# A note on immersed 3-spheres in 4-space

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**Abstract**

There are two special immersed 3–spheres in 4–space, which generate regular homotopy classes of immersed 3–spheres in 4–space. They are constructed with a special sphere eversion called $FM$–eversion. In this note we use geometric approaches to show an immersed 3–sphere in 4–space constructed with $FM$–eversions are not liftable into 5–space.

## 1 Introduction

We assume that all manifolds and maps are smooth. Let $M^m$ and $N^n$ be manifolds with dimensions $m$ and $n$ respectively.

A map $f : M^m \to N^n$ $(m < n)$ is called an *immersion* if $df_x : T_xM^m \to T_{f(x)}N^n$ has full rank for each $x \in M^m$. Two immersions $f, g : M^m \to N^n$ are said to be *regularly homotopic* if there is a homotopy

$$H : M^m \times I \to N^n \quad (I = [0, 1])$$

such that setting $H_t(x) = H(x, t)$ for $t \in I$ and $x \in M^m$, then $H_0(x) = f(x)$, $H_1(x) = g(x)$ and $H_t$ is an immersion for $t \in I$. The homotopy $H$ is called a *regular homotopy* from $f$ to $g$. Regular homotopy is an equivalence relation. Let $Imm_r(M^m, N^n)$ denote the set of regular homotopy classes of immersions from $M^m$ to $N^n$.

except for a finite number of values of $t$ (see Remark 1.1).

The *crossing set* of an immersion $f : M^m \to N^n$ is the set

$$C(f) = \{x \in M^m | \#(f^{-1}(f(x))) > 1\}.$$

We assume that immersions have only transversal crossing sets.

For a point $x \in C(f)$, if $\#(f^{-1}(f(x))) = k$, then $f(x)$ is called a *k-tuple point.* If $M^m$ is compact, then $C(f)$ is the union of

$$C_k(f) = \{x \in M^m | \#(f^{-1}(f(x))) = k\}, \quad (k = 2, 3, \ldots, \ell).$$

We denote the collection of immersed manifolds in $C(f)$ by $\mathcal{C}(f)$:

$$C(f) = \bigcup_{C \in \mathcal{C}(f)} C. \tag{1}$$

**Remark 1.1.** *During a regular homotopy $\{H_t\}, t \in I$ there are a finite number of points in $I$ where $C(H_t)$ contains non–transversal crossing points.*

A regular homotopy $H : M^m \times I \to N^n$ induces a *track*

$$\hat{H} : I \times M^m \to I \times N^n$$

defined by $\hat{H}(t, x) = (t, H_t(x))$, which is an immersion.

For an immersion $f : M^m \to N^n$, a map $\tilde{f} : M^m \to N^n \times \mathbf{R}^k$ is called a *lift over $f$ via $p_1$* if $p_1 \circ \tilde{f} = f$, where $p_1 : N^n \times \mathbf{R}^k \to N^n$ is the projection. If $\tilde{f}$ is an embedding, then $\tilde{f}$ is called an *embedded lift* and $f$ is said to be *liftable*. Otherwise, $f$ is said to be *non–liftable.*

## 1.1 Overview of results

In this note we introduce properties of regular homotopy from an immersed surface to an embedded surface in a $\mathbf{R}^3$. Using these properties we show that a special track over the sphere is not a projection of an isotopy of the two-sphere in $\mathbf{R}^4$.

A sphere eversion is a regular homotopy $H : S^2 \times I \to \mathbf{R}^3$ from the inclusion $i : S^2 \to \mathbf{R}^3$ to the reflected inclusion $\chi \circ i$, where $\chi : \mathbf{R}^3 \to \mathbf{R}^3$ is the reflection on the $xy$-plane.

Froissart and Morin introduced a special eversion so called the *$FM$-eversion.* Max and Banchoff drew pictures of some images of the intermediate maps of the $FM$-eversion (see [MB]), and proved that this eversion has one quadruple point.

We can construct an immersed 3-sphere in $\mathbf{R}^4$ from an FM-eversion track $\hat{H}$ capping off the boundary components with 3-balls. We denote this immersion by $K : S^3 \to \mathbf{R}^4$. The immersion $K$ contains the $FM$-eversion track $\hat{H} : I \times S^2 \to I \times \mathbf{R}^3$, which is not covered by an isotopy track (Corollary 5.1). Therefore, we have the following.

**Theorem 1.1.** *$K$ does not have an embedded lift into $\mathbf{R}^5$.*

It is known that

$$Imm_r(S^3, \mathbf{R}^4) \overset{1:1}{\leftrightarrow} \pi_3(S^3) \times \pi_3(\mathrm{SO}(3)) \cong \mathbf{Z} \oplus \mathbf{Z}, \tag{2}$$

where $1:1$ means an one-to-one correspondence. Also it is known that there is an invariant (see [Sma]).

$$I : Imm_r(S^3, \mathbf{R}^4) \to \mathbf{Z} \oplus \mathbf{Z}. \tag{3}$$

Generators of $Imm(S^3, \mathbf{R}^4)$ are obtained from $FM$-eversion (see [MB], [Hug]). Let $G : S^2 \times I \to \mathbf{R}^3$ be a regular homotopy constructed by the $FM$-eversion followed by its mirror image. This gives a track $\hat{G}$ and using the track and capping off with 3-balls, we obtain an immersion

$$E : S^3 \to \mathbf{R}^4. \tag{4}$$

These two immersions $K$ and $E$ generate

$$Imm_r(S^3, \mathbf{R}^4) \cong \mathbf{Z} \oplus \mathbf{Z}, \tag{5}$$

(see [Hug]). The followings are known [Hug].

$$I(K) = (-1, 0), \tag{6}$$

$$I(E) = (0, 1). \tag{7}$$

Note that the invariant $(0,0) \in \mathbf{Z} \oplus \mathbf{Z}$ corresponds to the regular homotopy class of the inclusion $i_3 : S^3 \to \mathbf{R}^4$.

Let $\chi_4 : \mathbf{R}^4 \to \mathbf{R}^4$ be the reflection on the $(x, y, z)$-plane in $\mathbf{R}^4$. Then $r = \chi_4 \circ i_3$ is called a *reflected inclusion*.

An immersed 3-sphere in $\mathbf{R}^4$ is given by $K$ and $E$. From (6) and (7), the standard embedding and the reflected inclusion of $S^3$ do not contain $FM$-eversion tracks.

This implies the following.

**Theorem 1.2.** *Let $f : S^3 \to \mathbf{R}^5$ be a 3-knot and let $p : \mathbf{R}^5 \to \mathbf{R}^4$ be a projection such that $p \circ K^3$ is an immersion. Then $p \circ f$ is regularly homotopic to the standard embedding or the reflected inclusion.*

**Remark 1.2.** *The reflected inclusion does not contain a $FM$-eversion track but some immersions which are regularly homotopic to the reflected inclusion may contain a $FM$-eversion track.*

## 1.2 Immersed 3-manifolds in $\mathbf{R}^4$

Let $M^3$ be an orientable closed 3–manifold. It is known that for every orientable closed 3–manifold is immersible in $\mathbf{R}^4$ [Hir]. Let $f : M^3 \to \mathbf{R}^4$ be an immersion and let $p_1 : \mathbf{R}^4 \to \mathbf{R}$ be the projection onto the first coordinate. We may assume that $h = p_1 \circ f$ is a Morse function on $M^3$. Take an interval $[c, d]$ such that $[c, d]$ does not contain any critical values of $h$. Then $f(h^{-1}([c, d]))$ can be expressed by images of some regular homotopy tracks in $\mathbf{R}^4$. Thus $f(M^3)$ may contain these tracks as immersed submanifolds. Then the following holds.

**Theorem 1.3.** *Let $f : M^3 \to \mathbf{R}^4$ be an immersion. Then if $f(M^3)$ contains the image of $FM$-eversion tracks, then $f$ does not have an embedded lift into $\mathbf{R}^5$.*

**Corollary 1.1.** *For every orientable closed 3–manifold $M^3$, there is a non–liftable immersion from $M^3$ into $\mathbf{R}^4$*

*Proof.* We know that immersions $K$ and $E$ are not liftable. For every orientable closed 3–manifold $M^3$, we can construct a liftable immersion $f : M^3 \to \mathbf{R}^4$ [Yas1]. Then we can define a connected sums of immersions:

$$f\#K, f\#E : M^3 \to \mathbf{R}^4. \tag{8}$$

Then they are not liftable. □

**Remark 1.3.** *In the above construction, the regular homotopy class of the immersion $f$ is changed. However, we can also construct such a non–liftable immersion without changing the regular homotopy class (see* [Yas3]*).*

# 2 Elementary deformations

A homotopy on an immersed closed surface in a compact 3-manifold can be expressed by a sequence of six types of local deformations [HN1][HN2] shown in form Figure 1 to Figure 6. The deformation from the left column of the arrow to the right column is denoted by type $*^+$ $h$-move otherwise denoted by type $*^-$ $h$-move, $(* = \mathrm{I}, \ldots, \mathrm{V})$. Note that type VI $h$-move is not distinguished by the directions.

Type $\mathrm{I}^\pm$ $h$-moves consist of a pair of disks and they create or eliminate a simple double circle.

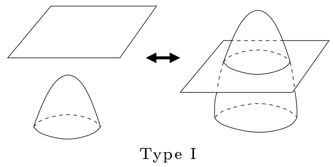
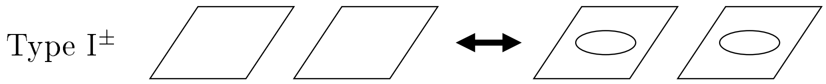



Figure 1: Type I $h$-move

The pre-images of type $\mathrm{I}^\pm$ moves consist of a pair of trivial circles on $F^2$.




Type $\mathrm{II}^\pm$ $h$-moves consist of three disks and they create or eliminate a pair of double circles intersecting at a pair of triple points.

For type $\mathrm{II}^\pm$ moves, they consist of a pair of disks with double segments and a disk. These deformations create or eliminate a circle on each disk of the pair of disks and a union of circles on the other disk.

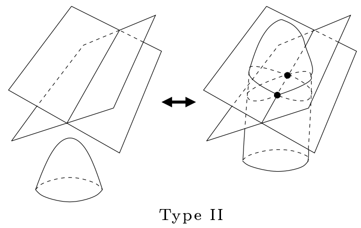
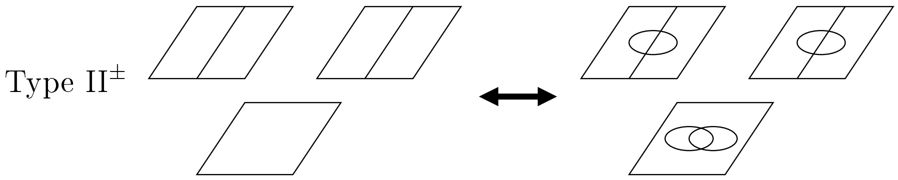



Figure 2: Type II $h$-moves




Type III$^\pm$ $h$-moves consist of a neighbourhood of a triple point and a disk so they have a triple of disks and a disk. They create or eliminate a triple of double circles intersecting at six triple points around the triple point.

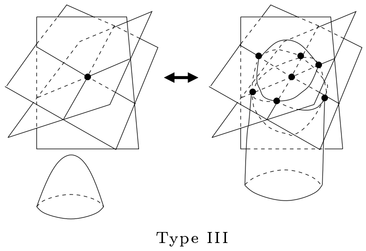
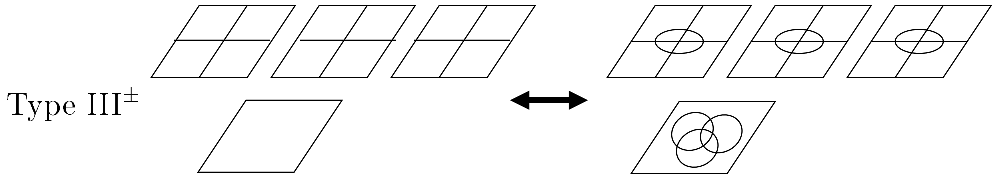
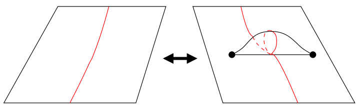



Figure 3: Type III $h$-moves

For type III$^\pm$ moves, they consist of a triple of disks with a three pairs of arcs crossing at the middle of arcs and a disk. These deformations create or eliminate a circle on each disk of the triple and three circles on the other disk as below.



Type IV$^\pm$ $h$-moves consist of only one disk and they create or eliminate a pair of branch points and a double segment terminated by the pair of branch points.

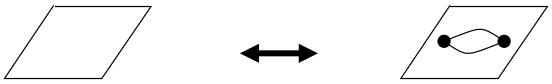


Type IV

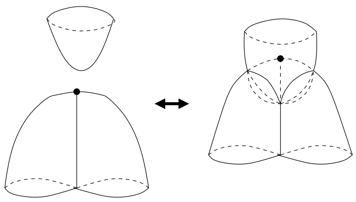
Figure 4: Type IV $h$-moves

For type IV$^\pm$ moves, they consist of a disk. These deformations create or eliminate a pair of arcs terminated by a pair of branch points.

Type IV$^\pm$



Type V$^\pm$ $h$-moves consist of a pair of disks and they create or eliminate a triple point and two looped double circles based on the triple point.

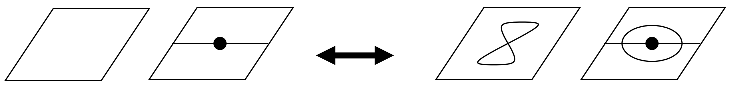


Type V

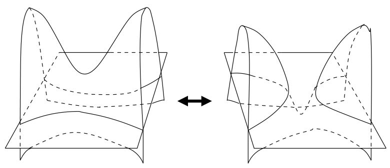
Figure 5: Type V $h$-moves

For type V$^\pm$ moves, they consist of a pair of disks. These deformations create or eliminate a curve of the figure of eight on one disk and a circle on the other disk.

Type V$^\pm$



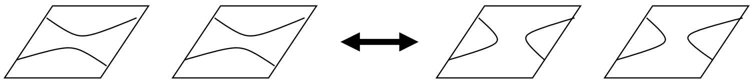
Type VI $h$-moves consist of a pair of disks and they interchange a pair of double arcs.



Type VI

Figure 6: Type VI $h$-moves

For type VI moves, they consist of a pair of disks. The deformation interchanges the connection of a pair of arcs.

Type VI 

Intermediate maps of the type VI $h$-move can be expressed as a model consisting of a disk $D_t^2 = \{(x, y, t) || x| \leq 1, |y| \leq 1\}$ for $t \in [0, 1]$ and a surface $D_2^2 = \{(x, y, x^2 - y^2 + \frac{1}{2}) || x| \leq 1, |y| \leq 1\}$. $D_0^2$ intersects $D_2^2$ and the double curves are proper curves represented by $x^2 - y^2 = -\frac{1}{2}$. $D_{\frac{1}{2}}^2$ intersects $D_2^2$ with double curves $x^2 - y^2 = 0$, which is a pair of lines with a crossing point at the origin. $D_1^2$ intersects $D_2^2$ with proper double curves $x^2 - y^2 = \frac{1}{2}$.

The following theorems can be found in [HN1, HN2].

**Theorem 2.1.** *Let $f, g : F^2 \to N^3$ be two immersions from a compact surface into a 3-manifold. If $f \simeq g$, then there is a sequence of $h$-moves realising this homotopy.*

**Theorem 2.2.** *Let $f, g : F^2 \to N^3$ be two immersions from a compact surface into a 3-manifold $N^3$. Then there is a sequence of $h$-moves which consist of type* I, II, III *and* VI *that deform the immersed surface $f(F^2)$ to $g(F^2)$ if and only if $f \simeq_r g$.*

## 2.1 Pre-images of $h$-moves

Pre-images of multiple points of an immersed surface $F^2$ in $\mathbf{R}^3$ form a set of immersed circles on $F^2$. For a regular homotopy between two immersed surfaces in $\mathbf{R}^3$ deforms the set of immersed circles to the other. In the following we discuss about pre-images of each type of $h$-moves.

Note that an elimination of immersed circles or arcs appear at type $\mathrm{I}^{\pm}$ or Type $\mathrm{IV}^{\pm}$ $h$-moves.

**Remark 2.1.** *Each pre-image of $h$-moves contains simple closed curve. However, these $h$-moves may be used as a combination with type* VI *$h$-moves so that they are joined as an immersed circle on the surface.*

In conclusion, the crossing set of the initial map $H_0$ is deformed by the homotopy $H_t, (t \in I)$. The homotopy deformation forms a sequence of the deformations of crossing sets on $F^2$. Thus the following is obtained.

**Lemma 2.1.** *Let $f, g : F^2 \to \mathbf{R}^3$ be homotopic immersions. If $g$ is an embedding, then there is a sequence of $h$-moves expressing the regular homotopy from $f$ to $g$. Then the last step of the sequence consist of type $\mathrm{I}^-$ or $\mathrm{IV}^-$ $h$-moves.*

*Proof.* The homotopy from $f$to $g$ is described by a sequence of $h$-moves of type $\mathrm{I}^{\pm}$, $\mathrm{II}^{\pm}$, $\mathrm{III}^{\pm}$, $\mathrm{IV}^{\pm}$, $\mathrm{V}^{\pm}$ and VI (Theorem 2.1).

There is a small number $\varepsilon > 0$ such that $H_{1-\varepsilon} : F^2 \to \mathbf{R}^3$ is an immersion or a map with branch points. The deformation from $H_{1-\varepsilon}$ to $H_1 = g$ eliminates the multiple points or branch points from $H_{1-\varepsilon}$. Such $h$-moves are only type $\mathrm{I}^-$ or $\mathrm{IV}^-$ $h$-moves. □

Then we have the following.

**Lemma 2.2.** *Let $F^2$ be an orientable closed surface. Let $f : F^2 \to \mathbf{R}^3$ be an immersion and let $g : F^2 \to \mathbf{R}^3$ be an embedding.*

*If $f$ is regularly homotopic to $g$ by a regular homotopy $H$, then a set of immersed circles in $C(f)$ are deformed to a set of trivial circles on $F^2$ by the homotopy $H_t$ $(0 \leq t \leq 1 - \varepsilon)$.*

*Proof.* Since $H_1(x) = g(x)$, the terminal map is an embedding so $C(H_1) = C(g) = \emptyset$. Thus from Lemma 2.1 the last $h$-moves in the sequence are Type $\mathrm{I}^-$ moves. Obviously, Type I $h$-move eliminates a trivial double circle. Thus $C(f)$ is deformed to a set of double circles which are trivial on $F^2$. □

# 3 Constructing immersed surfaces

Let $f : F^2 \to \mathbf{R}^3$ be an immersion from an orientable surface. It is known that there is an embedding $g : F^2 \to \mathbf{R}^3$ and an immersion $\gamma : S^1 \to F^2$ such that $f$ is regularly homotopic to an immersion obtained from $\gamma$ and $g$ [Hug][**?**]. In this section we describe the construction.

## 3.1 Bug constructions

### 3.1.1 Bugs

Let $D^2$ be a disk. Applying type $\mathrm{IV}^+$ $h$-move on the disk, we obtain a singular disk with a couple of branch points (see Figure **??**). We call this resulting part a *bug*.

Take an immersion $\gamma : S^1 \to F^2$. Create a bug on $\gamma$, where branch points are on $g \circ \gamma$ locating the double segment of the bug on $\gamma$ and the double segment does not meet the crossing points of $\gamma$. Then lengthen it along $g \circ \gamma$ on $g(F^2)$. We call one of branch point a *head* the other a *tail*. The head can pass through a part of the bug itself and it creates a pair of triple points. Finally, the head will reach the tail. We can eliminate the pair of branch points with a combination

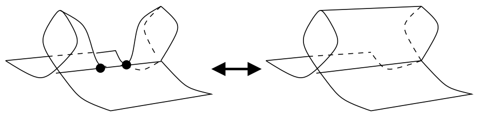


Figure 7: Eliminations and creations of branch points

of type VI and type IV$^-$ $h$-moves [HN1][HN2] (see Figure 7). This modification gives an immersed surface. We denote this by $g_\gamma$.

It is known that for every immersion $f : F^2 \to \mathbf{R}^3$, $f$ is regularly homotopic to $g_\gamma$ [HH] [Yas1].

**Note 3.1.** *It is also known that the regular homotopy class of $g_\gamma$ does not depend on the choice of the position of the first bug, that is even if we put a bug inside an embedded surface, then we obtain the same regular homotopy class* [**?**].

**Examples 1.** *Let $i : S^2 \to \mathbf{R}^3$ be the inclusion and let $8 : S^1 \to S^2$ be an eight-immersion. Then $i_8$ has a pair of triple points, two pairs of double loops and two double arcs.*

### 3.1.2 Pipes

We provide some notations to describe a modification of an immersed surface. Let $D^2$ be a disk homeomorphic to $I \times I$. An immersed disk which has a proper double arc is called a pipe (Figure 8).

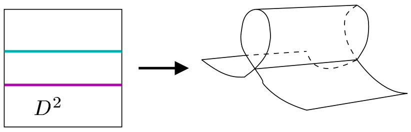



Figure 8: A pipe

**Note 3.2.** *A pipe can be split into two pipes with branch points. This is done by the reverse operation of one used for eliminating a pair of branch points (Figure 7).*

## 4 Liftabilities

In this section we discuss about conditions when an immersed orientable surface in $\mathbf{R}^3$ has an embedded lift in $\mathbf{R}^4$. Carter and Saito [CS] gave several conditions which detect whether an immersed surface has an embedded lift into $\mathbf{R}^4$.

We focus on relations between the liftability of an immersed surface and its deformations. We look at a deformation on an annulus and we describe some properties of it. In the following we assume that $F^2$ is closed and orientable.

### 4.1 Triple points

Let $f : F^2 \to \mathbf{R}^3$ be an immersion. Let $p$ be a triple point of $f(F^2)$ and let $U(p)$ be a neighbourhood of $T$ in $\mathbf{R}^3$ so that $U(p) \cap f(F^2)$ consists of three disks (see Figure 9), in which each pair of disks intersect at a segment in each disk.

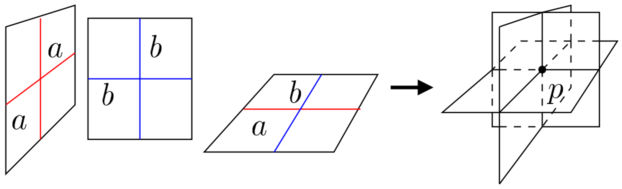



Figure 9: Three disks form a triple point

It is clear that $C(f) \subset F^2$ is a union of immersed circles in $F^2$. Since $F^2$ is orientable, there are pairs of immersed circles $\{C_a, C_b\}$ in $\mathcal{C}(f)$ such that $f(C_a) = f(C_b)$. Carter and Saito introduced the following conditions. The crossing set $C(f)$ is *colourable* [CS] if

- $(CS1)$ each pair of immersed circles $\{C_a, C_b\}$ so that $f(C_a) = f(C_b)$ has two colours $a$ for $C_a$ and $b$ for $C_b$ and
- $(CS2)$ for each triple point $p$, three disks of $U(p) \cap f(F^2)$ have a triple of pairs of colours, $\{(a, a), (a, b), (b, b)\}$.

**Note 4.1.** *The condition $(CS2)$ will be called a $(3, 2)$-condition. Similarly, we can define a $(4, 3)$-condition, where each case of four has one of these triples of colours, $\{(a, a, a), (a, a, b), (a, b, b), (b, b, b)\}$. Later this condition will be sued for an immersed 3-manifolds in $\mathbf{R}^4$.*

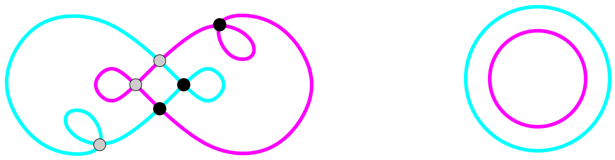
The following is known [CS]

**Theorem 4.1 (Carter and Saito).** *Let $F^2$ be a closed surface and let $f : F^2 \to \mathbf{R}^3$ be an immersion. Then $C(f)$ is colourable if and only if $f$ has an embedded lift into $\mathbf{R}^4$.*

We will call those conditions of liftability *$CS$-conditions.*

**Examples 2.** *$i_8(S^2)$ and $i_{\bigcirc}(S^2)$ have embedded lifts into $\mathbf{R}^4$.*

Their pre-images are shown in Figure 10. It is obvious that $i_8$ and $i_{\bigcirc}$ have embedded lifts into $\mathbf{R}^4$ since they are colourable.



$C(i_8)$ $C(i_{\bigcirc})$

Figure 10: Pre-images of $i_8$ and $i_{\bigcirc}$

#### 4.1.1 Pre-image conditions

The following is a natural consequence of Theorem 4.1.

**Lemma 4.1.** *Let $f, g : F^2 \to \mathbf{R}^3$ be immersions and let $H : F^2 \times I \to \mathbf{R}^3$ be a regular homotopy between $f$ and $g$. If $H$ is covered by an isotopy $\hat{H} : F^2 \times I \to \mathbf{R}^4$, then there is a pair $C_{a_t}$, $C_{b_t} \in \mathcal{C}(H_t)$ such that $H_t(C_{a_t}) = H_t(C_{b_t})$. Then for every $C_{a_t}$, $C_{b_t} \in \mathcal{C}(H_t)$, $C_{a_t}$ cannot connect with $C_{b_t}$ by type* VI *h-move.*

*Proof.* From the assumption for each $t \in I$, $H_t$ has an embedded lift $\hat{H}_t$ so that $\hat{H}_t$ is an intermediate map of the isotopy $\hat{H}$. From Theorem 4.1 $C(H_t)$ satisfies $CS$-conditions and it is preserved for all $t$ from 0 to 1. This implies that $C_{a_t}$ cannot contact with $C_{b_t}$. □

## 4.2 Annuli twists

There is a regular homotopy on an annulus from an annulus to an immersed annulus with a simple closed double circle along a essential curve on the annulus fixing the boundary. In this section we construct the deformation using a sphere eversion. We show that such a deformation is not covered by an isotopy on the annulus.

#### 4.2.1 Sphere eversions

Let $\chi : \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection in the $xy$-plane and let $i : S^2 \to \mathbf{R}^3$ be an inclusion. There is a regular homotopy $K : S^2 \times I \to \mathbf{R}^3$ from $i$ to $\chi \circ i$. $K$ is called a sphere eversion. The existence of a sphere eversion comes from the fact found by Smale [Sma]:

$$Imm_r(S^2, \mathbf{R}^3) \cong \pi_2(\mathrm{SO}(3)) = \{e\}.$$

The sphere $i(S^2)$ is deformed to $\chi \circ i(S^2)$ by $K$ (see Figure 11). In Figure 11, short arcs represent normal vectors. Take two points $p_- = (0, 0, -2)$ and $p_+ = (0, 0, 2)$ and take arcs $\gamma_1$ from $p_-$ to the South pole of $S^2$ and $\gamma_2$ from $p_+$ to the North pole.

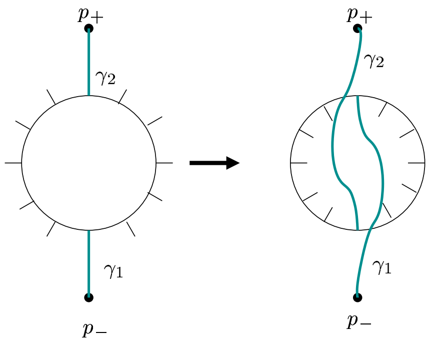



Figure 11: A sphere eversion and arcs.

We fix $p_\pm$ during the sphere eversion. Thicken these arcs to obtain pipes $P_1$ and $P_2$ along $\gamma_1$ and $\gamma_2$ respectively. Remove small neighbourhoods of the North pole and the South pole from $S^2$ to obtain an annulus $A_*$. Glue $P_1$ and $P_2$ to the boundary of $Cl(A_*)$. Thus we obtain a cylinder $A = P_1 \cup Cl(A_*) \cup P_2$.

The final stage of the deformation has an immersed annulus. We can modify the immersed annulus to an immersed annulus with a simple closed double curve along an essential curve on $A$ so that the modification does not create any triple points. Whole modifications above form a regular homotopy

$$H : A \times I \to \mathbf{R}^3.$$

We call this modification an *annulus twist* (see Figure 12).

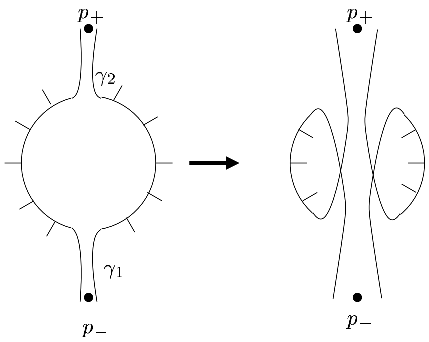



Figure 12: An annulus twist

The annulus twist $H$ induces a track called an *annulus twist track*.

$$\hat{H} : I \times S^2 \to I \times \mathbf{R}^3.$$

We obtain the following.

**Theorem 4.2.** *An annulus twist track $\hat{H}$ is not covered by an isotropy track $\tilde{H} : I \times A \to I \times \mathbf{R}^4$.*

*Proof.* Assume that the annulus twist track $\hat{H}$ is covered by an isotopy track $\tilde{H} : I \times A \to I \times \mathbf{R}^4$. Thus each immersed surface $(t, H_t(A))$ is a projection of an embedded surface $(t, H_t(A)) \subset \{t\} \times \mathbf{R}^4$. If we fix a colouring on $C(H_t)$ with $\{a, b\}$, then the colouring is preserved through the isotopy track. This implies a contradiction as follows.

Set $G_t = H_{1-t}$ and let $\tilde{G} = (t, G_t(x))$ for $x \in F^2, t \in I$. Then $G = \{G_t\}$ is a regular homotopy and $\tilde{G}$ is an isotopy track covering $G$.

From the definition of an annulus twist, $C(G_0)$ is a pair of essential circles $\{C_{a_0}, C_{b_0}\}$ on $A \cong S^1 \times [-2, 2]$. Obviously, we may assume that $C_{a_0}$ has a colour $a$ and $C_{b_0}$ has a colour $b$. Thus $G_0$ has an embedded lift into $\mathbf{R}^4$ from Theorem 4.1.

From Lemma 2.2 $\{C_{a_0}, C_{b_0}\}$ is deformed to a set of trivial circles in $A$ by $G$. We denote these trivial circles by $C_a$ and $C_b$. Remembering that orientations of immersed circles do not affect to connecting operation, each of $\{C_a, C_b\}$ represents a class of the first homology group $H_1(A; \mathbf{Z}_2) \cong \mathbf{Z}_2$. We denote these classes by $[C_{a_0}]$ and $[C_{b_0}]$ and $[C_a] = [C_b] = 0$. On the other hand, $C_{a_0}$ and $C_{b_0}$ represent non-trivial classes of $H_1(A; \mathbf{Z}_2)$ so $[C_{a_0}] + [C_{b_0}] = 0$.

We cannot join $C_{a_t}$ and $C_{b_t}$ if $G_t(C_{a_t}) = G_t(C_{b_t})$ from Lemma 4.1. This consequences that $C_{a_0}$ cannot be modified to a trivial circle with modifying $C_{a_0}$ itself.

A trivial circle in $A$ is obtained from a pair of non-trivial circles in $A$. The non trivial circle $C_{a_0}$ is not a pair in $A$ so pairs of non-trivial circles or trivial circles may be added to $C_{a_0}$ during the deformation. However, the resulting set of immersed circles in $A$ represents the same class of $C_{a_0}$.

Therefore, to change the class $[C_{a_0}]$ $C_{a_t}$ connects to a different colour component $C_{b_t}$. Thus for a $C_{a_t}$, there is a $C'_b$ such that $C'_b + C_a$ is trivial.

This means that $C_{a_0}$ has to contact with some $C'_b$ by type VI $h$-moves. This implies that the $CS$-conditions of $C(G_0)$ is not preserved by the regular homotopy $G$. □

# 5 The $FM$-eversion

Let $H : S^2 \times I \to \mathbf{R}^3$ be the $FM$-eversion. That is a regular homotopy from the inclusion $i : S^2 \to \mathbf{R}^3$ to $\chi \circ i$. The sphere eversion is expressed with a sequence of $h$-moves (Theorem 2.2). The final move of this sequence is Type $\mathrm{I}^-$ $h$-move. We denote the time of this stage by $1 - \varepsilon$, where $\varepsilon > 0$.

We will prove the following proposition.

**Proposition 5.1.** *There are two points $\{p, q\} \subset S^2$ such that*

$$\{p, q\} \cap C(H_t) = \emptyset, \qquad \textit{if } t < 1 - \varepsilon. \tag{9}$$

The proof is given in section 5.1. This proposition implies that $C(H_t)$ is contained in an open annulus $S^2 \setminus \{p, q\}$ for every $t < 1 - \varepsilon$. Therefore, $FM$-eversion is constructed by a deformation from an embedded annulus to an immersed annulus with an essential double curve on the annulus. This means that the eversion contain an annulus twist. Thus we have

**Corollary 5.1.** *The $FM$-eversion $H : S^2 \times I \to \mathbf{R}^3$ is not covered by an isotopy $\tilde{H} : S^2 \times I \to \mathbf{R}^4$.*

## 5.1 Details of the $FM$-eversion

For the proof of Proposition 5.1 we need to describe details of the $FM$-eversion. A sequence of intermediate maps of the $FM$-eversion can be found in [MB]. In the following, we follow those pictures in [MB] as well as we describe the details with the bug constructions.

Recall that a regular homotopy class of an immersed orientable surface is obtained by a bug construction with an embedded surface and an immersed circle on it (see section 3.1). In the $FM$-eversion we found an immersed sphere obtained by a bug construction as its intermediate map. The immersion is constructed with the inclusion $i : S^2 \to \mathbf{R}^3$, the reflection $\chi : \mathbf{R}^3 \to \mathbf{R}^3$, a trivial embedding $\bigcirc : S^1 \to S^2$ and an eight-immersion $8 : S^1 \to S^2$, where 8 is an immersion with only one crossing point (the figure of eight).

In this section we prove Proposition 5.1 as well as the following proposition.

**Proposition 5.2.** *The $FM$-eversion is a sequence of deformations from $i$ to $\chi \circ i$ passing through four immersions $(\chi \circ i)_\bigcirc$, $(\chi \circ i)_8$, $i_8$ and $i_\bigcirc$:*

$$i \to (\chi \circ i)_\bigcirc \to (\chi \circ i)_8 \to i_8 \to i_\bigcirc \to \chi \circ i. \tag{10}$$

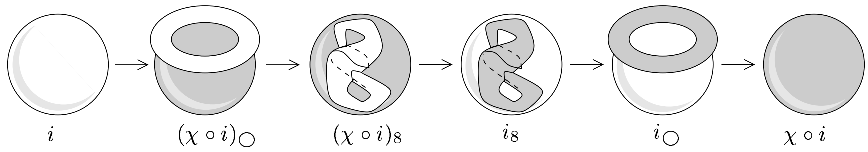



Figure 13: The sequence of the intermediate maps

**Remark 5.1.** *The deformation from $(\chi \circ i)_8$ to $i_8$ is the heart of the $FM$-eversion. This part creates a quadruple point* [MB].

### 5.1.1 Deformations

Take two points $p$ and $q$ on $S^2$. In the following figures, we denote $p$ and $q$ by $\otimes$ and $\oslash$ respectively We follow the $FM$-eversion and detect the orbit of $\{p, q\}$ to show the orbit does not intersect the set of tuple points at each $t \in [0, 1 - \varepsilon]$.

The $FM$-eversion starts from the inclusion $i : S^2 \to \mathbf{R}^3$ that deforms $i(S^2)$ with type $\mathrm{I}^+$ $h$-move to create a simple closed double curve (see Figure 14). It is not difficult to see that both points of $\{p, q\}$ are able to avoid the double curve. In the figure the result of the deformation is depicted as a cut surface. In this

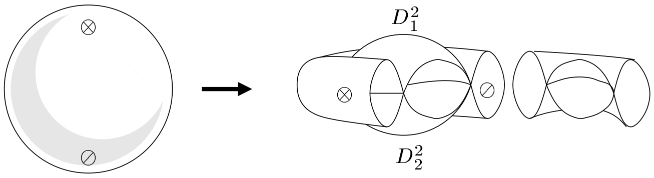



Figure 14: The first stage of the $FM$-eversion

picture there are two disjoint disks $D_1^2, D_2^2 \subset S^2$ such that $H_{t_1}(D_1^2) \cap H_{t_1}(D_2^2) = H_{t_1}(\partial D_1^2) = H_{t_1}(\partial D_2^2)$ is the double circle in the image. We again apply type $\mathrm{I}^+$ $h$-move on the disk $D_1^2$ so that a simple double curve is created in $D_1^2$. Then

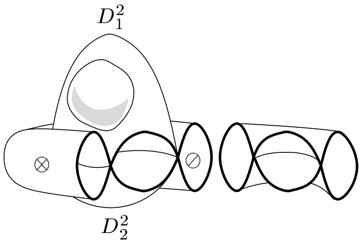



Figure 15: The second stage

we move the disk $D_2^2$ toward the simple closed double curve passing through it. This is done by type $\mathrm{II}^+$ $h$-move. Again $\{p, q\}$ can miss the self-intersections.

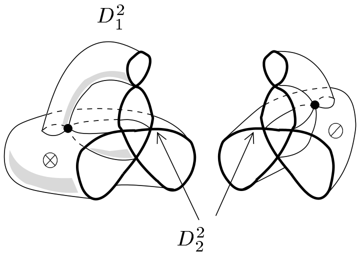



Figure 16: The third stage

We view the immersed sphere in Figure 16 from the bottom to obtain pictures in Figure 17. The set of tuple points consist of four double arcs, two double loops and two triple points. We have the double curve in the centre as zigzag so that we can apply type VI $h$-moves. In the right picture shadow disks depict descendent disks for type VI $h$-moves.

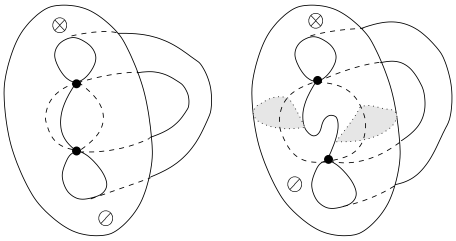


Figure 17: The view from the bottom

When we apply type VI $h$-moves at these descendent disks, we obtain an immersed sphere with a symmetry as shown in Figure 18. Type VI $h$-moves modify double arcs to create new double loops (see Figure 18). This modification joins a region containing the point $q$ with the other region so that $q$ can be moved to the other region.

Recall the definition of 'pipes' (see section 3.1). In this stage we can find some pipes in the immersed sphere. The tuple point set consists of two triple points, two pairs of double loops and two double arcs between triple points.

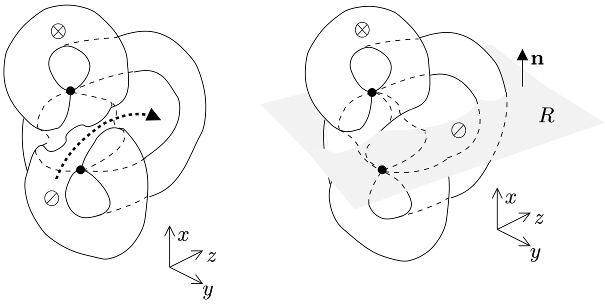



Figure 18: The symmetric pattern

#### 5.1.2 Constructing intermediate maps

We introduce some notations for these pipes, double point sets and triple points in an intermediate immersed sphere of the $FM$-eversion.

We will describe some intermediate maps of the $FM$-eversion with bug construction. Let $S$ be the intermediate immersed sphere in Figure 18. We may assume that $S$ has a rotational symmetry around the $z$-axis (see [MB]). Take a plane $R$ with a fixed unit normal vector $\mathbf{n} = (\mathbf{0}, \mathbf{1}, \mathbf{0})$ at the origin of $\mathbf{R}^3$. The space $\mathbf{R}^3$ is divided into two parts $\mathbf{R}^3_-$ and $\mathbf{R}^3_+$:

$$\mathbf{R}^3_- = \{x \in \mathbf{R}^3 | \mathbf{n} \cdot x < 0\}$$
$$\mathbf{R}^3_+ = \{x \in \mathbf{R}^3 | \mathbf{n} \cdot x \geq 0\},$$

where $\cdot$ is the inner product. Then $Cl(\mathbf{R}^3_- \cap S) = S_-$ $\mathbf{R}^3_+ \cap S = S_+$. $S_-$ is depicted in Figure 19. Then we obtain the picture in Figure 19.

The boundary of of $S_-$ is an immersed circle with two loops in $R$. We choose an inner loop then we shrink the loop to a point to create a branch point. In the whole picture this procedure creates a pair of branch points.

We can move one of branch point along a double curve. This is the reverse deformation of a bug construction. Finally, we obtain a singular sphere with two branch points, that is a bug. This is that an early stage of a bug construction. In fact, we can construct the immersed sphere $S$ using the bug. Tracing the movement of the branch point along $t$, we can find an immersed circle on $S^1$. This is the eight-immersion $8 : S^1 \to S^2$ Thus this immersed sphere is exactly $(\chi \circ i)_8$, that is that we have deformed $(\chi \circ i)_{\bigcirc}(S^2)$ to $(\chi \circ i)_8(S^2)$.

The constructed immersion $(\chi \circ i)_8$ has two triple points, two pairs of double loops and two double arcs joining distinct triple points. It is not difficult to see that a symmetry exist in $(\chi \circ i)_8(S^1)$ since the eight-immersion has a symmetry.

The crossing set $C((\chi \circ i)_8)$ consists of two pairs of loops based on triple points, two arcs between these triple points. $(\chi \circ i)_8(S^2) \setminus (\chi \circ i)_8(C((\chi \circ i)_8))$ consists of 8 disks. We take a small neighbourhood of each disk so that they are also disks and some of them form pipes in the immersed sphere. Observing these pipes we can find two pipes and two pipes with triple points in $(\chi \circ i)_8(S^2)$.

There are three types of pipes. One is a pipe which contains two triple points as the self-intersections of its double curve. We denote this by $P$. These two are seen inside $P$ and outside $P$. We denote these by $Q_1$ and $Q_2$ respectively.

Also we denote double loops of $P \cap S_+$ and $P \cap S_+$ by $a_+$ and $b_+$ and by $a_-$ and $b_+$ respectively. We denote the double arcs of $Q_1$ and $Q_2$ by $d_1$ and $d_2$ respectively.

### 5.1.3 Around the quadruple point

The main idea of the $FM$-eversion is to swap positions of $Q_1$ and $Q_2$. In this stage we can see some symmetries of the immersed sphere. The tuple point set also has a symmetry, shown in Figure 19. The $FM$-eversion has only one quadruple point [MB]. We describe around this point.

From Figure 20 to Figure 22, gray arcs shows double curves. We move up $Q_1$ toward the $d_2$, which is the double curve of $Q_2$. When $Q_1$ reaches $d_2$, we can simplify the situation of two pipes as shown in Figure 20. In the first stage, $a_+$ contacts $a_-$. Then $a_+$ and $a_-$ create a pair of triple points. Loops $b_+$ and $b_-$ are modified to be arcs between triple points.

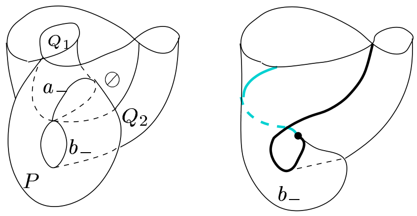



Figure 19: $S_-$ and its modification

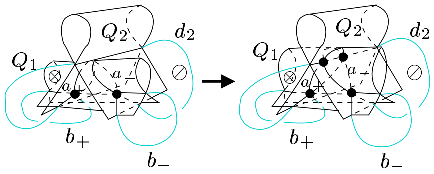



Figure 20: Swapping pipes

The left picture in Figure 21 shows that this immersed sphere has a quadruple point. Loops $a_+$ contacts $a_-$ and then they split and reconnected as new loops. As $Q_1$ moves, the double arc $d_1$ of $Q_1$ creates a pair of loops. We denote one of these loops by $d_1$ again. This means that new triple points are created. At this stage, the point $q$ is bounded by the loop $d_2$. Clearly, it has not intersected any double curves or triple points.

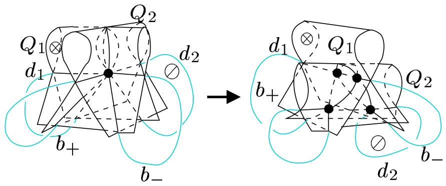



Figure 21: $Q_1$ and $Q_2$ create a quadruple point

Two of triple points will be eliminated when $d_1$ leaves $Q_2$. Finally, we obtain two pairs of loops and two arcs between triple points. This is similar situation to the beginning. At this stage the whole picture is similar to the first picture. In fact, if we rotate the resulting immersed sphere by 90 degree, then it has the same proportion of the original immersed sphere except the relation between inside and outside the surface. This implies that the immersed sphere is $(\chi \circ i)_8$.

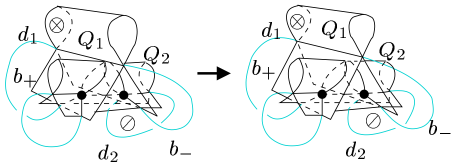



Figure 22: The deformation changes the double curves

Now we compare the original immersed sphere $(\chi \circ i)_8$ and $i_8$ (see Figure 23). The points $p$ and $q$ appear different places. We can follow the reverse deformation from this stage toward the immersion with a simple closed double curve, that is $i_{\bigcirc}$. This completes the proof of Proposition 5.2.

The immersed sphere $i_{\bigcirc}(S^2)$ has two disks $D_1^2$ and $D_2^2$ bounded by the double curve. It is not difficult to see that the each point of $\{p, q\}$ are on each disk. This implies that these points have avoided the self-intersections during the deformation until $t = 1 - \varepsilon$. This completes the proof of Proposition 5.1.

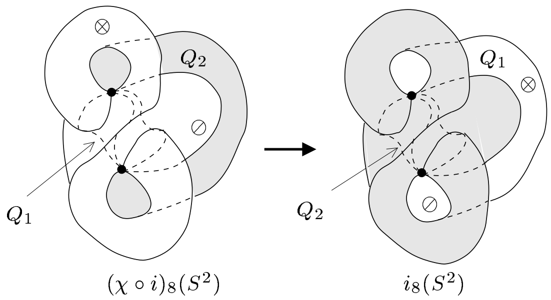



Figure 23: $(\chi \circ i)_8$ and $i_8$

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