QUASI-RELATIVISM, NARROW-GAP PROPERTY AND FORCED DYNAMICS OF ELECTRONS IN SOLIDS: FEW SOLVABLE MODELS.

B.S.Pavlov[1,2], **A.A.Pokrovski**[1], **A.V.Strepetov**[3]

[1] *Laboratory of Complex Systems Theory, Institute for Physics, St.-Petersburg State University, Ulyanovskaya 1, St.Petersburg 198904, Russia*
[2] *Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand*
[3] *Department of the Applied Mathematics, State University for Airspace Instrument-making, St.-Petersburg, 190000, Russia*

1 Introduction

Based on Landauer formula [6] for quantum conductance one may reduce the description of the quantum current [11] in quasi – one-dimensional nano-wires to solution of relevant one-dimensional scattering problem and calculation of the corresponding transmission coefficient $\overleftarrow{\mathcal{T}}(E)$ which connect the amplitudes in front of the asymptotically plain waves constituting the scattered waves $\psi(x,E)$ at $\pm\infty$

$$\psi(x,E) \approx \begin{cases} e^{-i\sqrt{E}x} + \overrightarrow{\mathcal{R}} e^{i\sqrt{E}x} & , x \to +\infty, \\ \overleftarrow{\mathcal{T}} e^{-i\sqrt{E}x} & , x \to -\infty. \end{cases}$$

The averaging of the conductance over the Fermi distribution, [1]

$$F(E, E_F) = \frac{CV\sqrt{E}}{\exp(\frac{E-E_F}{kT}) + 1}$$

gives another version of the Landauer formula - the current-voltage dependence $j(U)$ - for higher temperature and nonzero voltage applied to the wire :

$$j(U) = \frac{e^2}{h} \int \frac{|\overleftarrow{\mathcal{T}}(E)|^2}{1 - |\overleftarrow{\mathcal{T}}(E)|^2} [F(E, E_F + eU) - F(E, E_F - eU)] dE \quad , \tag{1}$$

here E_F is the Fermi level, U - applied voltage. Assuming that electrons in reservoirs on both ends of the quantum wire are almost free, we may express the density of states in (1) in terms of momentum as $dE = 2kdk$. For sufficiently small values of the transmission coefficient $\mathcal{T}(E)$ the formula (1) may be reduced to the formula:

$$j(U) = \frac{e^2}{h} \int |\mathcal{T}(E)|^2 [F(E, E_F + eU) - F(E, E_F - eU)] dE \quad , \tag{2}$$

We discuss below *ballistic* electron's transmission problem in the nanowire neglecting at this stage the electron-electron and electron-phonon interaction and considering only the one-body problem. But we pay a special attention to the calculation of the effective mass [7] of electron in the lattice. In fact the magnitude of the effective mass defines the mobility of electrons : the free path and the De-Broghlie wavelength. The assumption about the small effective mass serves a base of application of methods of the scattering theory to the description of ballistic electrons in nano-electronic networks, see [8], [4], [3], [5]. We analyze here two possible mechanisms of appearing of a small effective mass in super-lattices: the mechanism based on a strong interaction of blocks in the superlattice and the mechanism based on a weak interaction between sub-lattices. Respectively two examples are considered which show that electrons in strongly connected and weakly-connected superlattices may exhibit a quasi-relativistic behaviour and may be described by an analog of a one-dimensional Dirac equation, which *is derived from the first principles* under assumption of low temperature and weak external electric field.

Basing on the derived Dirac equation we consider the time-periodic perturbations in form of running wave and a localized singular potential. In both cases the transmission coefficients exhibit spectral band structure. This phenomenon may be used for manipulation of quantum current in quantum networks constructed of the narrow-gap superlattices.

2 A superlattice with a small effective mass.

In this section we construct a solvable model of a quantum waveguide with a small effective mass of charge-carriers caused by splitting between energy levels of electrons in the basic block of the corresponding lattice and strong interaction between blocks.

Consider an hermitian 2×2 matrix $Q : E \to E$ with real eigenvalues q_1, q_2. Introducing the mean value $q_0 := \dfrac{q_1 + q_2}{2}$ and splitting $\delta := \dfrac{q_1 - q_2}{2} > 0$, we may present Q as

$$q_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\delta}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

We use Q as a block for construction of a periodic lattice operator $\mathcal{Q}$ in the Hilbert space $\mathcal{E} := l_2 \times E = l_2(E)$ of infinite sequences $\vec{\mathcal{F}} = \left\{\vec{f}_l\right\}_{l=-\infty}^{\infty}$, $\quad \vec{f}_l \in E$.

Denote in this section by $\mathcal{T}$ the shift operator in $\mathcal{E}$

$$\mathcal{T}\left\{\vec{f}_l\right\}_{l=-\infty}^{\infty} = \left\{\vec{f}'_l\right\}_{l=-\infty}^{\infty}, \quad \vec{f}'_l = \vec{f}_{l-1}$$

and by $\mathcal{I}$ the unity in l_2. Introduce the operators P_1, P_2 projecting respectively onto the first and second components of each vector $\vec{f}_l$:

$$P_1 \vec{\mathcal{F}} = \left\{ \begin{array}{c} f_{l,1} \\ 0 \end{array} \right\}_{l=-\infty}^{\infty},$$

$$P_2 \vec{\mathcal{F}} = \left\{ \begin{array}{c} 0 \\ f_{l,2} \end{array} \right\}_{l=-\infty}^{\infty},$$

and by π_{12} the operators swoping the components of each vector $\vec{f}_l$,for instance

$$\pi_{12} \left\{ \begin{array}{c} f_{l,1} \\ f_{l,2} \end{array} \right\}_{l=-\infty}^{\infty} = \left\{ \begin{array}{c} f_{l,2} \\ f_{l,1} \end{array} \right\}_{l=-\infty}^{\infty}.$$

The required one-body lattice Hamiltonian is defined as

$$\mathcal{Q} = Q \times \mathcal{I}_{L_2} - i\alpha \pi_{12} P_2 \mathcal{T} + i\alpha \pi_{12} P_1 \mathcal{T}^+. \tag{4}$$

It is presented by an infinite periodic Jacobian matrix with 2×2 blocks Q on the main diagonal and the interaction between blocks introduced by the shifts:

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & -i\alpha & 0 & \dots & \dots \\ i\alpha & q_0 + \frac{\delta}{\sqrt{2}} & \frac{\delta}{\sqrt{2}} & \dots & \dots \\ 0 & \frac{\delta}{\sqrt{2}} & q_0 - \frac{\delta}{\sqrt{2}} & -i\alpha & \dots \\ \dots & \dots & i\alpha & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Fourier transform

$$\mathbf{F}\vec{\mathcal{F}} \longrightarrow \sum_{l=-\infty}^{\infty} e^{il\varphi} \vec{f}_l := F(\varphi),$$

$$\mathbf{F}\mathcal{T}\left\{\vec{f}_l\right\}_{l=-\infty}^{\infty} = e^{i\varphi}F(\varphi)$$

defines the similarity of the operator $\mathcal{Q}$ to the multiplication in $L_2((-\pi,\pi),E)$ by the matrix- function

$$Q(\varphi) = \begin{pmatrix} q_0 + \frac{\delta}{\sqrt{2}} & \frac{\delta}{\sqrt{2}} + i\alpha e^{-i\varphi} \\ \frac{\delta}{\sqrt{2}} - i\alpha e^{i\varphi} & q_0 - \frac{\delta}{\sqrt{2}} \end{pmatrix}. \tag{5}$$

The spectrum σ_Q of the operator $\mathcal{Q}$ is defined by dispersion equation

$$\det\left(Q(\varphi) - \lambda E\right) = (\lambda - q_0)^2 - \delta^2 - \alpha^2 + \alpha\delta\sqrt{2} - \alpha\delta\sqrt{2}(1 - \sin\varphi) = \tag{6}$$

$$(\lambda - q_0)^2 - \delta^2 - \alpha^2 + \alpha\delta\sqrt{2} - \alpha\delta\sqrt{2}\left(\sin\varphi/2 - \cos\varphi/2\right)^2 = 0, \ -\pi < \varphi < \pi,$$

and consists of two intervals - upper and lower *spectral bands* with the spectral gap between them :

$$\left(q_0 - [\delta^2 + \alpha^2 - \alpha\delta\sqrt{2}]^{1/2},\ q_0 + [\delta^2 + \alpha^2 - \alpha\delta\sqrt{2}]^{1/2}\right)$$

The effective mass

$$m_\delta = \left[\frac{d^2\lambda}{d\varphi^2}\right]^{-1}$$

is calculated from the dispersion function $\lambda(\varphi)$ defined by the dispersion equation (6). At the bottom of the upper spectral band $\varphi = \pi/2, \lambda = q_0 + [\delta^2 + \alpha^2 - \alpha\delta\sqrt{2}]^{1/2}$ it is positive and equal to

$$\frac{2[\delta^2 + \alpha^2 - \alpha\delta\sqrt{2}]^{1/2}}{\alpha\delta\sqrt{2}}.$$

It is as small as an inverse of the minimal of two (large) parameters α, δ is.

Note that the multiplication operator (5) has a uniformly convergent asymptotic expansion for small deviations of quasimomentum φ from the lowest point of the upper spectral band $\varphi = \pi/2$. This expansion is defined by the Taylor series of exponentials after the change of variables $\alpha(\varphi - \pi/2) \to p$

$$Q(\varphi) = \begin{pmatrix} q_0 + \frac{\delta}{\sqrt{2}} & \frac{\delta}{\sqrt{2}} + i\alpha \\ \frac{\delta}{\sqrt{2}} - i\alpha & q_0 - \frac{\delta}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 & -ip \\ ip & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha\sum_{l=2}^{\infty}\frac{(-ip)^l}{l!\alpha^l} \\ \alpha\sum_{l=2}^{\infty}\frac{(ip)^l}{l!\alpha^l} & 0. \end{pmatrix} \tag{7}$$

After the corresponding transformation of the spectral variable $\lambda \to \mu := \lambda - q_0$ we obtain a new operator

$$\begin{aligned} \mathcal{D} = Q - q_0E = \ & \sigma_y p + \frac{\delta}{\sqrt{2}}\sigma_z + \left(\frac{\delta}{\sqrt{2}} + \alpha\right)\sigma_x + \\ & + \begin{pmatrix} 0 & \alpha\left(e^{-ip/\alpha} - 1 + ip/\alpha\right) \\ \alpha\left(e^{ip/\alpha} - 1 - ip/\alpha\right) & 0 \end{pmatrix} = \\ = \ & \mathcal{D}_0 + \begin{pmatrix} 0 & \alpha\left(e^{-ip/\alpha} - 1 + ip/\alpha\right) \\ \alpha\left(e^{ip/\alpha} - 1 - ip/\alpha\right) & 0 \end{pmatrix}. \end{aligned}$$

The main part of it $\sigma_y p$ has the spectrum σ^D centered at the origin :

$$\sigma_D = \{\mu : q_0 + \mu \in \sigma_Q\}.$$

One can see that the last term in the expression for the operator D may be estimated near the origin as p^2. In fact small values of the p play an important role in conductance processes for low temperature and weak potential U. Really, assuming that the potential U is small and the temperature T is low we may introduce a new norm in the space of the spectral representation of the operator D :

$$\langle u,\, v\rangle = \int u\overline{v}\,[F(E, E_F + U) - F(E, E_F - U)]\, dE.$$

The weight function in this norm which is combined of Fermi distributions is almost a step- function Π_Δ under the above assumptions. It is localized on a small interval $\Delta, |\Delta| \approx \kappa T + 2eU$ near the Fermi -level. In particular assuming that the Fermi-level coincides with the bottom of the upper spectral band

$$E_F = Q_0 + [\delta^2 + \alpha^2 - \alpha\delta\sqrt{2}]^{1/2},$$

and introducing the quasimomentum φ instead of the energy $\lambda = E$ we see that the restriction of the operator D onto the spectral subspace corresponding to the step-function Π_Δ differs from the $\mathcal{D}_0$ – Fourier image of the Dirac operator – only by the small addend

$$\begin{pmatrix} 0 & \alpha\left(e^{-ip/\alpha} - 1 + ip/\alpha\right) \\ \alpha\left(e^{ip/\alpha} - 1 - ip/\alpha\right) & 0 \end{pmatrix}$$

with the norm not greater than $|\alpha|^{-1}\Delta^2$. Summarizing the above observations we obtain the following important assertion

Theorem 1 . *The dynamics of electrons in the quasi-one-dimensional lattice combined of strongly interacting two-dimensional blocks with large splitting of energy levels is described for low temperatures and weak fields by the quasi-relativistic equation, if the Fermi – level sits at the bottom of the upper spectral band.*

This fact is in natural agreement with physical observation of quasi-relativistic behavior of electrons in some semiconductors. But the same effect for the effective mass may arise in totally different situation, see the next section.

2.1 Narrow-gap superlattice.

Consider a double superlattice combined of two parallel weakly interacting branches with equal periods. Each branch is constituted by equivalent quantum dots (atoms or quasi-molecular complexes) with vacant normalized orbitals $\psi_l = \psi(x-le)$, $l = 0, \pm 1, \pm 2\ldots$; $\varphi_m = \phi(x-me)$, $m = 0, \pm 1, \pm 2\ldots$. We use the following assumption about the overlapping integrals:

$$\langle \psi_l,\, \psi_{l'}\rangle = \langle \varphi_l,\, \varphi_{l'}\rangle = 0, \text{ if } |l - l'| \geq 2,$$

$$\langle \varphi_l,\, \psi_{l'}\rangle = 0, \text{ if } l \neq l',$$

$$\begin{aligned} \langle \varphi_l,\, \varphi_{l+j}\rangle &= a_j, \text{ if } j = 0, \pm 1,\ a_0 = 1,\ a_1 = a_{-1} > 0;\ \langle \psi_l,\, \psi_{l+j}\rangle = b_j, \\ \text{if } j &= 0, \pm 1,\ b_0 = 1,\ b_1 = b_{-1} < 0;\ \langle \psi_l,\, \varphi_l\rangle = \alpha = \bar{\alpha}. \end{aligned} \quad (8)$$

We assume, that the one-electron Hamiltonian of the superlattice is presented as a sum of the two weakly interacting chains of quantum dots:

$$\mathcal{H}u = A\sum_{l=-\infty}^{+\infty} \varphi_l\langle u,\, \varphi_l\rangle + A\sum_{l'=-\infty}^{+\infty} \psi_{l'}\langle u,\, \psi_{l'}\rangle. \quad (9)$$

The formal eigenvalue A which corresponds to the vacant orbitals is assumed to be equal to the Fermi level of the total superlattice. We may assume that $A = 1$ since it gives just an additional factor in the description of the spectrum. The interaction α between the branches of the superlattice is assumed to be weak, $|\alpha| << 1$ compared with the Fermi level and the overlapping integrals a_l, b_m.

We find the spectrum of the superlattice and calculate the corresponding Bloch waves and effective mass. We shall use the ansatz for the Bloch waves in form :

$$u = \sum_{l=-\infty}^{\infty} u_l^{\varphi} \varphi_l + u_l^{\psi} \psi_l.$$

Substitution of the above ansatz into the constructed Hamiltonian gives the following infinite system of linear equations :

$$\left\{ \begin{array}{lllll} a_1 u_{l-1}^{\varphi} + & a_1 u_{l+1}^{\varphi} + & a_0 u_l^{\varphi} + & \alpha u_l^{\psi} = & \lambda u_l^{\varphi} \\ b_1 u_{l-1}^{\psi} + & b_1 u_{l+1}^{\psi} + & b_0 u_l^{\psi} + & \alpha u_l^{\varphi} = & \lambda u_l^{\psi}. \end{array} \right. \tag{10}$$

The Bloch-type solution of it is defined by the property

$$\left(\begin{array}{c} u_l^{\varphi} \\ u_l^{\psi} \end{array} \right) := \vec{u}_l = \Theta \vec{u}_{l-1},$$

which gives the following equation for the basic vector $\vec{u}_0$:

$$\left\{ \left(\begin{array}{cc} a_1 & 0 \\ 0 & b_1 \end{array} \right) \left[\Theta + \Theta^{-1} \right] + \left(\begin{array}{cc} a_0 & \alpha \\ \alpha & b_0 \end{array} \right) + \lambda I \right\} \vec{u}_0 = 0. \tag{11}$$

The Bloch exponent Θ may be found from the condition

$$\det \left\{ \left(\begin{array}{cc} a_1 & 0 \\ 0 & b_1 \end{array} \right) \left[\Theta + \Theta^{-1} \right] + \left(\begin{array}{cc} a_0 & \alpha \\ \alpha & b_0 \end{array} \right) + \lambda I \right\} = 0.$$

We accomplish the calculation assuming that $a_1 = 1 = -b_1$ and $a_0 = b_0$. Then we obtain for quasimomentum p, $\Theta = e^{ip}$ the simple algebraic equation

$$(a_0 - \lambda)^2 - 4(\cos p)^2 = \alpha^2.$$

The spectrum of the Hamiltonian (9) consists of two spectral bands separated by the gap $(-\alpha, \alpha)$. The gap is centered at the value of the quasimomentum $p_0 = \pi/2$, $\lambda_0^{\pm} = a_0 \pm \alpha$, so the gap is *narrow* if the interaction between the branches of the superlattice is *weak*. On the other hand the direct calculation of the effective mass gives the following result :

$$2(\lambda_0^{\pm} - a_0)\lambda_{pp} = 8(\sin p_0)^2, \tag{12}$$

hence $m_{eff} = (\lambda_{pp})^{-1} = \frac{\alpha}{4}$ is small. This result is in good agreement with observation of the narrow-gap property for weakly-connected superlattices, in particular for non-stable semiconductors like $HgTe$, [5]. Analysis of the operator in the invariant subspace corresponding to the small neighbourhood of the Fermi level at the bottom of the upper spectral band reveals the quasi-relativistic properties of the dynamics of electrons subject to the condition $\kappa T + 2eU << 1$, similarly to the case discussed in the previous section.

3 Dirac equation with running potential

The quasi-relativistic property of the dynamics of electrons observed in the previous sections permits to suggest a method of manipulation of the quantum current through the wire with use of the running wave, similar to the method suggested in [9] and [10].

Consider a non-stationary Dirac operator with running potential $\epsilon_1 \cos \epsilon_2 (x - vt)$.

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i\frac{\partial \psi}{\partial x} + \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi + (q_0 + \epsilon_1 \cos \epsilon_2 (x - vt)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \psi. \tag{13}$$

Following [10] we apply the Lorentz transform

$$\frac{x - vt}{\sqrt{1 - v^2}} = s, \; \frac{t - vx}{\sqrt{1 - v^2}} = \tau \tag{14}$$

and obtain a new equation

$$\begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{iv}{\sqrt{1-v^2}} \\ -\frac{iv}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} i\frac{\partial \psi}{\partial \tau} + \begin{pmatrix} \frac{v}{\sqrt{1-v^2}} & \frac{-i}{\sqrt{1-v^2}} \\ -\frac{i}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} \end{pmatrix} i\frac{\partial \psi}{\partial s} +$$

$$+\delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi + \left(q_0 + \epsilon_1 \cos \epsilon_2 \sqrt{1 - v^2} s\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \psi. \tag{15}$$

which admits separation of variables. Introducing the spectral variable Ω representation ψ as $\psi = e^{i\Omega\tau}\varphi_\Omega$, we obtain the spectral problem :

$$\begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{iv}{\sqrt{1-v^2}} \\ -\frac{iv}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \Omega\varphi_\Omega = \begin{pmatrix} \frac{v}{\sqrt{1-v^2}} & \frac{-i}{\sqrt{1-v^2}} \\ -\frac{i}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} \end{pmatrix} i\frac{\partial \varphi_\Omega}{\partial s} +$$

$$+\delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \varphi_\Omega + \left(q_0 + \epsilon_1 \cos \epsilon_2 \sqrt{1 - v^2} s\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varphi_\Omega := L\varphi_\Omega \tag{16}$$

The differential operator L which corresponds to this spectral problem in the corresponding weighted space of all square integrable function has periodic coefficients and band spectrum σ_L. Hence the equation (13) has also running solutions of the form $e^{i\Omega\frac{t-vx}{\sqrt{1-v^2}}}\varphi_\Omega(\frac{x - vt}{\sqrt{1 - v^2}})$ for $\Omega \in \sigma_L$, parametrized by the spectral points Ω .

4 Time-periodic Dirac equation

Another possibility of manipulation quantum current may be achieved via localized periodic excitations. Consider the Dirac operator with a singular time- periodic potential defined as an extension of the non-perturbed Dirac operator restricted (from D to D_0) onto the class of all W_2^1-smooth vector-functions vanishing near the origin $x = 0$. The adjoint operator D_0^+ is defined on the piecewise continuous W_2^1-smooth vector - functions $\vec{\psi} = (\psi_1, \psi_2)$ which may have a jump at the origin, i.e. the left limit $\vec{\psi}(-0) := \left(\psi_1^-, \psi_2^-\right)$ is not equal to the right limit $\vec{\psi}(+0) := \left(\psi_1^+, \psi_2^+\right)$. The straightforward calculation of the boundary form of the operator D_0^+ gives the following result for any elements $\vec{\psi}$, $\vec{\phi}$ from the domain of the adjoint operator:

$$< D_0^+\vec{\psi}, \vec{\phi} > - < \vec{\psi}, D_0^+\vec{\phi} >= i\left(\psi_1^-\bar{\phi}_2^- + \psi_2^-\bar{\phi}_1^-\right) - i\left(\psi_1^+\bar{\phi}_2^+ + \psi_2^+\bar{\phi}_1^+\right). \tag{17}$$

This implies the following description of boundary conditions defining the selfadjoint extensions D_Γ of D_0 parametrized by 2×2 hermitian matrices Γ,see[2]:

$$\begin{pmatrix} i\psi_1^- \\ \psi_2^+ \end{pmatrix} = \Gamma \begin{pmatrix} \psi_2^- \\ -i\psi_1^+ \end{pmatrix}. \tag{18}$$

In what follows we omit the vector notations $\vec{\psi}$, having in mind that each time the lower indices as in $\psi_{1,2}$ are absent, we have a 2-vector, that is $\psi \equiv \vec{\psi}$.

One can easily calculate the spectral properties of operators D_Γ in terms of the matrix Γ. Really, the solutions of the non-perturbed homogeneous equation may be presented in form

$$\psi^{\pm} = \begin{pmatrix} \pm i\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}}\, e^{i\pm\sqrt{(\lambda-q_0)^2-\delta^2}x} \\ e^{i\pm\sqrt{(\lambda-q_0)^2-\delta^2}x} \end{pmatrix}. \tag{19}$$

Solutions of the perturbed equation which satisfy the boundary conditions (18) are combined of $\psi^{\pm}$ as

$$\overrightarrow{\psi} = \begin{cases} \psi^+ + \overleftarrow{R}\psi^- & \text{for } x<0 \\ \overrightarrow{T}\psi^+ & \text{for } x>0, \end{cases}$$

$$\overleftarrow{\psi} = \begin{cases} \psi^- + \overrightarrow{R}\psi^+ & \text{for } x>0 \\ \overleftarrow{T}\psi^- & \text{for } x<0 \end{cases} \tag{20}$$

Then the scattering matrix of the operator D_Γ may be defined as

$$S_\Gamma = \begin{pmatrix} \overleftarrow{R} & \overleftarrow{T} \\ \overrightarrow{T} & \overrightarrow{R} \end{pmatrix}. \tag{21}$$

The values of transmission coefficients $\overset{\rightleftarrows}{T}$ do not depend on the normalization constants, i.e. they remain invariant with respect to replacement of $\psi^{\pm}$ by $|\alpha|e^{\pm i\varphi}\psi^{\pm}$. For special choice of the normalizing constants $|\alpha|e^{\pm i\varphi}$ the scattering matrix is unitary. But for our aims it is sufficient to calculate the transmission coefficients only. The following statement is true:

Theorem 2 *The scattering matrix of the operator D_Γ with respect to solutions $\psi^{\pm}$ has the form*

$$S_\gamma = \left\{ \begin{pmatrix} \sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} & 0 \\ 0 & 1 \end{pmatrix} - \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} \end{pmatrix} \right\}^{-1} \times$$

$$\times \left\{ \begin{pmatrix} \sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} & 0 \\ 0 & 1 \end{pmatrix} + \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} \end{pmatrix} \right\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

Proof may be obtained by the straightforward calculation of the transmission and reflection coefficients from the boundary conditions (18) for $\overset{\rightleftarrows}{\psi}$:

$$\begin{pmatrix} \overleftarrow{T}\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} \\ 1+\overrightarrow{R} \end{pmatrix} = \Gamma \begin{pmatrix} \overleftarrow{T} \\ -\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} + \overrightarrow{R}\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} \end{pmatrix},$$

$$\begin{pmatrix} -\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} + \overleftarrow{R}\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} \\ \overrightarrow{T} \end{pmatrix} = \Gamma \begin{pmatrix} 1+\overleftarrow{R} \\ \overrightarrow{T}\sqrt{\frac{\lambda-q_0+\delta}{\lambda-q_0-\delta}} \end{pmatrix}$$

In particular the last pair of equations may be solved as

$$\begin{pmatrix} \overleftarrow{R} \\ \overrightarrow{T} \end{pmatrix} = \left\{ \begin{pmatrix} \sqrt{\frac{\lambda - q_0 + \delta}{\lambda - q_0 - \delta}} & 0 \\ 0 & 1 \end{pmatrix} - \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\lambda - q_0 + \delta}{\lambda - q_0 - \delta}} \end{pmatrix} \right\}^{-1}$$

$$\left\{ \begin{pmatrix} \sqrt{\frac{\lambda - q_0 + \delta}{\lambda - q_0 - \delta}} & 0 \\ 0 & 1 \end{pmatrix} + \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\lambda - q_0 + \delta}{\lambda - q_0 - \delta}} \end{pmatrix} \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solution of the first pair of equations may be represented in a similar way. The combination of them gives the announced result.

□

We consider now the Dirac equation with time-dependent singular potential as a selfadjoint operator in space- time Hilbert space $L_2(R) \times L_2(R, E)$ which is obtained as an extension of the "non-perturbed" operator

$$i\frac{\partial \psi}{\partial t} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i\frac{\partial \psi}{\partial x} + \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi := \mathcal{D}, \tag{23}$$

restricted $\mathcal{D} \to \mathcal{D}_0$ onto the linear set of all W_2^1 functions vanishing near the t- axis on the phase space $R_2 = R_t \times R_x$. The corresponding adjoint operator is defined on the splitted space $W_2^1(R_2, t < 0) \oplus W_2^1(R_2, t > 0)$, and the boundary form of it may be written as the integral of the boundary form (17) of the operator D_0^+ over R_t:

$$< \mathcal{D}_0^+ \vec{\psi}, \vec{\phi} > - < \vec{\psi}, \mathcal{D}_0^+ \vec{\phi} >=$$

$$\int_{-\infty}^{\infty} \left\{ i \left(\psi_1^- \bar{\phi}_2^- + \psi_2^- \bar{\phi}_1^- \right) - i \left(\psi_1^+ \bar{\phi}_2^+ + \psi_2^+ \bar{\phi}_1^+ \right) \right\} dt. \tag{24}$$

Existence of the boundary values $\psi_{1,2}^{\pm} \in L_2(R_t)$ follows from standard embedding theorems. To define the selfadjoint extension $\mathcal{D}_\Gamma$ we submit the boundary values $\psi_{1,2}^{\pm}$ to the *local* time-dependent boundary conditions:

$$\begin{pmatrix} i\psi_1^- \\ \psi_2^+ \end{pmatrix} = \Gamma \cos \Omega t \begin{pmatrix} \psi_2^- \\ -i\psi_1^+ \end{pmatrix} \tag{25}$$

with a constant Hermitian matrix Γ . The solutions of the homogeneous equation

$$i\frac{\partial \psi}{\partial t} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i\frac{\partial \psi}{\partial x} + \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi = \Lambda \psi \tag{26}$$

serve the eigenfunctions of the operator $\mathcal{D}_\Gamma$. One can show that the whole spectrum of this operator is absolutely continuous and consists of two branches : the branch with infinite multiplicity, which corresponds to the scattered waves combined of solutions of the non-perturbed homogeneous equations

$$i\frac{\partial}{\partial t}\psi_\Omega(t)\partial t = \Omega \psi_\Omega(t),$$

$$D\psi_\lambda^{\pm} = \lambda \psi_\lambda^{\pm}$$

for $\Lambda = \Omega + \lambda$ as

$$\overrightarrow{\Psi}_{\Lambda,\Omega} = \psi_\Omega(t) \times \begin{cases} \psi_\lambda^+ + \overleftarrow{\mathcal{R}}\psi^-, & \text{for } x < 0 \\ \overrightarrow{\mathcal{T}}\psi^-, & \text{for } x > 0, \end{cases},$$

$$\overleftarrow{\Psi}_{\Lambda,\Omega} = \psi_\Omega(t) \left\{ \begin{array}{ll} \psi^- + \overrightarrow{\mathcal{R}}\psi^+ & \text{for } x > 0 \\ \overleftarrow{\mathcal{T}}\psi^- & \text{for } x < 0 \end{array} \right. , \tag{27}$$

and the waveguide branch which is constructed of outgoing solutions on x-axis:

$$\Psi_\Lambda(x,t) = \psi_\Omega(t) \left\{ \begin{array}{ll} \psi^+(x) & \text{for}\, x > 0 \\ \mathcal{K}\psi^-(x) & \text{for}\, x < 0 \end{array} \right. , \tag{28}$$

The transmission and reflection coefficients $\overleftrightarrow{\mathcal{T}}$, $\overleftrightarrow{\mathcal{R}}$ for given λ still depend on Ω and may be defined from the boundary conditions (25). One may construct of them the scattering matrix as

$$\mathcal{S}_\Gamma(\Lambda,\Omega) = \left(\begin{array}{cc} \overleftarrow{\mathcal{R}} & \overleftarrow{\mathcal{T}} \\ \overrightarrow{\mathcal{T}} & \overrightarrow{\mathcal{R}} \end{array} \right). \tag{29}$$

The amplitudes K are defined uniquely for given λ from the additional condition of quasiperiodicity with respect to the time-variable: $\Psi_\Lambda(x, t+\frac{2\pi}{\Omega}) = \Theta\Psi_\Lambda(x,t)$ which may be imposed just on the boundary values $\Psi_\Lambda(\pm 0, t)$. The quasimomentum exponential $\Theta(\Lambda)$ may play the role of an alternative spectral variable for the waveguide branch of an absolutely- continuous spectrum. Similar spectral variable may be introduced also for the scattered waves $\Psi^{\rightleftarrows}_{\Lambda,\Omega}$ to substitute one of parameters Λ, Ω.

We concentrate now on the analysis of eigenfunctions of absolutely continuous spectra which correspond to the spectral point $\Lambda = 0$. The corresponding eigenfunction satisfy the equation $\mathcal{D}_\gamma\Psi = 0$. In particular the waveguide-type eigenfunction of this type (if exist) may correspond to some sort of bound states which depend on time harmonically. For investigation of the conductance of the quantum channel of the narrow-gap semiconductor manipulated by the localized time-periodic excitation an important role is played by the scattered waves $\overleftrightarrow{\Psi}_{\Lambda,\Omega}$. Precisely, the conductance for given Ω is calculated from the corresponding transmission coefficients via Landauer formula as $\dfrac{e^2}{h}\dfrac{|\mathcal{T}(0,\Omega)|^2}{1-|\mathcal{T}(0,\Omega)|^2}$. The investigation of the behavior of the transmission coefficient $\overrightarrow{\mathcal{T}}(0,\Omega)$ with respect to frequency Ω of the excitation is an important ongoing problem.

Let us estimate the frequency necessary for effective control of electrons dynamics in a narrow-gap semiconductor using this method. It is clear that the required frequency must be of the same order as the frequency of the ballistic electrons. When the Fermi level coincides with the bottom of the upper spectral band the frequency of electrons ν_{el} at average thermal energy kT (for room temperature $T = 300K$) is

$$\nu_{el} = 2\pi\frac{kT}{h} \approx 3\cdot 10^{13} Hz,$$

which corresponds to IR wavelengths $\lambda \approx 10\mu m$.

References

[1] L.A.Girifalco. Statistical Physics of Materials. J.Wiley&Sons, New York- London-Sydney-Toronto,1973,376p.

[2] B.S.Pavlov. The theory of extensions and Exactly Solvable Models. Uspekhi Mat. Nauk, V.42, 1987.

[3] B.S. Pavlov, G.Roach, A.Yafyasov. Resonance Scattering and Design of Quantum Gates.

[4] B.S. Pavlov, G.P. Miroshnichenko. Patent 2062530, (Russia) 12.03.1992.

[5] Antononiou, B. Pavlov, A. Yafyasov. Quantum electronic devices, based on metal-dielectric transmission in low-dimensional quantum structures. In: D. S. Bridges, C. Calude, J. Gibbons, S. Reeves, I. Witten (eds.). Combinatorics, Complexity, Logic, Proceedings of DMTCS'96, Springer-Verlag, Singapore, 1996, 90-104.

[6] R. Landauer. Spatial variation of currents and fields due to localized scatterers in metallic constructions. IBM Jornal for Research and Development, 1 (1957), 223-231.

[7] E.Titchmarsh. Eigenfunctions Expansions Associated with Second - Order Differential Equations. Clarendon Press, Oxford, 1946.

[8] B.Pavlov, N.Smirnov. Resonance scattering by one-dimensional crystal and thin film. Vestnik of the Leningrad University, V.13, No.3, 1977 (in Russian)

[9] B.S.Pavlov, A.A.Pokrovski, L.V.Prokhorov. Light Propagation in Truncated Periodic Waveguide. Progress Report (July 1, 1998 – December 31, 1998) ESPRIT Project 28890 NTCONGS.

[10] B.S.Pavlov, A.A.Pokrovski, L.V.Prokhorov. Quasi-stationary analysis of the effective fiber acoustooptic equation. Progress Report (January 1, 1999 – June 31, 1999) ESPRIT Project 28890 NTCONGS.

[11] C.W.J.Beenakker, H. van Houten. Quantum transport in semiconductor nanostructures. In: H.Ehrenreich, D.Turnbull (Eds.) Solid State Physics. Advances in Research and Applications. V. 44 (1991). Academic Press, San Diego. P. 1-228.