The Majoritarian Compromise in Large Societies

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Abstract: First, we dwell on the definition of the Majoritarian Compromise in the case of an odd number of alternatives. Then, assuming the Impartial Culture hypothesis we calculate the average maximum welfare achievable by the Majoritarian Compromise procedure and show that this social choice rule is asymptotically stable with the proportion of the number of unstable profiles to the total number of profiles being in the order of $O\left(1/\sqrt{n}\right)$,[1] where n is the total number of agents.

Running title: Majoritarian compromise.

Key words: Majoritarian compromise, maximum majority welfare, asymptotic stability, asymptotic strategy-proofness.

[1]We write $g(n) = O(f(n))$ in case there is a positive constant C such that $|g(n)| \leq Cf(n)$ for all sufficiently large values of n

1. Basic Concepts and Definitions

Let $A = \{a_1, \ldots, a_m\}$ be an m-element set whose elements will be called alternatives. By $\mathcal{L}(A)$ we denote the set of all linear orders on A; they represent the preferences of agents over A. The elements of the cartesian product

$$\mathcal{L}(A)^n = \mathcal{L}(A) \times \ldots \times \mathcal{L}(A) \quad (n \text{ times})$$

are called n-profiles or simply profiles. They represent the collection of preferences of an n-element society of agents $\mathcal{N} = \{1, 2, \ldots, n\}$. A typical profile $\overrightarrow{R} = (R_1, \ldots, R_n)$ is an ordered n-tuple of linear orders, where a linear order $R_i \in \mathcal{L}(A)$ represents the preferences of the i-th agent. By aR_ib, where $a, b \in A$, we denote that the ith agent prefers a to b.

In this paper we assume the so-called Impartial Culture (IC) conjecture under which every voter chooses a linear order from a uniform distribution on the set of linear orders and all voters are independent.

A family of correspondences $F = \{F_n\}$,

$$F_n \colon \mathcal{L}(A)^n \to \mathcal{P}(A),$$

where n is a positive integer and $\mathcal{P}(A)$ is the power set of A, we will call a social choice rule (SCR). Normally it is assumed that F represents a certain algorithm which, on accepting a positive integer n and an n-profile $R \in \mathcal{L}(A)^n$, outputs a subset $F_n(R)$ of A.

One of the SCRs is the majoritarian compromise procedure, which was suggested by Sertel circa 1986 and has been widely discussed recently. The goal of this paper is to study the behavior of the majoritarian compromise in large societies. We first define it following Sertel and Yilmaz (1999).

Suppose a profile $\overrightarrow{R} = (R_1, \ldots, R_n) \in \mathcal{L}(A)^n$ is given, where $R_i \in \mathcal{L}(A)$ is the linear order which represents preferences of the voter $i \in \mathcal{N}$ over the alternatives from A. According to Sertel and Yilmaz, for the ith voter the alternative $a \in A$ obtains an ordinal "utility"

$$\pi_i(a) = card\{b \in A \mid aR_ib\},$$

which is a positive integer such that $1 \leq \pi_i(a) \leq m$. For a coalition $K \subseteq \mathcal{N}$ any alternative $a \in A$ provides an ordinal "welfare"

$$\pi_K(a) = \min_{i \in K} \pi_i(a),$$

and

$$\pi_K(\overrightarrow{R}) = \max_{a \in A} \pi_K(a)$$

is the maximal welfare achievable for the coalition K. A coalition K is called a majority in $\mathcal{N}$ iff $card(K) \geq card(\mathcal{N} \setminus K)$. The set of all majorities is denoted as $\mathcal{M}$. Then

$$\pi(\overrightarrow{R}) = \max_{K \in \mathcal{M}} \pi_K(\overrightarrow{R})$$

is the maximal majority welfare achievable. Let

$$K(\overrightarrow{R}, a) = \{i \in \mathcal{N} \mid \pi_i(a) \geq \pi(\overrightarrow{R})\}.$$

be the coalition of voters whose welfare at a is at least the maximal majority welfare. Then we define

$$MC(\overrightarrow{R}) = \{a \in A \mid K(\overrightarrow{R}, a) \in \mathcal{M}\}.$$

Definition 1. *The social choice correspondence*

$$MC\colon \overrightarrow{R} \to MC(\overrightarrow{R})$$

is called the Majoritarian Compromise.

If we need a social choice function, then we may try to consider the following refinement of the Majoritarian Compromise by choosing the set

$$MC_1(\overrightarrow{R}) = \{a \in MC(\overrightarrow{R}) \mid card(K(\overrightarrow{R}, a)) \text{ is maximal}\}.$$

This also does not guarantee that the choice set will be a singleton. Eventually we will have to consider using one of the tie-breaking rules.

2. Maximum Majority Welfare Achievable

Let us put the Majoritarian Compromise in the algorithmic setting. Given a profile $\overrightarrow{R}$ we say that an alternative $a \in A$ gains kth degree of approval from a coalition K if $\pi_K(a) \geq m-k+1$. In the first round we look for alternatives which gain first degree approvals from their respective majorities (there may be no more than two of them). If such alternatives exist, the algorithm stops and outputs these alternatives. In other words, in the first round we test the hypothesis that $\pi(\overrightarrow{R}) = m$.

In the second round we test the hypothesis that $\pi(\overrightarrow{R}) = m-1$. We look if there is an alternative (or several of them) which gains second degree approval from a majority. If such alternatives exist, the algorithm stops and outputs these alternatives.

If the first two rounds do not reveal the winner(s), we continue to test hypotheses $\pi(\overrightarrow{R}) = k$ for $k = m, m-1, \ldots$. If $m = 2p$ is even, then at most p rounds would be necessary and the last hypothesis to test is $\pi(\overrightarrow{R}) = p+1$. Indeed, there are a total of $\frac{mn}{2}$ approvals of pth degree, hence at least one alternative will get $\geq \frac{n}{2}$ approvals.

If $m = 2q+1$ is odd, then q rounds might not be enough. From the Lemma below we will see that the probability of this event tends to 1. The same Lemma shows that the probability that all alternatives gain $(q+1)$th degree approval is also tends to 1.

Lemma 1. *Let $a \in A$ and $\ell < m/2$. Then there exist $0 < \alpha < 1$ such that the probability of the event*

$$card\{i \in \mathcal{N} \mid \pi_i(a) \geq m - \ell + 1\} \geq \left\lfloor \frac{n-1}{2} \right\rfloor$$

is in the order of $O(\alpha^n)$, when $n \to \infty$.[2]

Proof. We consider that n is even, in which case $\left\lfloor \frac{n-1}{2} \right\rfloor = n/2-1$; when n is odd, some minor and obvious changes should be made. It is convenient to view any profile $\overrightarrow{R} = (R_1, \ldots, R_n)$ as a table in which the ith column represent R_i so that aR_ib iff a is higher in this column than b. In these terms we now need to estimate the probability ν of the event that in a random profile $\overrightarrow{R}$ an alternative $a \in A$ is found in the upper ℓ rows of the table $s \geq n/2-1$ times.

Let $i = m/2 - \ell \geq 1/2$. Then the probability ν_s that an alternative $a \in A$ is found in $\overrightarrow{R}$ exactly s times in the upper ℓ rows of the table is equal to

$$\nu_s = \frac{((m-1)!)^n}{((m!)^n} \binom{n}{s} (m/2-i)^s (m/2+i)^{n-s} = \frac{1}{m^n} \binom{n}{s} (m/2-i)^s (m/2+i)^{n-s}.$$

When $s \geq n/2$, then $n - s \leq n/2 \leq s$, and we get

$$\nu_s = \frac{1}{2^n} \binom{n}{s} \left(1 - \frac{2i}{m}\right)^s \left(1 + \frac{2i}{m}\right)^{n-s} = \frac{1}{2^n} \binom{n}{s} \left(1 - \frac{4i^2}{m^2}\right)^{n-s} \left(1 - \frac{2i}{m}\right)^{2s-n}$$

[2] $\lfloor x \rfloor$ stands for the largest integer which do not exceed x.

$$\leq \frac{1}{2^n}\binom{n}{s}\left(1-\frac{1}{m^2}\right)^{n/2} = \frac{1}{2^n}\binom{n}{s}\alpha^n,$$

where $\alpha = \sqrt{(1-1/m^2)}$. When $s = n/2-1$, then $n-s = n/2+1$, and

$$\nu_s = \frac{1}{2^n}\binom{n}{s}\left(1-\frac{2i}{m}\right)^s\left(1+\frac{2i}{m}\right)^{n-s} = \frac{1}{2^n}\binom{n}{s}\left(1-\frac{4i^2}{m^2}\right)^s\left(1+\frac{2i}{m}\right)^2$$

$$\leq \frac{C}{2^n}\binom{n}{s}\left(1-\frac{1}{m^2}\right)^{n/2} = \frac{C}{2^n}\binom{n}{s}\alpha^n$$

for some $C > 1$. The probability $\nu = \nu_{n/2-1} + \nu_{n/2} + \cdots + \nu_n$ in question, then, can be estimated as

$$\nu \leq \frac{C}{2^n}\sum_{s=n/2-1}^{n}\binom{n}{s}\alpha^n \leq C\alpha^n.$$

The lemma is proved. □

So how to fix this irregularity for the odd $m = 2q+1$ case? In this paper we suggest that simple majority should not be used in the last round. Instead, for the last round a subset $K \subseteq \mathcal{N}$ should be considered a majority iff $card(K) \geq \frac{q+1}{m}n$. The reason for that is simple. We have a total of $(q+1)n$ approvals of degree $q+1$ with the average number of approvals being $\frac{q+1}{m}n$. Therefore only those alternatives should be selected which have more than average number of approvals. For example, if $m = 3$, then the $2/3$ majority rule must be used in the second round, and if $m = 5$, then in the third round we should use $3/5$ majority rule.

We summarise our results in the following theorem

Theorem 1. *As n approaches infinity, then the probability that the majoritarian compromise procedure terminates in the ℓ-th round, for some $\ell < m/2$, is in the order of $O(\alpha^n)$, for some $0 < \alpha < 1$. In particular, the probability that the maximum majority welfare achievable is different from $\lfloor \frac{m}{2} \rfloor$ is exponentially small.*

3. The Asymptotic Stability

Definition 2. *Let F be a SCR and let $\overrightarrow{R} = (R_1, \ldots, R_n)$ be a profile. We say that the profile $\overrightarrow{R}$ is unstable for F if there exists a linear order $R_i^{'}$ such that for a profile $\overrightarrow{R}^{'} = (R_1, \ldots, R_i^{'}, \ldots, R_n)$, where $R_i^{'}$ replaces R_i, we have $F_n(R^{'}) \neq F_n(R)$.*

In other words the profile is unstable if one of the voters is pivotal and can change the result. Let us define the index of instability of F by the formula

$$L_F(n, m) = \frac{e_F(n, m)}{(m!)^n}, \tag{1}$$

where $e_F(n, m)$ is the total number of all unstable profiles. We note that under assumption that $\mathcal{L}(A)^n$ is a discrete probability space with the uniform distribution, the index $L(m, n)$ becomes the probability of drawing at random an unstable profile.

Definition 3. *We say that a SCR F is asymptotically stable if for any number m of alternatives $L_F(n, m) \to 0$ as $n \to \infty$.*

Theorem 2. *The majoritarian compromise is asymptotically stable with the probability $L_{MC}(n, m)$ of drawing an unstable profile being in the order of $O\left(1/\sqrt{n}\right)$.*

Proof: By Theorem 1 we may prove the statement conditional on the event that the procedure terminates in the last round.

In general, we have to consider four cases depending on the parities of n and m. The parity of n is not really important although the formulae will differ slightly for the odd and the even case. We will assume that n is even. It will be absolutely clear what changes should be made for the odd case. The parity of m is more important since the last round is different for even and odd m. Thus we will consider two cases.

a) m is even. Then a profile is unstable if and only if, for some alternative a,

$$card\{i \in \mathcal{N} \mid \pi_i(a) \geq k + 1\} = \frac{n-1}{2} \pm \frac{1}{2}.$$

Representing again profiles as tables, this is the same to say that there are either $n/2$ or $n/2-1$ entries of a in the upper half of the table. Using the well-known inequality

$$\binom{n}{n/2} < \frac{2^n}{\sqrt{n}}, \tag{2}$$

we obtain

$$L_{MC}(n,m) \leq m \frac{\binom{n}{n/2}(m/2)^{n/2}(m/2)^{n/2}\,((m-1)!)^n}{(m!)^n} +$$

$$m \frac{\binom{n}{n/2-1}(m/2)^{n/2-1}(m/2)^{n/2+1}\,((m-1)!)^n}{(m!)^n} \leq 2m\binom{n}{n/2}\frac{1}{2^n} < \frac{2m}{\sqrt{n}}.$$

b) m is odd. Then a profile is unstable if and only if, for some alternative a, there are either $\left\lceil \frac{(m+1)n}{2m} \right\rceil$ or $\left\lceil \frac{(m+1)n}{2m} \right\rceil - 1$ entries of a in the upper k rows of the table.[3] For simplicity of notation we will consider that $\frac{(m+1)n}{2m}$ is an integer. Some minor changes should be made in the general case. We will need the following asymptotic formula for the binomial coefficients (see, for example, Peterson and Weldon (1972)):

$$\binom{n}{\alpha n} \sim (2\pi\alpha(1-\alpha)n)^{-1/2}2^{nh(\alpha)}, \tag{3}$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the entropy function and $f(n) \sim g(n)$ means that $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$. Thus, using (3)

$$L(n,m) \leq m\frac{((m-1)!)^n}{(m!)^n}\binom{n}{\frac{(m+1)n}{2m}}\left(\frac{m+1}{2}\right)^{\frac{(m+1)n}{2m}}\left(\frac{m-1}{2}\right)^{\frac{(m-1)n}{2m}} +$$

$$m\frac{((m-1)!)^n}{(m!)^n}\binom{n}{\frac{(m+1)n}{2m}-1}\left(\frac{m+1}{2}\right)^{\frac{(m+1)n}{2m}-1}\left(\frac{m-1}{2}\right)^{\frac{(m-1)n}{2m}+1}$$

$$\leq \frac{C_1}{m^n}\binom{n}{\frac{(m+1)n}{2m}}\left(\frac{m+1}{2}\right)^{\frac{(m+1)n}{2m}}\left(\frac{m-1}{2}\right)^{\frac{(m-1)n}{2m}}$$

$$\sim \frac{C_2}{\sqrt{n}}2^{n\left(h\left(\frac{(m+1)}{2m}\right)-1\right)}\left[\left(1+\frac{1}{m}\right)^{\frac{m+1}{2m}}\left(1-\frac{1}{m}\right)^{\frac{m-1}{2m}}\right]^n = \frac{C_2}{\sqrt{n}}.$$

[3] $\lceil x \rceil$ stands for the smallest integer which is greater than or equal to x.

The last equality holds because $h\left(\frac{m-1}{2m}\right) = h\left(\frac{m+1}{2m}\right)$ and

$$\log_2\left[\left(1+\frac{1}{m}\right)^{\frac{m+1}{2m}}\left(1-\frac{1}{m}\right)^{\frac{m-1}{2m}}\right] = 1 - h\left(\frac{m-1}{2m}\right).$$

The latter can be proved by a simple calculation. Hence $L_{MC}(n,m) = O\left(1/\sqrt{n}\right)$ as required. □

4. Asymptotic Strategy-Proofness

The well-known impossibility theorem of Gibbard and Satterthwaite states that every non-dictatorial singleton valued SCR is manipulable (Gibbard (1973), Satterthwaite (1975); see also Pattanaik (1976)). This result is also valid for arbitrary social choice rules with an appropriate concept of manipulability (see, for example, Ching and Zhou (1999) and the literature there). The difficulty in the non-singleton valued SCRs stems from the fact that to define a manipulability of such SCR F one has to know which changes of the choice set $F_n(R)$ are advantageous for the manipulating agent and hence one has to rank all subsets of alternatives one way or another. The latter can be done in many different ways. This is addressed in Barberà et al (2001).

Nevertheless, for most classical rules it has been proved (Slinko (2002a), (2002b)) that the probability of possibility to manipulate tends to zero as the number of agents grows. Such SCRs are called asymptotically nonmanipulable or asymptotically strategy-proof. Clearly every manipulable profile, no matter how the manipulability is defined, must be unstable, therefore the asymptotic stability implies the asymptotic strategy-proofness of any kind. Hence we can state:

Theorem 3. *The majoritarian compromise is asymptotically strategy-proof for any definition of manipulability.*

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