbf{R}^3$ be an embedding. Take a point $x \in F^2$ and set $y \in f(x)$. Let $V(y)$ be a small neighbourhood of y in $\mathbf{R}^3$ and assume that $V(y) \cap f(F^2)$ is a disc. Applying type IV^+ h–move on the disc, we obtain a singular disc with a couple of branch points (see Figure 4). We call this resulting part a *bug* [Yas2].

Take an oriented immersion γ: $S^1 \to F^2$. Create a bug on γ, where branch points are on $g \circ \gamma(S^1) \subset F^2$ locating the double segment of the bug along γ and the double segment does not meet the crossing points of γ. Then lengthen it along $g \circ \gamma$ on $g(F^2)$. We call one of branch point a *head* and the other a *tail*. The head can pass through a part of the bug itself creating a pair of triple points (see Figure 7). Finally, the head will reach the tail. Then we can eliminate the pair of branch points with a combination of type VI and type IV$^-$ h–moves [HN1][HN2] (see Figure 5). This modification gives an immersed surface. We denote the

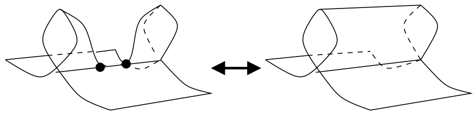


Figure 5: Eliminations and creations of branch points

immersion by g_γ.

It is known that for every immersion $f\colon F^2 \to \mathbf{R}^3$, f is regularly homotopic to g_γ [HH] [Yas1].

Note 5.1. *It is known that the regular homotopy class of g_γ does not depend on the choice of the position of the first bug; that is, even if we put a bug on the other side of the embedded surface, then we obtain an immersed surface representing the same regular homotopy class* [Yas1].

5.1.2 Pipes

We provide some notations to describe a modification of an immersed surface. Let D^2 be a disc defined by $I \times I$. An immersed disc which has a proper double arc is called a pipe (Figure 6).

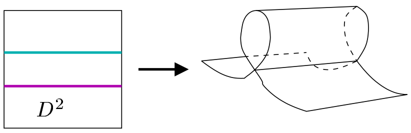


Figure 6: A pipe

Examples 2. *Let $i\colon S^2 \to \mathbf{R}^3$ be the inclusion and let $8\colon S^1 \to S^2$ be an eight–immersion (the figure of eight). Then i_8 has a pair of triple points, two pairs of double loops and two double arcs.*

Note 5.2. *A pipe can be split into two pipes with branch points. This is done by the reverse operation of one used for eliminating a pair of branch points (*Figure 5*).*

5.2 Crossing pipes

Hass and Hughes [HH] introduced 'kinky box' as an immersed disc, which appears in an immersed surface obtained by a bug construction with an embedded surface and an immersed circle. At a crossing point of the immersed circle the bug construction creates a neighbourhood, where of a point in an immersed surface in a 3–manifold containing an immersed disc, which is obtained with a bug construction along the immersed circle We call this *crossing pipes*. Let $y \in \gamma(S^1)$ be a crossing point of γ on $g(F^2)$. Then let $U(y)$ be the neighbourhood of y in $\mathbf{R}^3$. The pre–image of $U(y) \cap g_\gamma(F^2)$ and the image are shown in Figure 7.

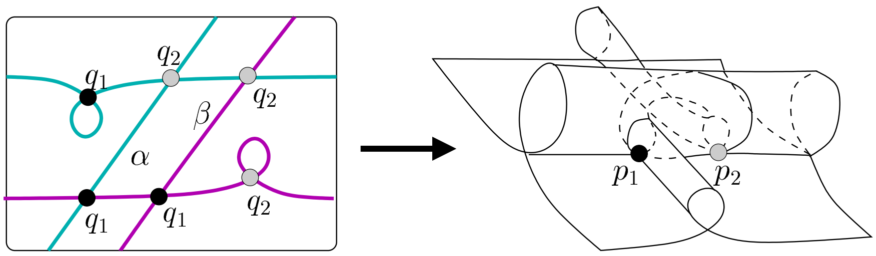


Figure 7:

In the pre–image in Figure 7 we label two arcs with α and β coloured with a and b. It is not difficult to check that the pre–image satisfies the CS–conditions even if colours of α and β are swapped. Hence the immersed disc has an embedded lift in $\mathbf{R}^4$. Thus we obtain an embedded lift in $\mathbf{R}^4$ so the following holds.

Lemma 5.1. *Let $g\colon F^2 \to \mathbf{R}^3$ be an embedding and let $\gamma\colon S^1 \to F^2$ be an immersion. Then the immersion g_γ has an embedded lift $\widetilde{g}_\gamma$ into $\mathbf{R}^4$.*

6 Swapping pipes

Let D_1^2, D_2^2 be discs. Let $Q_1\colon D_1^2 \to \mathbf{R}^3$ and $Q_2\colon D_2^2 \to \mathbf{R}^3$ be pipes which are intersecting each other as depicted in Figure 8. Thus it has two triple points. In this subsection we describe a deformation of interchanging these pipes. This deformation swaps these triple points.

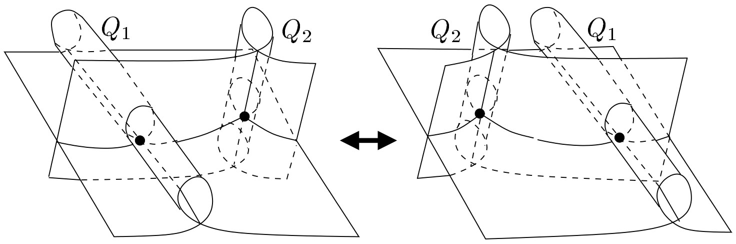


Figure 8: Swapping pipes.

Note that swapping pipe move increase the number of triple points and then decrease the same number of increased triple points. As a result, the original number of triple points is preserved by this move.

A regular homotopy

$$H\colon (D_1^2 \cup D_2^2) \times I \to \mathbf{R}^3 \tag{12}$$

is defined as a interchanging of these pipes. This is done by a sequence of regular homotopy moves. We will call this modification a *swapping pipes*. We will show this process in the following.

The regular homotopy induces a track:

$$\widehat{H}\colon I \times (D_1^2 \cup D_2^2) \to I \times \mathbf{R}^3 \tag{13}$$

defined by $\widehat{H}(t,x) = (t, H_t(x))$ for $t \in I$ and $x \in (D_1^2 \cup D_1^2)$.

Lemma 6.1. *The track $\widehat{H}$ has an embedded lift into $\mathbf{R}^5$.*

Proof. The pre–images of the initial map and the terminal map of the swapping pipes are depicted in Figure 9. There are two triples of points, which are pre–images of two triple points. Obviously, these pipes have embedded lifts into $\mathbf{R}^4$. We will observe the deformation covering the regular homotopy track $\widehat{H}$ that is an isotopy deformation. This is done

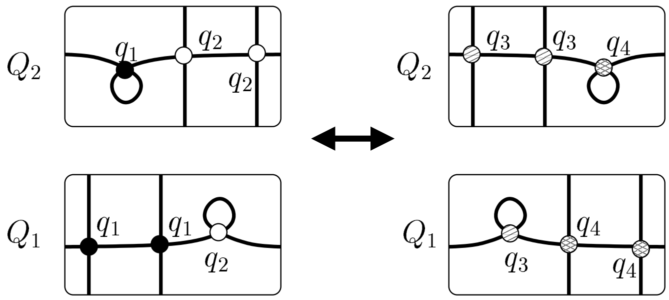


Figure 9: Pre–images of terminal maps

by observing pre–images of intermediate maps of the swapping pipe deformation. We put two colours on arcs in pre–images so that we can check the CS–condition for whether these intermediate maps have embedded lifts or not.

The deformation starts from moving pipes toward each other. When one pipe intersects the other, the pre–images are shown in the left picture in Figure 10. This creates a trivial circle on each pipe (see the left picture of Figure 10). Continuing the movement, then we obtain a couple of new triple points (see the right picture of Figure 10). Next stage we will have

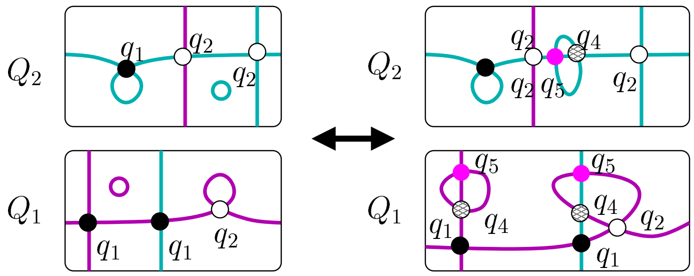


Figure 10: Creating new self–intersections.

six triple points and we denote each pre–images of these triple points in figures by $q_1, q_2, q_3, q_4, q_5, q_6$. Type VI h–move is applied to join immersed curves in the image. In Figure 11 dashed curves indicate arcs mapped onto the boundary of a descendent disc in $\mathbf{R}^3$. Type VI h–move joins a pair of parts of crossing curves. The left picture in Figure 11 contains four triangles; $\langle q_1, q_2, q_3\rangle$, $\langle q_1, q_4, q_3\rangle$, $\langle q_4, q_2, q_3\rangle$ and $\langle q_4, q_2, q_1\rangle$. In the image it is obvious that these four triangles form a complete 4–graph. Each vertex in the image forms a triple point and then it forms a triple curves in the deformation track. In each triangle vertices are moving and shrink

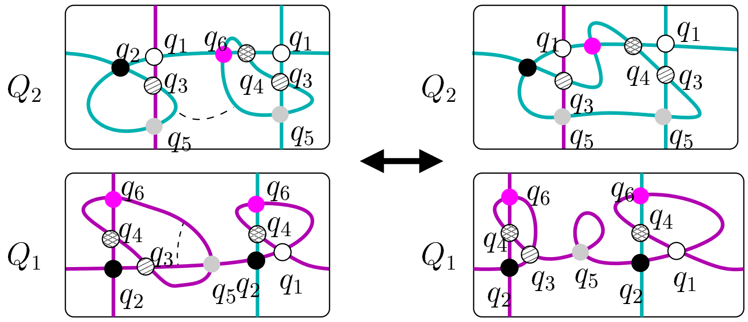


Figure 11: There are four triangles.

the triangle then swap two of them to produce a triangle with the same vertices (see from Figure 11 to Figure 12). During this deformation in the image these triple points form triple curves and they intersect at one point which is a quadruple point.

In Figure 12 dashed lines indicate places where a type VI h–move is applied and curves in the image will be joined.

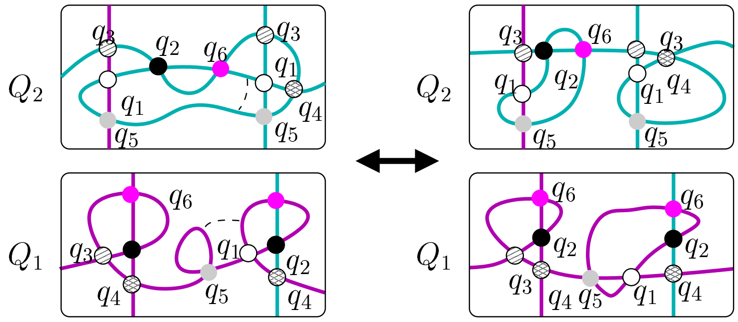


Figure 12: Applying type VI h–move to obtain the second picture.

Some triple points in the left picture in Figure 13 are eliminated. Finally, we obtain the intersecting pipes with only two triple points.

It is not difficult to see that these triple curves satisfy the condition (Y2) in Theorem 1.2. If we put two colours on $C(H_0)$, then we can read from these diagram that the colouring can be consistent with the colouring of $C(H_1)$. This means we can lift this deformation track $\widehat{H}$ into $\mathbf{R}^5$. $\square$

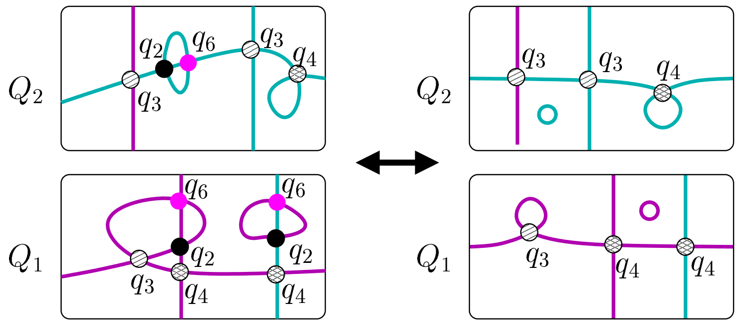


Figure 13: Eliminating points q_2 and q_6.

Note 6.1. *Triple points in the initial immersed surface seem to be eliminated and new triple points are created at the terminal immersed surface but in the image of the track, they are connected as triple curves. In fact, a triple curve formed by q_1 (respectively q_2) connects with a triple curve formed by q_5 (respectively q_6), which is connected with a triple curve formed by q_3 (respectively q_4). Thus there are two triple curves are playing in the image of the track and they form a quadruple point in the image of the track.*

6.1 Proof of Theorem 1.3

Proof. We will deform the crossing pipe into two disjoint pipes. We can do this with a sequence of local deformations. In this section, we use liftable track to realise this particular deformation. First, we will describe such a deformation then we will show that the deformation is covered by an isotopy in $\mathbf{R}^4$.

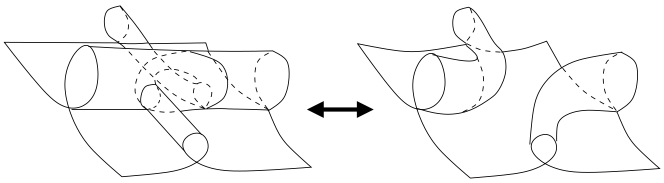


Figure 14: The deformation of crossing pipes

We will use type VI h–moves to deform the crossing pipes. To do this, we have to find a descendent disc D^2 in $\mathbf{R}^3$. We can find such a disc as the shadow disc in the left picture of Figure 15. Pre–images of

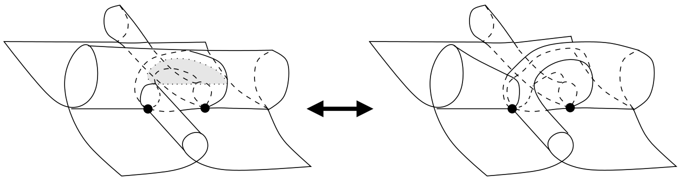


Figure 15: The descendent disc and applying a type VI h-move

this deformation are shown in Figure 16. In the figure dashed lines are representing a pair of arcs which are mapped onto the boundary of the descendent disc.

In Figure 16 if the dashed arc joins different coloured curves, then the track cannot have an embedded lift into $\mathbf{R}^5$ with fixing end maps. There are two loops with distinct colours and if they are exchanged, then the liftability is obtained. We can interchange these loops in the pre–

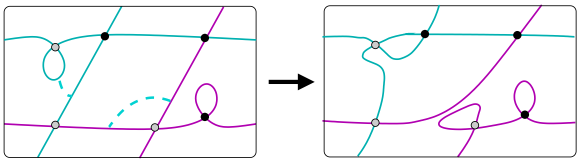


Figure 16: Applying Type VI h–move to modify double curves

image using a swapping pipe deformation so that the colouring becomes appropriate one to obtain a liftable deformation track. Further, we apply a type VI h–move on the deformed crossing pipes (see Figure 17). In figure 17, the VI h–move produces four proper arcs and a pair of circles (see the right hand picture of Figure 17).

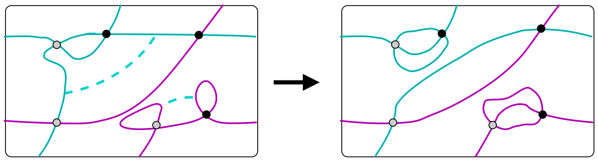


Figure 17: Applying Type VI h–move

Two of these proper arcs intersect each other at two points and each of other curves are passing through each of circles. The rest of the deformations are just an elimination of these crossings. Note that this deformation track does not contain any quadruple point.

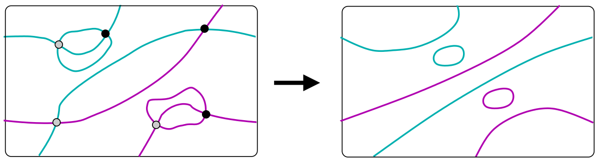


Figure 18: Eliminating a pair of triple points

Note 6.2. *The author constructed another local deformation which contains a sphere eversion track* [Yas2]*, which is not covered by an isotopy in* $\mathbf{R}^4$ [Yas1].

6.1.1 Smoothings

Let F^2 be a surface and let $\gamma\colon S^1 \to F^2$ be an oriented immersion. For each crossing point p of $\gamma(S^1)$ we have a small neighbourhood $U(p)$ in F^2 such that $U(p)$ contains two proper arcs crossing at the middle points of the arcs. A *smoothing* is a deformation which change the crossing in $U(p)$ to a pair of proper arcs without creating any crossing point. This operation preserves the orientation of the immersed circle.

We apply an operation called a *smoothing* to the immersed circle to obtain a set of simple closed circles on F^2. The smoothing is depicted in Figure 19.

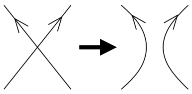


Figure 19: The smoothing operation.

Let $g\colon F^2 \to \mathbf{R}^3$ be an embedding. We obtain an immersion g_γ with the bug construction. Applying the above deformations to g_γ, we obtain a new immersion g_δ where δ is a set of simple closed circles on F^2 which is obtained by smoothing γ.

The pre–image of g_γ has neighbourhoods as Figure 16 around the pre–images of crossing pipes. Thus the crossing set $C(g_\gamma)$ is colourable as well as $C(g_\delta)$ is colourable. It is not difficult to see that colouring conditions are consistent during the deformation. This implies that the regular homotopy track from g_γ to g_δ has an embedded lift into $\mathbf{R}^5$ by Theorem 1.2. Thus the lift gives an isotopy from the lift $\widetilde{g}_\gamma$ to $\widetilde{g}_\delta$ in $\mathbf{R}^4$. Therefore, Theorem 1.3 is proved. $\square$

7 Constructing Immersed 3–spheres in $\mathbf{R}^4$

As we have seen in the previous sections the crossing pipes includes six double curves consisting of two loops based at triple points, two arcs bounded by triple points and four arcs joining the boundary of the immersed disc and triple points. We ignore loops and an arc bounded by triple points. Then we obtain a diagram as the left picture of Figure 20. The deformation separates the diagram as in Figure 20 (compare with Figure 14).

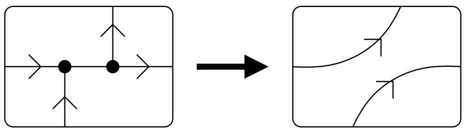
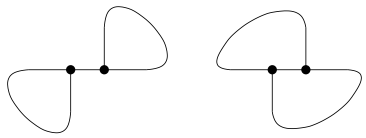


Figure 20:

Let g: $S^2 \to \mathbf{R}^3$ be an embedding and let $\bigcirc$: $S^1 \to S^2$ be an embedding. Considering the immersion $g_8(S^2)$, there are two types of immersed sphere represented by the following diagrams. The deformation of



(a) (b)

the crossing pipes deforms $g_8(S^2)$ of type (a) to $g_\bigcirc(S^2)$ and it deforms $g_8(S^2)$ of type (b) to $g_{\{\bigcirc\cup\bigcirc\}}(S^2)$, where $\bigcirc\cup\bigcirc$ is a disjoint union of two copies of the embedding $\bigcirc$. The former track does not have an embedded lift in $\mathbf{R}^5$, while the later track does. Therefore, we can construct liftable immersed 3–sphere in $\mathbf{R}^4$ with these tracks. This construction will be used for a proof of Theorem 1.4.

7.1 Proof of Theorem 1.4

Proof. Using the deformation of crossing pipes, we can construct a non–liftable immersed 3–sphere in $\mathbf{R}^4$ without quadruple points into $\mathbf{R}^5$.

We use the same notations in Example 1 and previous subsection. Let $\chi\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection on xy–plane. Then we deform $i(S^2)$ to the immersed sphere $(\chi \circ i)_{\bigcirc}(S^2)$. This is done by the type I^+ h–move applying to $i(S^2)$.

We can deform $(\chi \circ i)_{\bigcirc}(S^2)$ of type (a) to $(\chi \circ i)_8(S^2)$ with the reverse deformation of the deformation on crossing pipes. Crossing sets of both maps are shown in Figure 21. At this stage we can assume that $(\chi \circ i)_8(S^2)$

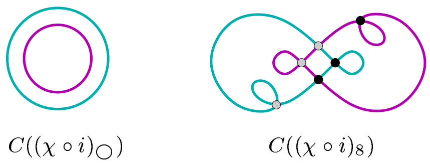


Figure 21: Crossing sets of $(\chi \circ i)_{\bigcirc}(S^2)$ and $(\chi \circ i)_8(S^2)$

is of type (a). Then we deform $(\chi \circ i)_8(S^2)$ to $(\chi \circ i)_{\bigcirc}(S^2)$ with the deformation of crossing pipes. Finally, we return to $i(S^2)$.

Let

$$H\colon S^2 \times I \to \mathbf{R}^3 \tag{14}$$

be a regular homotopy from i to i passing through above deformations between $(\chi \circ i)_{\bigcirc}(S^2)$, $(\chi \circ i)_8(S^2)$ and $(\chi \circ i)_{\bigcirc}(S^2)$. Then H induces a regular homotopy track

$$\hat{H}\colon I \times S^2 \to I \times \mathbf{R}^3 \subset \mathbf{R}^4 \tag{15}$$

Obviously, $\hat{H}$ is an immersion without quadruple points. We cap off both ends of this track with 3–balls so that we obtain an immersion

$$f\colon S^3 \to \mathbf{R}^4 \tag{16}$$

Note that the immersed 3–sphere $f(S^3)$ does not contain quadruple points. The crossing set $C((\chi \circ i)_8)$ consists of two immersed circles (see Figure 21). During the deformation from $(\chi \circ i)_8(S^2)$ to $(\chi \circ i)_{\bigcirc}(S^2)$ these immersed circles are joined by type VI h–moves. On the other hand, the deformation from $(\chi \circ i)_8(S^2)$ to $(\chi \circ i)_{\bigcirc}(S^2)$ joins two immersed components of $C(H_t)$ with a type VI h–move. This implies that $C(f)$ has one component, which is an immersed surface. This shows that f does not satisfy the condition (Y1) of Theorem 1.2 thus we have proven Theorem 1.4.

$\square$

Note 7.1. *We can construct a non–liftable immersed 3–sphere with satisfying (Y1) but (Y2) (see* [Yas3]*).*

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