Ranking Committees, Words or Multisets

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Abstract: We investigate the ways in which a linear order on a finite set A can be consistently extended to a linear order on a set $\mathcal{P}_k(A)$ of multisets on A of fixed cardinality k. We show that for $card(A) = 3$ all linear orders on $\mathcal{P}_k(A)$ are additive and classify them by means of Farey fractions. For $card(A) \geq 4$ we show that there are non-additive consistent linear orders of $\mathcal{P}_k(A)$, we prove that they cannot be extended to a linear order of $\mathcal{P}_K(A)$ for K sufficiently large. We give the lower bounds for the number of additive linear orders in $\mathcal{P}_2(A)$ and the total number of consistent linear orders in $\mathcal{P}_2(A)$.

Key words: multiset, additive linear orders, expected utility structure

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1. Introduction

There are several viewpoints from which to regard what we purport to do in this paper. The main motivation is politico-economic: we wish to consistently extend preferences over political parties to committees (e.g., parliaments) composed of their representatives. We assume that any two members of the same political party sitting on a committee pursue the same policy and are, hence, indistinguishable. Another interpretation of the problem is the extension of a total order over an "alphabet" of "letters" to the free commutative semigroup of "words," where the order of letters in the word is disregarded and the product of any two words is their concatenation.

Indeed, taking any non-empty set A as our set of political parties or, respectively, our alphabet, we are interested in the committees or, respectively, the words which can be formed of members of A, and in any case we regard any two committees, or resp., words, u and v as equivalent when, for each member a of A the number of times a occurs is the same in u as in v. Given a linear order on A, the set of committees can be viewed as the set of all non-decreasing finite sequences of elements of A. It is in this form that the question of the consistent extension of preferences on the set of political parties to the set of committees was proposed by Sertel [1] in a series of lectures, as later also published in Sertel and Kalaycıoğlu [2].

Another set of motivating applications stems from the fact that, in different branches of mathematics and computer science, rankings of words constructed of elements of a certain set A have been in use for quite some time. The set of all such words, commonly called "*multisets*" on A, will be denoted as $\mathcal{P}(A)$. The concept of a multiset generalizes the concept of a set, since a word may contain several identical elements of A. Mathematically speaking, a *multiset* M on a set A is a pair $M = (A, \mu)$, where $\mu\colon A \to \mathbb{N}$ is a function, from A to the set $\mathbb{N}$ of all nonnegative integers, which for each element $a \in A$ gives the *multiplicity* (the number of occurrences) $\mu(a)$ of a. The usual notions for sets can be carried over to multisets in a natural way. Assuming henceforth that A is finite with $A = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$, the cardinality of a multiset $M = (A, \mu)$ can be defined as the sum $\operatorname{card}(M) = \mu(a_1) + \mu(a_2) + \ldots + \mu(a_n)$. We say that a multiset $M_1 = (A, \mu_1)$ is a subset of a multiset $M_2 = (A, \mu_2)$ and write $M_1 \subseteq M_2$ if $\mu_1(a) \leq \mu_2(a)$ for all $a \in A$. Rankings of $\mathcal{P}(A)$ and its subsets which extend a given ranking of A have been instrumental in proofs of program termination, in equational reasoning algorithms based on term rewriting systems, in computer algebra, the theory of invariants, and the theory of partitions. We refer the reader to the two surveys by Martin [3] and Dershowitz [4] on these topics.

Returning to economic motivations, we note that the outcome of a lottery played several times is a multiset of prizes. And although it can be argued that the "real" preferences are over outcomes and not lotteries, von Neumann and Morgenstern [5] avoided the difficulty of comparing multisets of prizes by defining preferences over lotteries directly. Nevertheless practical observations revealed the so-called "paradoxes of utility theory," which show that von Neumann and Morgenstern theory cannot explain certain experimental results [6, 7]. In this respect it is interesting to investigate what kind of preferences might individuals have on the set of all multisets of prizes and whether all these preferences have the so-called "expected utility" structure.

Another incentive to consider extensions of rankings from A to $\mathcal{P}(A)$ and its subsets is that it might provide a common unifying framework to a number of topics in the economics literature which may otherwise appear as unrelated. On the one hand we have an extensively studied topic of ranking subsets of A, given a ranking on A. The reader is referred for the literature on this topic to the survey by Barberá, Bossert and Pattanaik [8]. On the other hand there are a number of papers in the representation theory of measurements, mostly by Fishburn [9, 10, 12] on ranking finite Cartesian products $A_1 \times A_2 \times \ldots \times A_n$. Both can be interpreted in terms of rankings of multisets but the connection between rankings of Cartesian products and rankings of multisets needs further investigation.

Following [8], any reflexive, complete and transitive relation will be called an *order* and any antisymmetric order will be called a *linear order*. Orders on $\mathcal{P}(A)$ will typically be denoted as $\succeq$. In this case $\succ$ will be the strict preference relation of $\succeq$, i.e. $M \succ N$ will mean that $M \succeq N$ holds but $N \succeq M$ fails, and $M \sim N$ will be the indifference relation of $\succeq$, i.e. $M \sim N$ will mean that both $M \succeq N$ and $N \succeq M$ hold.

Orders of the following type play an exceptional role in statistics [13, 14], the representational theory of measurement and decision making [10], as well as computer science [11] and related other areas.

Definition 1. *Let $\mathcal{P}$ be a family of multisets on A. An order $\succeq$ on $\mathcal{P}$ is said to be additively representable (or simply additive) iff there exist nonnegative real numbers $w_1, \ldots, w_m$ such that, for all $M = (A, \mu)$, $N = (A, \nu)$ belonging* *to $\mathcal{P}$,*

$$M \succeq N \Longleftrightarrow \sum_{i=1}^{n} \mu(a_i) w_i \geq \sum_{i=1}^{n} \nu(a_i) w_i. \tag{1}$$

Knuth and Bendix [11] ordered the term algebra by replacing every term by the multiset consisting of functional symbols occurring in this term and then assigning weights to each functional symbol. They then calculated the total weight of every term and ranked the terms accordingly. Such an additive order of the term algebra is now known as a Knuth-Bendix order.

We will also refer to the numbers $w_1, \ldots, w_n$ as the *weights* of $a_1, \ldots, a_n$. In the literature these numbers are often referred to as "utilities" or, when $w_1 + w_2 + \ldots + w_n = 1$, intuitive probabilities [13, 14]. When we fix weights then we naturally determine an order not only on A but also its extension to an order of $\mathcal{P}(A)$.

A similar concept of additive representability was also defined for orders on a Cartesian product $A_1 \times A_2 \times \ldots \times A_n$. Here the utility of the tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is defined as $u(\mathbf{a}) = u_1(a_1) + u_2(a_2) + \ldots + u_n(a_n)$, where u_i is a utility function for A_i.

There is an extensive literature on the axiomatic characterization of additive orders. The condition presented below in Definition 2 was suggested by de Finetti [13] and plays a central role in this literature. To formulate it we need to define the operation of multiset union. To this end, given any two multisets $M_1 = (A, \mu_1)$ and $M_2 = (A, \mu_2)$ on A, by their union $M_1 \cup M_2$ we mean the multiset (A, μ), for which $\mu(a) = \mu_1(a) + \mu_2(a)$ for all $a \in A$. We note that the union defined in this way corresponds to the product of multisets viewed as elements of the free semigroup on A.

Definition 2. *Let $\mathcal{P}$ be a family of multisets on A. We say that an order $\succeq$ on $\mathcal{P}$ is preserved under the operation of multiset union iff, for any U, V and W belonging to $\mathcal{P}$ with $U \cup W$ and $V \cup W$ also in $\mathcal{P}$, we have*

$$U \succeq V \Longleftrightarrow U \cup W \succeq V \cup W \tag{2}$$

whenever $U \cup W$ and $V \cup W$ are both defined, i.e. they are also elements from $\mathcal{P}$.

The strength of this condition can vary depending on the structure of $\mathcal{P}$ under the partial order of inclusion of multisets. Indeed, if $\mathcal{P}$ is a set of all multisets of fixed cardinality, then no two subsets U and V belonging to $\mathcal{P}$

have their union again in $\mathcal{P}$ and this condition does not restrict orders on $\mathcal{P}$ at all. On the other hand, if $\mathcal{P}$ contains all singleton subsets of A, then this condition is rather strong.

De Finetti and a number of other researchers investigated orders on the set $\mathcal{P} = 2^A$ of all subsets of a finite set A. For subsets of A the multiset union of two sets is an ordinary set-theoretic union if the sets are disjoint and it is not defined otherwise. It was de Finetti who noticed that for an order $\succeq$ on the set $\mathcal{P} = 2^A$ to be additive it is necessary that this order is preserved under the operation of multiset union. On the other hand, Kraft et al [15] constructed an order on subsets of a set A consisting of five elements which is preserved under the operation of multiset union but is, nevertheless, non-additive. (Kraft et al [15] referred to orders preserved under the operation of multiset union as "additive" (with a meaning quite distinct from our meaning for this term). This condition was also called "strong extended independence" in [8].

In the computer science literature, this property of an order on a family of multisets is called "tameness." It guarantees that there are no infinitely decreasing chains of multisets, which is helpful in proving program termination [4]. The interest of this property has decreased significantly since Martin [3] showed that when $\mathcal{P} = \mathcal{P}(A)$, i.e. $\mathcal{P}$ is the family of all multisets, all orders preserved under the operation of multiset union are additive (i.e. Knuth-Bendix orders). Therefore for these two natural choices of $\mathcal{P}$ - when $\mathcal{P}$ is the set of all subsets and when $\mathcal{P}$ is the set of all multisets - we get two different results. The reason is that in the latter case the operation of multiset union is always defined, making the condition stronger.

For many purposes Definition 2 is too restrictive. Indeed, suppose that we can assign a utility w_i to each element $a_i \in A$. This will determine a certain order on A. Then, for any multiset $M = (A, \mu)$ let us say that the value $u_t(M) = \sum_{i=1}^{n} \mu(a_i)w_i$ represents the "total goodness" of M. If we rank multisets in accordance with their total goodness, then we get the order (1) which is preserved under the operation of multiset union. But if we define the value

$$u_a(M) = \text{card }(M)^{-1} \sum_{i=1}^{n} \mu(a_i)w_i$$

representing the "average goodness" of M and order multisets accordingly, i.e.

$$M \succeq N \iff u_a(M) \geq u_a(N), \tag{3}$$

then this order will not satisfy the condition (2) although it agrees with the order on A determined by the utilities and extends this order to $\mathcal{P}(A)$ in some natural way. This kind of extension of an order on A to an order on $\mathcal{P}(A)$ can be useful in some applications.

In the sequel, we will often identify elements of A and the corresponding singleton subsets of A.

Definition 3. *Let A be an ordered set and $\mathcal{P}$ be a family of multisets on A which contains all singleton subsets. An order $\succeq$ on $\mathcal{P}$ is said to be a weakly consistent extension of the given order on A iff,*

1. *The order induced by $\succeq$ on singleton subsets coincides with the linear order on A;*

2. *for any two multisets U, V of equal cardinality and any W belonging to $\mathcal{P}$,*
$$U \succeq V \Longleftrightarrow U \cup W \succeq V \cup W \tag{4}$$
whenever $U \cup W$ and $V \cup W$ both belong to $\mathcal{P}$.

Now, whether we define an order on multisets on the basis of total goodness or average goodness, both orders of multisets will satisfy weak consistency.

What can we expect from weakly consistent orders? Of course, we cannot expect them all to be additive, as the order (3) shows. But what we can still hope for is that the following property of *weak additivity* holds.

Definition 4. *Let $\mathcal{P} \subseteq \mathcal{P}(A)$. An order $\succeq$ on $\mathcal{P}$ is said to be weakly additive iff there exist nonnegative real numbers $w_1, \ldots, w_n$ such that, for any two multisets $M = (A, \mu)$, $N = (A, \nu)$ of the same cardinality belonging to $\mathcal{P}$, we have*
$$M \succeq N \Longleftrightarrow \sum_{i=1}^{n} \mu(a_i) w_i \geq \sum_{i=1}^{n} \nu(a_i) w_i. \tag{5}$$

It is important to emphasize that the weights used to determine a weakly additive order do not provide us with any basis to compare multisets of different cardinalities.

In this paper we undertake an investigation of orders of three critical sets of multisets. Given the set A equipped with a linear order $\succeq$, first we will be interested in the extension of $\succeq$ to weakly consistent orders of the set $\mathcal{P}(A)$ of all multisets, and to such orders of the set $\mathcal{P}_{\leq k}(A)$ of all multisets of cardinality $\leq k$. We will also be interested in extending $\succeq$ to the set $\mathcal{P}_k(A)$ of all multisets of cardinality k. For example, if we want to compare two compositions of a parliament we need to compare multisets of a fixed cardinality. Below we discuss what it means for an order on $\mathcal{P}_k(A)$ to be consistent with a given order on A.

Definition 5. *We say that an order $\succeq$ on $\mathcal{P}_k(A)$ is consistent (with a given order on A) iff there exists a weakly consistent order on $\mathcal{P}_{\leq k}(A)$ which coincides with $\succeq$ on the multisets of cardinality k.*

Alternatively we could define a consistent order on $\mathcal{P}_k(A)$ recursively by the following conditions:

1. The given order on A induces the only consistent order on $\mathcal{P}_1(A)$;

2. Let $\succeq$ be an order on $\mathcal{P}_k(A)$, $k \geq 2$, and let $1 \leq j < k$. Then for any $W \in \mathcal{P}_j(A)$ we may define an order $\succeq_W$ on $\mathcal{P}_{k-j}(A)$ by setting, for any $U, V \in \mathcal{P}_{k-j}(A)$,

$$U \succeq_W V \iff U \cup W \succeq V \cup W. \tag{6}$$

   An order $\succeq$ of $\mathcal{P}_k(A)$, $k \geq 2$, is said to be consistent iff for every $j = 1, 2, \ldots, k-1$, the orders $\succeq_W$ for all W's of fixed cardinality j coincide and this common order is a consistent order of $\mathcal{P}_{k-j}(A)$.

In other words, any consistent order $\succeq$ of $\mathcal{P}_k(A)$ stipulates, for $i = 2, \ldots, k-1$, the existence of consistent orders on $\mathcal{P}_i(A)$, satisfying (6). This is, of course, equivalent to the existence of a weakly consistent order on $\mathcal{P}_{\leq k}(A)$ which coincides with $\succeq$ on $\mathcal{P}_k(A)$.

This condition of consistency is stronger than the analogue of Bossert's condition of *responsiveness* [16], which he studied for orders of subsets of fixed cardinality, but he also assumed a very strong neutrality condition which we do not assume here.

Let us now preview our results in this paper. Without loss of generality, we assume that the set $A = [n] = \{1, 2, \ldots, n\}$ and that

$$\{1\} \succ \{2\} \succ \ldots \succ \{n\}. \tag{7}$$

We assume that the order on A (or on its singleton subsets) is linear since the case when $\{i\} \sim \{j\}$ for some i and j is not interesting and can be easily reduced to (7) for smaller n. To simplify notation we will identify $\{i\}$ with i and abbreviate $\mathcal{P}([n])$, $\mathcal{P}_{\le k}([n])$ and $\mathcal{P}_k([n])$ to $\mathcal{P}[n]$, $\mathcal{P}_{\le k}[n]$ and $\mathcal{P}_k[n]$, respectively. We will also omit $[n]$ when this invites no confusion.

We prove that, for a 3-element set $A = [3]$, for all integers $k \ge 1$, all weakly consistent orders on $\mathcal{P}_{\le k}[3]$ are weakly additive and hence all consistent orders on $\mathcal{P}_k[3]$ are additive for all k. We classify additive orders on $\mathcal{P}_k[3]$ by means of Farey fractions and find their asymptotics. When $A = [4]$, we show that there exist 12 consistent linear orders on $\mathcal{P}_2[4]$, that two of them, namely A_4 and E_4 in Figure 1, are non-additive. Moreover, no weakly consistent linear order on $\mathcal{P}_{\le 2}[4]$ coinciding with A_4 or E_4 on multisets of cardinality 2 can be extended to a weakly consistent order on $\mathcal{P}_{\le 3}[4]$.

We prove that there exists a positive integer valued function $k \mapsto f(k)$ such that, for $n \ge 4$, a linear order $\succeq$ on $\mathcal{P}_{\le k}[n]$ can be extended to a weakly consistent linear order of $\mathcal{P}_{\le f(k)}[n]$ iff $\succeq$ is weakly additive on $\mathcal{P}_{\le k}[n]$. As a corollary, any weakly consistent linear order on $\mathcal{P}[n]$ is weakly additive, providing us with the analogue of a result due to Martin [3].

Finally, for an arbitrary positive integer n, we give the lower bound for the number of additive linear orders on $\mathcal{P}_2[n]$ and the lower bound for the total number of consistent linear orders on $\mathcal{P}_2[n]$.

2. Orderings of families of multisets on a set A of cardinality $n = 3$

In this section we fix a consistent order $\succeq$ on $\mathcal{P}_k[3]$. By the definition, it stipulates a weakly consistent order on $\mathcal{P}_{\le k}[3]$ which will also be denoted as $\succeq$ without any confusion.

We will start with the following example.

Example 1 (Sertel, [1]). [1] *Let us take $n = 3$ and $k = 2$. Then the order $1 \succ 2 \succ 3$ determines all relations between the pairs (multisets from $\mathcal{P}_2[3]$) except the one between $\{1,3\}$ and $\{2,2\}$. Thus, to construct a consistent order of $\mathcal{P}_2[3]$ we have one degree of freedom and hence we can have just two different consistent linear orders of pairs. We will show that both opportunities materialize.*

[1] Also given in [2].

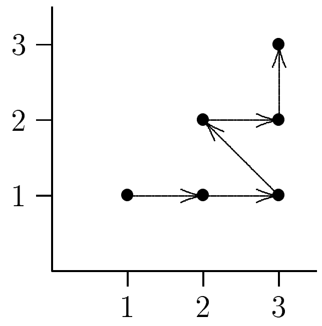
Assuming $\{1,3\} \succ \{2,2\}$ *we will obtain the "risk-loving" linear order,*

$$\{1,1\} \succ \{1,2\} \succ \{1,3\} \succ \{2,2\} \succ \{2,3\} \succ \{3,3\}, \tag{8}$$

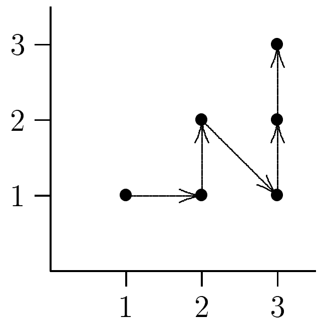
and assuming $\{2,2\} \succ \{1,3\}$ *we have the "risk-averse" one,*

$$\{1,1\} \succ \{1,2\} \succ \{2,2\} \succ \{1,3\} \succ \{2,3\} \succ \{3,3). \tag{9}$$

They are graphically represented as follows:



Risk loving



Risk averse

Assuming $\{1,3\} \sim \{2,2\}$ *we will obtain an order,*

$$\{1,1\} \succ \{1,2\} \succ \{1,3\} \sim \{2,2\} \succ \{2,3\} \succ \{3,3\}, \tag{10}$$

which we will call "risk-neutral."

Consistency of these orders is easy to check. We observe that all three orders of $\mathcal{P}_2[3]$*, the risk-loving, the risk-averse, and the risk-neutral, are additive with the weights* $(w_1, w_2, w_3) = (1, 1/2 - \epsilon, 0)$*,* $(w_1, w_2, w_3) = (1, 1/2 + \epsilon, 0)$*, and* $(w_1, w_2, w_3) = (1, 1/2, 0)$*, respectively, where* ϵ *is a small real positive number (less than 1/2).*

Lemma 1. *Let* $\succeq$ *be a weakly consistent order of* $\mathcal{P}_{\leq k}[n]$*.*

(a) If $U, V \in \mathcal{P}_\ell[n]$*,* $R, Q \in \mathcal{P}_h[n]$*, and* $U \succeq V$*,* $R \succeq Q$*,* $\ell + h \leq k$*. Then* $U \cup R \succeq V \cup Q$*.*

(b) If $U, V \in \mathcal{P}_\ell[n]$*, and* $U \succeq V$*,* $\ell h \leq k$*, for some integer* h*. Then*

$$\underbrace{U \cup U \cup \ldots \cup U}_{h} \succeq \underbrace{V \cup V \cup \ldots \cup V}_{h}.$$

Proof. (a) Due to weak consistency of $\succeq$, we get $U \cup R \succeq V \cup R \succeq V \cup Q$. (b) follows from (a) by induction. $\square$

In this section we prove a theorem that describes all consistent orders of $\mathcal{P}_k[3]$ for all k. First we prove that they are all additive and then we describe all additive orders in terms of the Farey fractions. We need to remind the reader of several definitions and concepts from Number Theory. The famous Farey sequence of fractions $\mathbf{F}_k$ is the increasing sequence of all fractions in lowest terms between 0 and 1 whose denominators do not exceed k. For example, the sequence $\mathbf{F}_6$ will be:

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}.$$

Let $\phi\colon \mathbb{N} \to \mathbb{N}$ be the Euler totient function, for which $\phi(1) = 1$ and, for $h \geq 2$, $\phi(h)$ is the number of positive integers not exceeding h and relatively prime to h. It is easy to see that passing from $\mathbf{F}_k$, to $\mathbf{F}_{k+1}$, there will be exactly $\phi(k+1)$ new Farey fractions added. Therefore there are exactly $\Phi(k) + 1$ fractions in $\mathbf{F}_k$, where $\Phi(k) = \sum_{h=1}^{k} \phi(h)$. (The standard reference for Farey fractions is [17]. See also [18]). It is a celebrated result by Franz Mertens (see, for example, [19], p. 59) that establishes the asymptotics of the sum $\Phi(k)$, namely,

$$\Phi(k) = \frac{3k^2}{\pi^2} + O(k \log k). \tag{11}$$

This means that $\Phi(k) - \frac{3k^2}{\pi^2} \leq Ck \log k$ for some constant C which does not depend on k.

Theorem 1. *All weakly consistent orders of $\mathcal{P}_{\leq k}[3]$ are weakly additive. For an arbitrary integer $k \geq 1$ there exist exactly $2\Phi(k) - 1$ different consistent orders of $\mathcal{P}_k[3]$ and $\Phi(k)$ of these are linear orders. All consistent orders of $\mathcal{P}_k[3]$ are additive.*

Proof. Let k be a positive integer and let $\mathbf{F}_k$ be the kth sequence of Farey fractions. We consider nonnegative real weights w_1, w_2, w_3 for $1, 2, 3 \in A$, respectively, with $w_1 \geq w_2 \geq w_3$. Hence w_2 lies on the segment $[w_3, w_1]$. For every Farey fraction $f = \frac{a}{b} \in \mathbf{F}_k$ we will consider the following weighted average of w_1 and w_3:

$$\hat{f} = \frac{aw_1 + (b-a)w_3}{b} = \frac{a}{b}w_1 + \frac{b-a}{b}w_3.$$

When $k = 2$, we have $\phi(2) = 1$, $\Phi(2) = 2$. We have

$$\mathbf{F}_2 = \{f_0 = 0, f_1 = 1/2, f_2 = 1\}$$

and respectively,

$$\hat{f}_0 = w_3,\ \ \hat{f}_1 = \frac{w_1 + w_3}{2},\ \ \hat{f}_2 = w_1.$$

The risk-loving linear order occurs precisely when $w_1 + w_3 > 2w_2$ or $w_2 \in (\hat{f}_0, \hat{f}_1)$, while the risk-averse linear order occurs precisely when $w_2 \in (\hat{f}_1, \hat{f}_2)$. We will have also three non-antisymmetric orders in each of the cases where $w_2 = \hat{f}_0$, $w_2 = \hat{f}_1$, and $w_2 = \hat{f}_2$, but the first and the last must be excluded as they give $1 \sim 2$ and $2 \sim 3$, respectively. Therefore we get three different orders, two of which are linear. This gives a basis for our induction.

Let us prove by induction that, for all k, there will be as many consistent orders of $\mathcal{P}_k[3]$ as the number of Farey fractions in $\mathbf{F}_k$ different from 0 and 1, plus the number of intervals between the neighboring fractions in $\mathbf{F}_k$, that these orders are all additive and each of them arises when $w_2 \in (\hat{f}_i, \hat{f}_{i+1})$ for some two neighboring Farey fractions f_i and f_{i+1} (in which case we obtain a linear order) or when $w_2 = \hat{f}_i \notin \{w_1, w_3\}$ (in which case the order is not antisymmetric).

Assuming that this is true for $\mathcal{P}_k[3]$, let us consider $\mathcal{P}_{k+1}[3]$. By the induction hypothesis, all consistent orders of $\mathcal{P}_k[3]$ are additive and therefore each has at least one extension to an order of $\mathcal{P}_{k+1}[3]$, in particular the one with the same weights. Let us explore where we can obtain more than one extension for an order $\succeq$ of $\mathcal{P}_{\le k}[3]$. We may assume that its weights are such that $w_2 \in (\hat{f}_i, \hat{f}_{i+1})$, for otherwise the extension is unique.

If any two multisets of $\mathcal{P}_{k+1}[3]$ contain an element in common, then their order in any weakly consistent extension is already determined by $\succeq$. The only thing which is not determined by $\succeq$ and the weak consistency of the extension is the position of the multiset $T = \{2, 2, \ldots, 2\}$ (with $\mu(2) = k + 1$ and $\mu(1) = \mu(3) = 0$) in the sequence of the following k multisets:

$$S_0 = \{\underbrace{3, 3, \ldots, 3}_{k+1}\} \prec S_1 = \{\underbrace{1, 3, \ldots, 3}_{k+1}\} \prec \ldots \prec S_{k+1} = \{\underbrace{1, 1, \ldots, 1}_{k+1}\}, \quad (12)$$

where S_i contains i digits of 1 and $k+1-i$ digits of 3. These multisets will be arranged in this particular order for every weakly consistent order $\succeq$ of $\mathcal{P}_{\le k+1}[3]$. Two different values of w_2 from the same interval $(\hat{f}_i, \hat{f}_{i+1})$ might position T in this sequence differently. This happens if and only if one of the fractions

$$\frac{1}{k+1},\ \frac{2}{k+1},\ \ldots,\ \frac{k}{k+1}, \tag{13}$$

falls into the interval (f_i, f_{i+1}). Indeed, if no such fraction falls into this interval, then for some $j \in \{1, \ldots, k-1\}$ we have

$$\frac{j}{k+1} \le f_i < f_{i+1} \le \frac{j+1}{k+1}.$$

This means that for every $w_2 \in (\hat{f}_i, \hat{f}_{i+1})$ we actually have

$$\frac{jw_1 + (k+1-j)w_3}{k+1} \le w_2 \le \frac{(j+1)w_1 + (k-j)w_3}{k+1}$$

and, so, every weakly consistent extension of $\succeq$ ranks T between S_j and S_{j+1}. Hence there is a unique extension of $\succeq$ to a linear order on $\mathcal{P}_{k+1}[3]$..

On the other hand, when one of the fractions (13), say $f = \frac{j}{k+1}$, falls into (f_i, f_{i+1}), there can be three different extensions of $\succeq$ to a weakly consistent order on $\mathcal{P}_{\le k+1}[3]$. One will occur when $w_2 \in (\hat{f}_i, \hat{f})$, another, when $w_2 = \hat{f}$, and the last one when $w_2 \in (\hat{f}, \hat{f}_{i+1})$. They will position T between S_{j-1} and S_j, force $T \sim S_j$, or position T between S_j and S_{j+1}, respectively. There can be no more than one new Farey fraction in (f_i, f_{i+1}) since it is known that, for any two neighboring Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have $|ad - bc| = 1$, which is not true for $\frac{a}{b} = \frac{j}{k+1}$ and $\frac{c}{d} = \frac{j+1}{k+1}$. This shows that we will have as many more consistent additive orders of $\mathcal{P}_{\le k+1}[3]$ in comparison to $\mathcal{P}_{\le k}[3]$ as twice the new Farey fractions which appear through the transition from $\mathbf{F}_k$ to $\mathbf{F}_{k+1}$, namely $2\phi(k+1)$ new orders, half of them being linear. Thus, $2\Phi(k) - 1$ weakly additive orders on $\mathcal{P}_{\le k}[3]$ become $2\Phi(k+1) - 1$ weakly additive orders on $\mathcal{P}_{\le k+1}[3]$.

Let us show now that every weakly consistent order on $\mathcal{P}_{\le k+1}[3]$ appears as an extension of one of the weakly consistent additive orders on $\mathcal{P}_{\le k}[3]$ in the way which has been just described. This will prove that all orders on $\mathcal{P}_{\le k+1}[3]$ are additive. Assume that $\succeq$ is a weakly consistent order of $\mathcal{P}_{\le k+1}[3]$. Then it induces a weakly consistent order $\succeq_i$ on $\mathcal{P}_{\le i}[3]$ for each $i \in \{1, 2, \ldots, k\}$. The induction hypothesis assumes that all orders $\succeq_i$ are additive relative to a common w_2 which either belongs to the interval $(\hat{f}_i, \hat{f}_{i+1})$ for some two neighboring terms f_i and f_{i+1} of $\mathbf{F}_k$ or coincides with $\hat{f}_i$ for some i. As Case 1et us assume, first, that $\succeq$ positions the multiset $T =$

$\{2,2,\ldots,2\}$ ($k+1$ elements) against the elements of the sequence (12) as follows: $S_j \prec T \prec S_{j+1}$.

Case 1a. $w_2 \in (\hat{f}_i, \hat{f}_{i+1})$. We know that $S_j \prec T \prec S_{j+1}$ for all additive orders with $w_2 \in \left(\widehat{\frac{j}{k+1}}, \widehat{\frac{j+1}{k+1}}\right)$. Hence we need to prove that

$$(f_i, f_{i+1}) \cap \left(\frac{j}{k+1}, \frac{j+1}{k+1}\right) \neq \emptyset, \tag{14}$$

for then w_2 can be adjusted, if needed, to obtain an additive order which coincides with $\succeq$. Suppose to the contrary that (14) does not hold and that the intersection there is empty. Without loss of generality, we assume that $f_{i+1} < \frac{j}{k+1}$. (The case of $f_i > \frac{j+1}{k+1}$ can be handled similarly.) Then $w_2 < \widehat{\frac{j}{k+1}}$, as $w_2 < \hat{f}_{i+1}$.

Suppose first that the fraction $\frac{j}{k+1}$ is not in its lowest terms, i.e. $1 < d = \gcd(j, k+1)$. Let $\ell = \frac{j}{d}$ and $h = \frac{k+1}{d}$. Then $w_2 < \widehat{\frac{\ell}{h}} = \widehat{\frac{j}{k+1}}$ and hence

$$\{\underbrace{2,2,\ldots,2}_{h}\} \prec_h \{\underbrace{1,1,\ldots,1}_{\ell}, \underbrace{3,\ldots,3}_{h-\ell}\}.$$

Since $hd = k+1$ and $\ell d = j$, this immediately implies, due to weak consistency of $\succeq$ and Lemma 1(b), that

$$T = \{\underbrace{2,2,\ldots,2}_{k+1}\} \prec S_j = \{\underbrace{1,1,\ldots,1}_{j}, \underbrace{3,\ldots,3}_{k+1-j}\},$$

which is a contradiction.

So assume that the fraction $\frac{j}{k+1}$ is in its lowest terms, i.e. $1 = \gcd(j, k+1)$. Then we consider the two neighboring Farey fractions $\frac{s}{t}$ and $\frac{\ell}{h}$ of $\frac{j}{k+1}$ in $\mathbf{F}_{k+1}$. We assume that

$$\frac{s}{t} < \frac{j}{k+1} < \frac{\ell}{h}. \tag{15}$$

Then one of the main theorems about Farey fractions states that in this case

$$s + \ell = j \quad \text{and} \quad t + h = k+1 \tag{16}$$

(see [17], Theorem 29 or [19] Theorem 9). In particular, $\ell < k+1$ and $h < k+1$.

As we assumed the contrary to (14) we have $w_2 < \hat{f}_{i+1} \le \widehat{\frac{s}{t}} < \widehat{\frac{\ell}{h}}$. Hence it follows that

$$\{\underbrace{2,2,\ldots,2}_{h}\} \prec_h \{\underbrace{1,1,\ldots,1}_{\ell},\underbrace{3,\ldots,3}_{h-\ell}\}, \tag{17}$$

$$\{\underbrace{2,2,\ldots,2}_{t}\} \prec_t \{\underbrace{1,1,\ldots,1}_{s},\underbrace{3,\ldots,3}_{t-s}\}. \tag{18}$$

Taking the union of the multisets on the left and the union of the multisets on the right, by Lemma 1(a) and (16) we then get $T \prec S_j$, which is a contradiction. So (14) holds.

Case 1b. $w_2 = \hat{f}_i$ for some i. We need to show that in this case

$$f_i \in \left(\frac{j}{k+1}, \frac{j+1}{k+1}\right). \tag{19}$$

The argument is very similar to the one above. Assuming $f_i < \frac{j}{k+1}$ we again take the two neighboring Farey fractions $\frac{s}{t}$ and $\frac{\ell}{h}$ of $\frac{j}{k+1}$ in $\mathbf{F}_{k+1}$ satisfying (15). As $f_i \le \frac{s}{t} < \frac{\ell}{h}$, we also get (16), (17) and (18) from which we deduce $T \prec S_j$ and again obtain a contradiction with $S_j \prec T$. This proves (19).

As Case 2, we take $T \sim S_j$ for some j. We again have two cases.

Case 2a. $w_2 \in (\hat{f}_i, \hat{f}_{i+1})$. We know that $S_j \sim T$ for the only additive order with $w_2 = \widehat{\frac{j}{k+1}}$. Hence we need to prove that

$$\frac{j}{k+1} \in (f_i, f_{i+1}), \tag{20}$$

for then w_2 can be adjusted, if needed, to obtain an additive order which coincides with $\succeq$. Suppose to the contrary that (20) does not hold. Without loss of generality, we assume that $f_{i+1} < \frac{j}{k+1}$. (The case of $f_i > \frac{j}{k+1}$ can be handled similarly.) Then $w_2 < \widehat{\frac{j}{k+1}}$, as $w_2 < \hat{f}_{i+1}$. This can be shown to contradict $T \sim S_j$ as in Case 1a.

Case 2b. $w_2 = \hat{f}_i$ for some i. We need to show that in this case $f_i = \frac{j}{k+1}$.

Assuming the contrary we may also assume (without loss of generality) that $f_i < \frac{j}{k+1}$. Then a contradiction can be obtained as in Case 1b.

So much proves that $\succeq$ is additive. Since there are no orders in $\mathcal{P}_k[3]$ which are not additive, the total number of them is the number of intervals between the Farey fractions of $\mathbf{F}_k$, i.e. $\Phi(k)$ plus the number of the Farey fractions in $\mathbf{F}_k$ minus one. In total we have $2\Phi(k) - 1$ orders, $\Phi(k)$ of them linear. $\square$

Note that the proof of Theorem 1 is algorithmic and, given a consistent order on $\mathcal{P}_k[3]$, it should be easy to write a computer program to construct weights for this order.

In section 4 we will show that, for any $n \geq 4$, there exist non-additive linear orders on $\mathcal{P}_2[n]$.

The following example shows that in Theorem 1 one cannot replace weakly consistent order with an order which is preserved under the multiset union and simultaneously weak additivity with additivity.

Example 2. *It can be checked that the linear order on $\mathcal{P}_{\leq 2}[3]$ with*

$$\{1,1\} \succ \{1,2\} \succ \{2,2\} \succ \{1,3\} \succ \{1\} \succ \{2,3\} \succ \{3,3\} \succ \{2\} \succ \{3\} \succ \emptyset$$

is preserved under the multiset union but is not additive and cannot be extended to an order of $\mathcal{P}_{\leq 3}[3]$ which is preserved under the multiset union.

3. Non-extendability of non-additive linear orders

Definition 6. *Given positive integers k, ℓ, n with $k < \ell$, we will say that a consistent order $\succeq_k$ on $\mathcal{P}_k[n]$ can be extended to a consistent order $\succeq_\ell$ on $\mathcal{P}_\ell[n]$ iff there is a weakly consistent order $\succeq$ on $\mathcal{P}_{\leq \ell}[n]$ such that its restriction on $\mathcal{P}_k[n]$ coincides with $\succeq_k$ and its restriction on $\mathcal{P}_\ell[n]$ coincides with $\succeq_\ell$.*

Theorem 2. *For any two positive integers $n \geq 4$ and k, there exists an integer $f(n,k) > k$ such that a linear order $\succeq$ on $\mathcal{P}_k[n]$ can be extended to a consistent order of $\mathcal{P}_{f(n,k)}[n]$ if and only if it is additive.*

Proof. It is clear that any additive order $\succeq$ on $\mathcal{P}_k[n]$ can be extended to an additive order of $\mathcal{P}_m[n]$ for all $m \geq k$. It is enough to take an order with the same weights.

Suppose now that $\succeq$ is not additive. Under this assumption, we will prove that $\succeq$ cannot be extended to a consistent order of $\mathcal{P}_{f(n,k)}[n]$ for $f(n,k) = k^{n+1}n^{n/2+1}$. This number is huge but sufficient for our purposes. We have not made any attempts to find a smaller one.

Let $N = \text{card}\,(\mathcal{P}_k[n])$. Let us enumerate all multisets X_i, $(i = 1, \ldots, N)$ of $\mathcal{P}_k[n]$ so that

$$X_1 \succ X_2 \succ \ldots \succ X_N. \tag{21}$$

Let μ_i be the multiplicity function of X_i. To each multiset (X_i, μ_i) we assign a linear form

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} \mu_i(j) x_j \qquad (i = 1, \ldots, N). \tag{22}$$

Then the following system of $N-1$ linear inequalities

$$\begin{aligned} f_1(\mathbf{x}) - f_2(\mathbf{x}) &> 0 \\ f_2(\mathbf{x}) - f_3(\mathbf{x}) &> 0 \\ &\ldots \\ f_{N-1}(\mathbf{x}) - f_N(\mathbf{x}) &> 0 \end{aligned}$$

cannot be consistent (otherwise we would be able to find weights for $\succeq$). The ith inequality $f_i(\mathbf{x}) - f_{i+1}(\mathbf{x}) > 0$ defines a half-space H_i in $\mathbb{R}^n$ determined by the corresponding hyperplane $f_i(\mathbf{x}) - f_{i+1}(\mathbf{x}) = 0$. For each $i \in \{1, \ldots, N-1\}$ we define the vector

$$\mathbf{v}_i = (\mu_i(1) - \mu_{i+1}(1), \ldots, \mu_i(n) - \mu_{i+1}(n))^t.$$

It is an inner normal vector of H_i, i.e. $\mathbf{x} \in H_i$ iff $(\mathbf{v}_i, \mathbf{x}) > 0$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$ have coefficients ranging from k to $-k$, and the sum of all coefficients is zero for each of them.

A standard linear-algebraic argument tells us that inconsistency of the system above is equivalent to the existence of a nontrivial linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$ with nonnegative coefficients which is equal to zero. Since the sum of all coefficients of every $\mathbf{v}_i$ is zero, they all lie in a subspace of lower dimension than n.

Another elementary linear-algebraic argument shows that there exist $m \leq n-1$ vectors $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ among $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$, which are linearly dependent with positive coefficients and such that no subset of $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}\}$ is linearly dependent. This means, in particular, that the linear dependency between $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ is unique up to a scalar multiple, that is if $a_1\mathbf{v}_{i_1} + \ldots + a_m\mathbf{v}_{i_m} = 0$ and $b_1\mathbf{v}_{i_1} + \ldots + b_m\mathbf{v}_{i_m} = 0$ are two nontrivial linear combination that vanish, then there exists $c \neq 0$ such that $a_i = cb_i$ for all $i = 1, 2, \ldots, m$. In particular any linear combination $b_1\mathbf{v}_{i_1} + \ldots + b_m\mathbf{v}_{i_m} = 0$ have all its coefficients of the same sign, either all positive or all negative.

Without loss of generality we may consider that $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ are the first m vectors of the system $\{\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}\}$. Let us consider the matrix $V = (\mathbf{v}_1 \mid \ldots \mid \mathbf{v}_m)$, whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Suppose, without loss of generality, that the first $m-1$ rows of V are linearly independent. Let A be a square $m \times m$ matrix whose m rows are the upper m rows of V. Let $A = (\mathbf{w}_1 \mid \dots \mid \mathbf{w}_m)$, where $\mathbf{w}_1, \dots, \mathbf{w}_m$ are the columns of A. It is clear that $b_1\mathbf{v}_1 + \dots + b_m\mathbf{v}_m = 0$ if and only if $b_1\mathbf{w}_1 + \dots + b_m\mathbf{w}_m = 0$. Since $\det A = 0$, we get $A_{11}\mathbf{w}_1 + A_{12}\mathbf{w}_2 + \dots + A_{1m}\mathbf{w}_m = 0$, where A_{ij} is the (i,j)-cofactor of matrix A. Since the entries of every cofactor are integers between $-k$ and k, the maximal value of such a determinant can be no greater than $k^m m^{m/2} \le k^n n^{n/2}$. (This immediately follows from an important theorem of Hadamard [20] which states that if A is any real $n \times n$ matrix with $-1 \le a_{ij} \le 1$, then $|\det A| \le n^{n/2}$.) As was mentioned above, this implies $A_{11}\mathbf{v}_1 + A_{12}\mathbf{v}_2 + \dots + A_{1m}\mathbf{v}_m = 0$, and all coefficients can be made positive due to the comment made earlier. So we assume that we have a linear combination

$$a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m = 0 \tag{23}$$

with nonnegative integer coefficients a_i such that $0 \le a_i \le g(n,k)$, where $g(n,k) = k^n n^{n/2}$.

Now we recollect that each $\mathbf{v}_k$ is the inner normal of the half-space H_k defined by the inequality

$$\sum_{j=1}^{n} \mu_{i_k}(j)x_j - \sum_{j=1}^{n} \mu_{i_k+1}(j)x_j > 0, \tag{24}$$

where μ_{i_k} and μ_{i_k+1} are the multiplicity functions of X_{i_k} and X_{i_k+1}, respectively. We denote $M_k = X_{i_k}$ and $N_k = X_{i_k+1}$ and note that $M_k \succ N_k$. Let us consider now the two multisets:

$$M = \bigcup_{i=1}^{m} \underbrace{M_i \cup \ldots \cup M_i}_{a_i}, \qquad N = \bigcup_{i=1}^{m} \underbrace{N_i \cup \ldots \cup N_i}_{a_i} \tag{25}$$

where a_i, $i = 1, \dots, m$, are the coefficients of (23). The common cardinality of M and N is no greater than $(\sum_{i=1}^{m} a_i)k \le kng(n,k) = f(n,k)$. Hence both M and N are from $\mathcal{P}_{\le f(n,k)}[n]$. If we assume that $\succeq$ can be extended to a weakly consistent order on $\mathcal{P}_{\le f(n,k)}[n]$, then Lemma 1 will imply that $M \succ N$. On the other hand, (23) implies that $M = N$. as these two sets consist of the same elements taken with the same multiplicities. This contradiction proves the theorem. □

Corollary 1. *For all n, any weakly consistent linear order of $\mathcal{P}[n]$ is weakly additive.*

Proof. If it is not weakly additive, then it induces a non-additive linear order on a certain $\mathcal{P}_k[n]$ for some k. But then it cannot be extended to a weakly consistent order of $\mathcal{P}_{\leq f(n,k)}[n]$, which is a contradiction. $\square$

4. Linear orders on multisets of cardinality $k = 2$

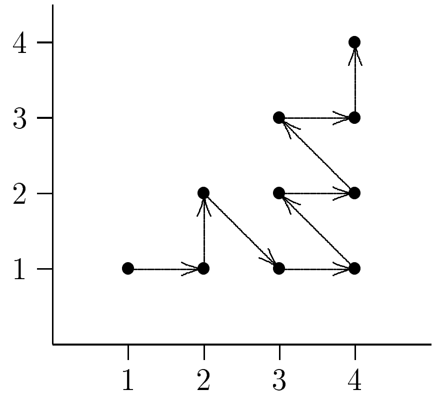
In order to describe consistent linear orders of multisets of cardinality two (pairs), even in the case of $\mathcal{P}_2[4]$, we need the concept of reducibility. In other words, we need to know which linear orders can be obtained by combining two or more simpler ones.

Definition 7. *Let R_1 and R_2 be two linear orders on $\mathcal{P}_2[n_1]$ and $\mathcal{P}_2[n_2]$, respectively. We define the linear order $R = R_1 \times R_2 \in \mathcal{P}_2[n_1 + n_2]$ as follows:*

1. $\{i,j\}R\{p,q\} = \{i,j\}R_1\{p,q\}$ *for all* $i,j,p,q \in [n_1]$*;*
2. $\{i,j\}R\{p,q\} = \{i - n_1, j - n_1\}R_2\{p - n_1, q - n_1\}$ *for all* $i,j,p,q \in [n_1 + n_2] \setminus [n_1]$*;*
3. $\{i,j\}R\{p,q\}$ *is always true when* $i,j \in [n_1]$ *and* $p \notin [n_1]$ *or* $q \notin [n_1]$*;*
4. *If* $i \neq j \in [n_1]$ *and* $p,q \in [n_1 + n_2] \setminus [n_1]$*, then* $\{i,p\}R\{j,q\}$ *if and only if* $i \succ j$*.*

The linear order $R_1 \times R_2$ will be called the product of R_1 and R_2.

Example 3. *Let us denote the only linear order on $\mathcal{P}_1[1]$ as I_1. Then the only consistent linear order of $\mathcal{P}_2[2]$ will be $I_1 \times I_1$. The two linear orders in Example 1 will be $I_1 \times (I_1 \times I_1)$ and $(I_1 \times I_1) \times I_1$, respectively. The linear order $(I_1 \times I_1) \times (I_1 \times I_1)$ of $\mathcal{P}_2[4]$ will have the diagram shown below:*



Example of a reducible order

Definition 8. *We will call a linear order R on $\mathcal{P}_2[n]$ irreducible if it cannot be represented as $R = L \times M$ for $L \in \mathcal{P}_2[n_1]$ and $M \in \mathcal{P}_2[n_2]$ for any positive integers n_1 and n_2 with $n_1 + n_2 = n$. Otherwise, it will be called reducible.*

Theorem 3. *If two linear orders $L \in \mathcal{P}_{n_1}$ and $M \in \mathcal{P}_{n_2}$ are additive, then their product $L \times M$ is also an additive linear order.*

Proof. First we notice that, for all k and n, an additive linear order of $\mathcal{P}_k[n]$ with the system of weights $w_1, \ldots, w_n$ will stay unchanged if for some two positive integers a and b, for all $i = 1, \ldots, n$, we undertake an affine transformation $w_i' = aw_i + b$ of its weights. Let $u_1, \ldots, u_{n_1}$ be a system of weights for L and $v_1, \ldots, v_{n_2}$ be a system of weights for M. Let V be the sum of all weights of the second system and a, b be two sufficiently large integers such that the new system of weights $u_1', \ldots, u_{n_1}'$, where $u_j' = au_j + b$, satisfies the following two conditions: $2u_{n_1}' > u_1' + V$ and $u_{k+1}' - u_k' > V$ for all $1 \leq k < n_1 - 1$. Then the system of weights $u_1', \ldots, u_{n_1}', v_1, \ldots, v_{n_2}$ will define $L \times M$. This proves the theorem. $\square$

As we saw, all linear orders of $\mathcal{P}_2[n]$ for $1 < n \leq 3$ are reducible. In $\mathcal{P}_2[4]$ we will have seven irreducible ones. They are all described in the following theorem.

Theorem 4. *There are 12 different linear orders in $\mathcal{P}_2[4]$:*

1. *The five reducible linear orders are*

$$\begin{aligned} R_{1,4} &= I_1 \times (I_1 \times (I_1 \times I_1)), \\ R_{2,4} &= I_1 \times ((I_1 \times I_1) \times I_1), \\ R_{3,4} &= (I_1 \times I_1) \times (I_1 \times I_1), \\ R_{4,4} &= (I_1 \times (I_1 \times I_1)) \times I_1, \\ R_{5,4} &= ((I_1 \times I_1) \times I_1) \times I_1, \end{aligned}$$

   all of which are additive;

2. *The seven irreducible linear orders are A_4, B_4, C_4, D_4, E_4, F_4, G_4, given by their diagrams in Figure 1. Five of them apart from A_4 and E_4 are additive. A_4 and E_4 are not additive.*

Proof. There are only five different arrangements of brackets that convert an associative word $x_1x_2x_3x_4$ of length four into a nonassociative word. These nonassociative words are $x_1(x_2(x_3x_4))$, $x_1((x_2x_3)x_4)$, $(x_1x_2)(x_3x_4)$, $(x_1(x_2x_3))x_4$, $((x_1x_2)x_3)x_4$. They are distinct elements of the free nonassociative monoid (see, for example, [21], Chapter 1). Accordingly, we can construct five reducible linear orders listed in the Theorem. They are all additive due to Theorem 2. It is easy to check directly that they are different (but later we will prove a general statement in this respect). There are no other reducible linear orders in $\mathcal{P}_2[4]$ since, if R is reducible, then $R = P \times Q$, where I_1, $I_1 \times I_1$, $I_1 \times (I_1 \times I_1)$, $(I_1 \times I_1) \times I_1$ are the only possibilities for P and Q.

The weights for B_4, C_4, D_4, F_4, G_4 can be chosen according to Figure 2. Since any affine transformation of the system of weights does not change the order, we normalize these weights so that $w_4 = 0$ and $w_1 = 1$. The boundary between $\{1,4\} \succ \{2,3\}$ and $\{2,3\} \succ \{1,4\}$ will be the line $w_2 + w_3 = 1$. The boundary between $\{2,2\} \succ \{1,3\}$ and $\{1,3\} \succ \{2,2\}$ will be the line $2w_2 - w_3 = 1$, etc. After drawing all such lines we get ten regions which correspond to all ten additive orders.

The order A_4 cannot be additive for the following reasons: we have $\{1,3\}A_4\{2,2\}$, $\{2,3\}A_4\{1,4\}$, and $\{2,4\}A_4\{3,3\}$. If a system of weights for A_4 existed, then we would have

$$w_1 + w_3 > 2w_2, \;\; w_2 + w_3 > w_1 + w_4, \;\; w_2 + w_4 > 2w_3.$$

This system is inconsistent since adding the first and the third inequality gives us $w_2 + w_3 < w_1 + w_4$, i.e. just the opposite to the second inequality.

Similarly, E_4 cannot be additive either. (Further, we are about to prove a stronger statement that A_4 and E_4 cannot be extended to a consistent order of $\mathcal{P}_3[4]$. $\square$

Theorem 5 (Irene Peng [22]). *Let $\succeq$ be an order of $\mathcal{P}_2[n]$ and suppose that there exist indices i, j, k, ℓ satisfying at least one of the following two conditions:*

(1) $\{i,i\} \succ \{j,k\}$, $\{j,\ell\} \succ \{i,k\}$ and $\{k,k\} \succ \{i,\ell\}$;

(2) $\{j,k\} \succ \{i,i\}$, $\{i,k\} \succ \{j,\ell\}$ and $\{i,\ell\} \succ \{k,k\}$;

Then the corresponding weakly consistent order $\succeq$ of $\mathcal{P}_{\leq 2}[n]$ cannot be extended to a weakly consistent order of $\mathcal{P}_{\leq 3}[n]$.

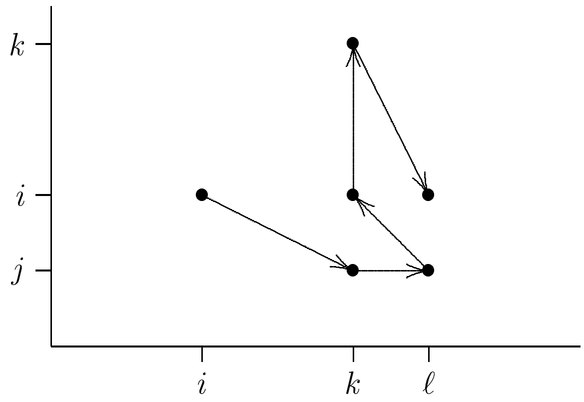


Illustration of the Condition (1)

Proof. We will show that the requirement of weak consistency for any extension leads to its intransitivity. Indeed, in the first case

$$\begin{aligned}
\{i,i\} \succ \{j,k\} &\implies \{i,i,\ell\} \succ \{j,k,\ell\} \\
\{j,\ell\} \succ \{i,k\} &\implies \{j,k,\ell\} \succ \{i,k,k\} \\
\{k,k\} \succ \{i,\ell\} &\implies \{i,k,k\} \succ \{i,i,\ell\},
\end{aligned}$$

so no extension can be transitive. The second case is similar. $\square$

Corollary 2. *A_4 and E_4 cannot be restrictions on $\mathcal{P}_2[4]$ of any weakly consistent order of $\mathcal{P}_{\leq 3}[4]$.*

Proof. Indeed, in both cases we can spot two arrows going in one direction and an arrow between them going in the opposite direction. Thus E_4 falls into the first condition with $i = 2$, $j = 1$, $k = 3$, while A_4 falls into the second for the same set of parameters. $\square$

We also need to remind the reader of the Catalan numbers (see, for example, [23], Ch. 20). We need them because, among other things, the nth Catalan number

$$c(n) = \frac{1}{n+1}\binom{2n}{n}.$$

describes the number of ways in which brackets (parentheses) can be placed in an associative word $x_1x_2\dots x_n$ of length n to determine the order in which the indeterminates must be multiplied. In other words $c(n)$ characterizes the number of different nonassociative words that can be defined on the associative word $x_1x_2\dots x_n$. Let us denote the set of all such nonassociative words by $\mathcal{W}$.

Let $w = (x_1x_2\dots x_n)_q$ be a nonassociative word belonging to $\mathcal{W}$ with the arrangement of brackets q. Then we can construct a linear order on $\mathcal{P}_2[n]$

$$w(I_1) = (I_1I_1\dots I_1)_q$$

where the operation is the product of linear orders given in Definition 7.

Lemma 2. *Let $w_1 = (x_1x_2\dots x_n)_{q_1}$ and $w_2 = (x_1x_2\dots x_n)_{q_2}$ be two nonassociative words belonging to $\mathcal{W}$ with the arrangements of brackets q_1 and q_2, respectively. Let $w_1(I_1)$ and $w_2(I_1)$ be the corresponding linear orders. Then $w_1(I_1) = w_2(I_1)$ if and only if $q_1 = q_2$.*

Proof. We will prove this statement by induction. As we saw in Example 1 for $n = 3$ we have only two different arrangements of brackets and they correspond to different linear orders. This gives us a basis for our induction.

As is known (see, for example, [21]) any nonassociative word has a unique representation as a product of two nonassociative words of smaller length. Suppose now that

$$\begin{aligned} w_1 &= (x_1\dots x_k)_{r_1}(x_{k+1}\dots x_n)_{r_2} \\ w_2 &= (x_1\dots x_m)_{s_1}(x_{m+1}\dots x_n)_{s_2} \end{aligned}$$

where r_1, r_2, s_1, s_2 are certain arrangements of brackets. We denote $\succ_1 = w_1(I_1)$ and $\succ_2 = w_2(I_1)$. If $k = m$, then either $r_1 \neq s_1$ or $r_2 \neq s_2$ and we may apply the induction hypothesis. Suppose now that $m > k$ (we can do this without loss of generality). Then x_{k+1} can be found in the first bracket. This will lead to $\{1, n\} \succ_1 \{k+1, k+1\}$ but $\{k+1, k+1\} \succ_2 \{1, n\}$. This shows that $\succ_1$ and $\succ_2$ are different. The theorem is proved. $\square$

Theorem 6. *There exist at least $c(n)$ additive reducible linear orders on $\mathcal{P}_2[n]$. In total, there are at least 2^{2n-5} consistent linear orders on $\mathcal{P}_2[n]$.*

Proof. By Lemma 2 we can produce as many additive reducible linear orders on $\mathcal{P}_2[n]$ as claimed using the trivial order I_1 and the product operation defined above. Indeed, we have one such linear order for any nonassociative arrangement of brackets on a word $x_1x_2\ldots x_n$ of length n.

Let us see now what we can do if we drop both additivity and reducibility. Let $i \in [n]$. By a "diagonal" let us agree to mean any set of pairs satisfying one of the two following properties:

1. $\{i, i\}, \{i-1, i+1\}, \ldots, \{1, 2i-1\}$, in case $2i - 1 \le n$,
2. $\{i, i\}, \{i-1, i+1\}, \ldots, \{2i-n, n\}$, in case $2i - 1 > n$.

For each diagonal we independently choose a direction of arrows and follow it through the whole diagonal. For example, we may choose

$$\{i, i\} \succ \{i-1, i+1\} \succ \ldots \succ \{1, 2i-1\}$$

or

$$\{i, i\} \prec \{i-1, i+1\} \prec \ldots \prec \{1, 2i-1\}.$$

By choice of directions on all diagonals a linear order is defined uniquely, and clearly, this will be a consistent order. Since we have $2n-5$ such diagonals and their directions are chosen independently, we can construct at least 2^{2n-5} consistent linear orders. The theorem is proved. $\square$

To compare the two bounds we note that asymptotically the nth Catalan number is

$$c(n) \sim \frac{1}{\sqrt{\pi}} \frac{2^{2n}}{n^{3/2}} = \frac{1}{\sqrt{\pi}} 2^{2n - \frac{3}{2}\log_2 n}.$$

This can be found in [24]. As we see the second bound is only slightly better than the first.

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Irreducible orders of $\mathcal{P}_2[4]$

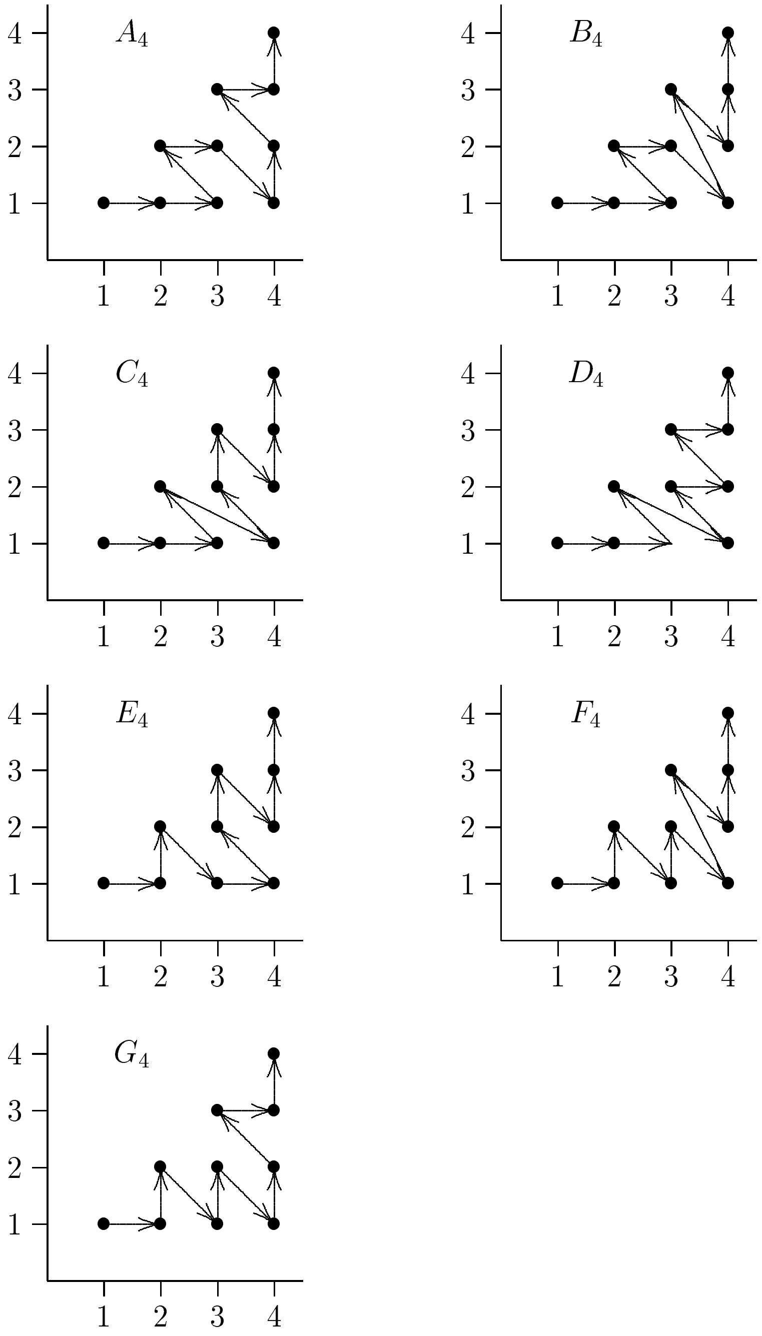

Figure 1

Classification of additive orders from $\mathcal{P}_2[4]$ according to their values of w_2 and w_3

($w_1 = 1$ and $w_4 = 0$)

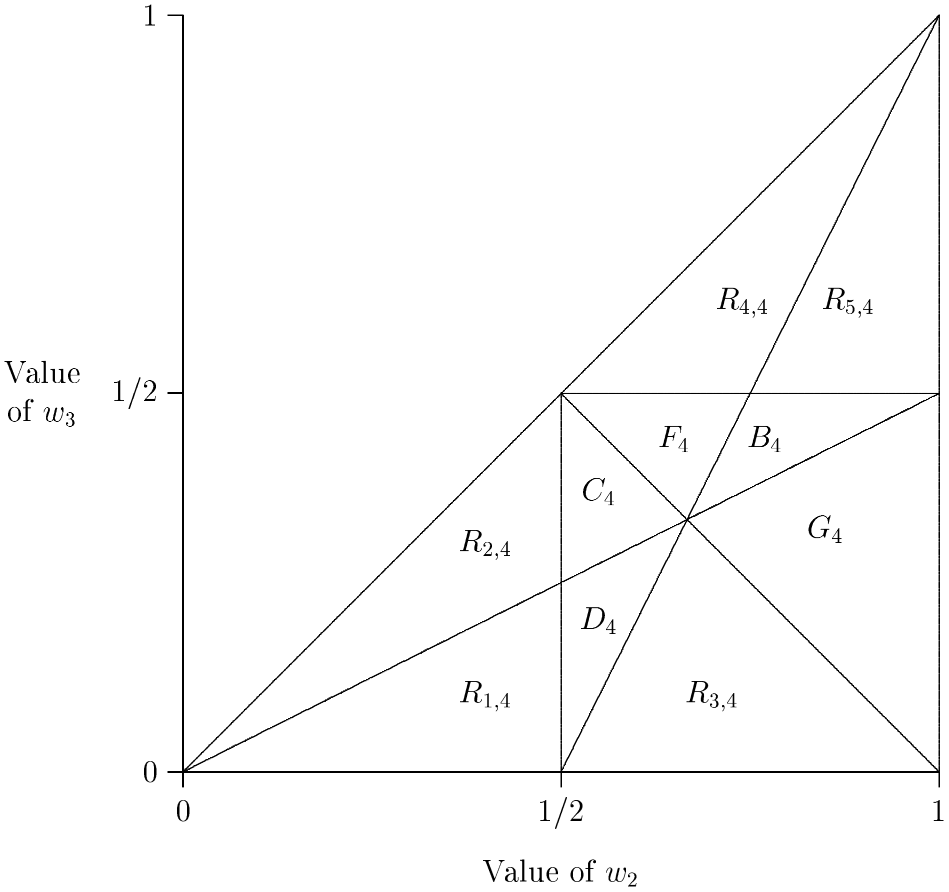

Figure 2

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