HALIN'S THEOREM FOR THE MÖBIUS STRIP

DAN ARCHDEACON, C. PAUL BONNINGTON, MARISA DEBOWSKY, AND MICHAEL PRESTIDGE

ABSTRACT. Halin's Theorem characterizes those locally finite infinite graphs that embed in the plane without accumulation points by giving a set of six topologically-excluded subgraphs. We prove the analogous theorem for graphs that embed in an open Möbius strip without accumulation points. There are 153 such obstructions under the ray ordering defined herein. There are 350 obstructions under the minor ordering. There are 1225 obstructions under the topological ordering. The relationship between these graphs and the obstructions to embedding in the projective plane is similar to the relationship between Halin's graphs and $\{K_5, K_{3,3}\}$.

1. INTRODUCTION

A fundamental result in graph theory is Kuratowski's Theorem [13], which says that a finite graph embeds in the plane if and only if it does not contain a subdivision of either K_5 or $K_{3,3}$. We are concerned here with embeddings of locally-finite infinite graphs. For the requisite background on embedding finite and infinite graphs see [14] and [6] respectively (see [7] for related work on embedding infinite graphs).

Halin [12] proved that a locally finite, infinite graph embeds in the plane without an accumulation point if and only if it does not contain a non-isomorphic subdivision of one of six graphs. These graphs are shown in Figure 1. In this figure the circled vertices are to be identified with the tails of disjoint one-way-infinite induced rays.

The graphs are closely related to those of Kuratowski's Theorem. In particular, for an infinite graph to embed in the plane without accumulation points it must first embed in the plane. Hence it cannot contain a K_5 or $K_{3,3}$ subgraph (see [9, 10]). The next two graphs are formed from K_5 and $K_{3,3}$ by deleting a single vertex and adding disjoint one-way-infinite paths attached to each vertex adjacent to the deleted one. Finally, the last two graphs are formed from K_5 and $K_{3,3}$ by first subdividing an edge, deleting the resulting degree-two vertices, and adding disjoint one-way-infinite paths attached to each vertex adjacent to the deleted one.

An equivalent form of Halin's Theorem says that the exclusion of these graphs characterize all locally-finite graphs that embed on a 2-sphere (the one-point compactification of the plane) with a single accumulation point.

Our goal in this paper is to give a similar forbidden subgraph characterization for those graphs that embed on the Möbius strip without an accumulation point. (We mean the *open* Möbius strip, that is, the one that is non-compact and without boundary.) In the spirit of the previous paragraph, this is equivalent to characterizing graphs that embed on the real projective plane with a single accumulation point.

Research supported by the Marsden Fund (Grant Number UOA825) administered by the Royal Society of New Zealand.
Submitted for publication, 15 August 2001.

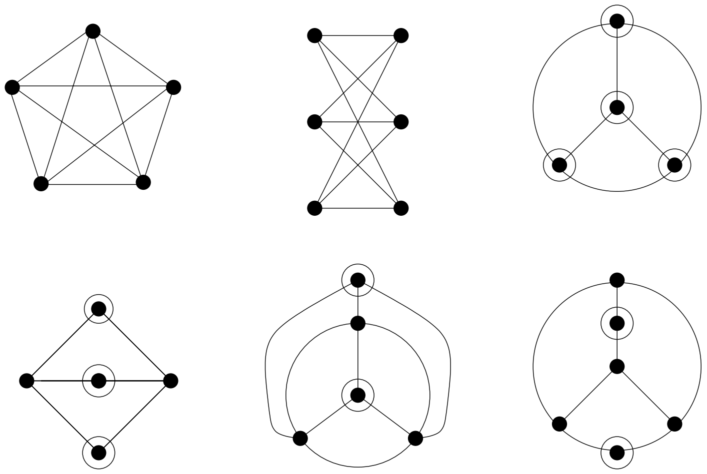

FIGURE 1. Halin's six obstructions (add rays to the circled vertices)

As we will show, these graphs are related to the obstruction set for finite graphs to embed on the projective plane. The latter set is known and we exploit this knowledge to prove our main theorem. The topological obstructions were first found by Glover, Huneke, and Wang [11]; their set was proved complete by Archdeacon [1, 2]. The latter two works implicitly give the minor obstructions.

In any "excluded subgraph" theorem it is important to specify the partial order. In general, you can consider any partial order that preserves the property—in this case that there is an embedding on the open Möbius band without accumulation points. The finer the partial order, the fewer the number of obstructions. We consider three respectively coarser partial orders: the ray order, the minor order, and the topological order. We determine the obstruction set for this property under all three orders. This turns out to be equivalent to characterizing rooted projective-planar graphs using three different partial orders.

In Section 2 we make the above more precise. In particular, we discuss embeddings of infinite graphs and their relationship with embedding rooted projective-planar graphs. We give the partial orderings on these two classes and prove various equivalences under these partial orders. In Section 3 we give the statements and proofs of our main theorems. In Section 4 we describe some double-checks on our results and discuss directions for future research.

2. THE RELATED GRAPHS AND THEIR PARTIAL ORDERS

In this section we give an equivalence between our desired characterization of infinite graphs on an open Möbius band without accumulation points and a characterization of rooted non-projective-planar graphs. We begin with three basic partial orders.

The *subgraph order* is the transitive closure of the following two elementary operations. Write $H \subset G$ if and only if you form H from G by:

1) deleting an isolated vertex, or
2) deleting an edge.

The *topological order* includes the following elementary operation, called *smoothing a degree-two vertex*:

3) Delete a degree-two vertex and add an edge joining its neighbors.

Kuratowski's Theorem [13] characterizes the planar graphs by giving two subgraphs excluded under the topological order. Halin's Theorem [12] characterizes the locally-finite (possibly infinite) planar graphs by giving six subgraphs excluded under the topological order.

The *minor order* is the transitive closure of the topological order and the following operation of *contracting an edge e*:

4) Delete the two vertices u, v incident with an edge e, and adding in a new vertex w and edges joining w to all verticies adjacent to u or v.

The inverse of Operation 4) above is called *splitting a vertex*. It is well-defined only if you specify which edges incident with w are to be incident with u, or equivalently with v. Wagner's Theorem [16] says that planar graphs are characterized by excluding K_5 and $K_{3,3}$ as minors. It is easy to see that Halin's Theorem characterizes planar graphs without accumulation points by excluding the six graphs under either the topological or the minor order. In general, the sets of excluded obstructions are different for these two orders.

In an infinite graph G, a proper topological subgraph or minor H may be isomorphic to G. For example, any vertex smoothing or edge contraction in an infinite path leaves an infinite path. A *strict* minor or topological subgraph is one that is not isomorphic to the original graph.

A *tail* of an infinite graph G is a one-way-infinite ray where every interior vertex is of degree 2 in G. The tail is *attached to* the unique vertex incident with one edge of the tail. The *residue* of G is formed by simultaneously deleting all tails. Call G *residually finite* if the residue is a finite graph, that is, if G is a finite graph with a set of tails attached.

Bonnington and Richter [6] have shown the following.

Proposition 2.1. *Let G be an infinite graph. Suppose that G does not embed on a surface S without accumulation points, but that every strict topological subgraph does so embed. Then G is residually finite and each vertex is attached to at most one tail.* □

Let G^∞ be a residually finite graph. Form a finite graph G^+ by deleting all tails from G^∞, adding a new vertex v^+, and edges from v^+ to all vertices attached to tails in G^∞. The new vertex v^+ is called the *root* vertex, and G^+ is called a *rooted graph*. Conversely, given a rooted graph G^+, we can form the associated infinite graph G^∞.

Bonnington and Richter [6] have also shown the following.

Proposition 2.2. *A residually finite graph G^∞ embeds on a closed surface with a single accumulation point if and only if G^+ embeds on that surface.*

We now describe two partial orderings on the class of rooted graphs G^+. We use operations similar to those defining the topological order, except that we replace 1) and 3) with the following restricted versions.

1r) delete an isolated vertex that is not the root,
3r) smooth a degree-two vertex that is not the root.

The partial order defined by Operations 1r), 2), and 3r) is called the *restricted topological ordering*. Also, define the operation:

4r) contract an edge not incident with the root.

Adding Operation 4r) to the restricted topological ordering makes the *restricted minor ordering.*

Lemma 2.3. *H^∞ is a strict topological subgraph of G^∞ if and only if H^+ is a restricted topological subgraph of G^+. Likewise, H^∞ is a strict minor of G^∞ if and only if H^+ is a restricted minor of G^+.*

Proof. Deleting an edge in the residue of G^∞ corresponds to deleting an edge in G^+ not incident with the distinguished vertex. Deleting an edge in a tail of G^∞ corresponds to deleting an edge in G^+ incident with root (we can delete the component that is a one-way-infinite path). Smoothing a degree 2 vertex in G^∞ not in a tail corresponds to smoothing the corresponding degree 2 vertex in G^+. Contracting an edge in G^∞ not in a tail corresponds to contracting the in G^+. □

The restrictions on smoothing the root vertex or contracting an edge incident are not entirely natural. Let's examine the effect on the corresponding infinite graphs if we remove these restictions on G^+.

Let G^+ be a graph with a root v^+ of degree 2, and let H^+ be the unrooted graph formed by smoothing v. Then H^∞ is formed from G^∞ by the following *Operation 5)*:

5) Delete the only two tails in G^∞ and add an edge joining the vertices where they were attached.

Similarly let G^+ be a graph with a root v^+ incident with an edge uv^+. Let $N^-(u)$ denote the neighbors of u except for v^+. Let H^+ be formed from G^+ by contracting the edge e, and making the new vertex the root. Then H^∞ is formed from G^∞ by the following *Operation 6)*:

6) Delete a tail attached to u, delete u and all its incident edges, and add new tails attached to each vertex in $N^-(u)$.

The *ray order* on residually finite graphs allows Operations 1)-6). We call H a *ray minor* of G.

Lemma 2.4. *H^∞ is a strict ray minor of G^∞ if and only if H^+ is a minor of G^+.*

The ray order is finer than the minor order on residually-finite graphs. However, the property of embedding on a surface with a single point is closed under this order. In other words, if H is a ray minor of G and G embeds in S with at most one accumulation point, then so does H. Hence it makes sense to look for the graphs that are strictly minimal under this order that do not have such embeddings.

We summarize the results of this section as follows.

Theorem 2.5. *Let G^∞ be a residually-finite graph and let G^+ be its associated rooted graph. Then:*

a) *G^∞ does not embed in a closed surface S with a only one accumulation point if and only if G^+ does not embed in S.*
b) *If every strict topological subgraph of G^∞ embeds in S with at most one accumulation point, then every restricted topological subgraph of G^+ embeds in S.*

c) *If every strict minor of G^∞ embeds in S with at most one accumulation point, then every restricted minor of G^+ embeds in S.*
d) *If every ray-minor of G^∞ embeds in S with at most one accumulation point, then every minor of G^+ embeds in S.*

3. The main results

In this section we give our main results: the analogues to Halin's Theorem for graphs that embed on the open Möbius band. These results rely on some lengthy calculations; in Section 4 we give some double checks on these calculations.

Theorem 3.1. *There are exactly 153 graphs that do not embed in the projective plane with at most one accumulation point, but such that every strict ray minor does so embed. Equivalently, excluding these graphs under the ray order characterizes graphs that embed in the open Möbius band.*

Proof. Any infinite graph embeds on the projective plane if and only if every finite subgraph does [9, 10]. For finite graphs the ray order is the same as the minor order. There are exactly 35 minor-minimal graphs that do not embed in the projective plane [1, 2, 11].

If the infinite graph does embed in the projective plane, then by Proposition 2.1 it is a residually finite G^∞. The corresponding graph G^+ is minor minimal for embedding in the projective plane. The proof now reduces to examining each of these 35 graphs and finding the possible roots. This is equivalent to finding the vertex orbits of each graph under the action of the automorphism group.

The number of vertex orbits for these graphs are given in Appendix 2. The 35 graphs are those marked with a star $\star$. The column $||V||$ gives the number of vertex orbits, or equivalently the number of ways to distinguish a root. The 35 graphs are also given in Appendix 1, where a representative of each vertex orbit has been circled.

There were exactly 118 vertex orbits in these minimal graphs. These 118 graphs plus the 35 finite obstructions give the 153 graphs as claimed. $\square$

Theorem 3.2. *There are exactly 350 graphs that do not embed in the projective plane with at most one accumulation point, but such that every strict minor does so embed. Equivalently, excluding these graphs under the minor order characterizes graphs that embed in the open Möbius band.*

Proof. Let G^∞ be the graph as described. If G^∞ does not embed in the projective plane, then it contains one of the 35 minor-minimal obstructions.

If G^∞ does embed in the projective plane, but only with more than one accumulation point, then G^+ does not embed in the projective-plane. Moreover, every minor does so embed, except that we cannot smooth the distinguished vertex or contract an edge incident with the distinguished vertex. Hence G^+ is topologically-minimal with respect to not embedding in the projective plane. Call an edge *contractible* if G^+/e is not projective planar.

If the distinguished vertex is of degree 2, then it must be incident with all contractible edges. In other words, the smoothed graph is topologically minimal, and there is no contractible edge other than possibly the one that contained the root. There are exactly 103 topologically-minimal non-projective graphs. We examined the edge orbits of each and identified the edges that were not disjoint from a contractible edge. The results are given in Appendix 2 under the column $||E_c||$

If the root vertex is of degree exceeding 2, then G^+ is topologically minimal and the root is incident with all contractible edges. Again, we examined the 103 graphs in turn and found all such vertex orbits. The results are given in Appendix 2 under the column $||V_c||$.

The 35 projective planar obstructions, the 155 graphs with roots of degree 2, and the 160 graphs with the roots of degree exceeding 2 give the 350 graphs as claimed. □

Theorem 3.3. *There are exactly 1235 graphs that do not embed in the projective plane with at most one accumulation point, but such that every strict topological subgraph does so embed. Equivalently, excluding these graphs under the topological order characterizes graphs that embed in the open Möbius band.*

Proof. The number comes from the 103 topologically minimal graphs, their 488 vertex orbits, and their 644 edge orbits. See Appendix 2 for details under the columns $||V||$ and $||E||$. □

4. Conclusion

The calculations of vertex- and edge-orbits required for our results are tedious, and in any such calculations errors may creep in. We describe some of the double-checks we employed to ensure our calculations are correct.

First and most importantly, every entry in Appendix 1 and 2 was calculated both by hand and by computer. For roughly 60% of the graphs the hand calculations were done independently by at least two of the authors. The numbers obtained agreed, and also agreed with some earlier work (see for example [5]). We used a well-tested computer list of the 35 minor-minimal non-projective-planar graphs. Together with a projective-planarity tester developed by the second author, we were able to generate the 103 topological-minimal non-projective-planar graphs, verifying the results of [11]. We then tested which such G and edges e had G/e projective-planar. It was then straightforward to calculate the entries in Appendix 2, and to check these results against the hand calculations. This independent computer verification gives us confidence that the calculations are correct.

The techniques of this paper easily generalize to other related results. However, we rely heavily on the known obstruction set for embedding in the projective plane. Complete sets of obstructions for embedding in other surfaces are not known.

There is another partial order that has received some attention. A $Y\Delta$ transformation on a graph deletes a vertex of degree three and adds 3 edges joining pairs of edges adjacent to that vertex. If G is a $Y\Delta$ transformation of H and H embeds in a surface, then G embeds in that same surface. It would be possible to find the relationship between $Y\Delta$ transformations and residually finite graphs, but we choose not to pursue this.

Halin's Theorem [12] characterizes connected locally-finite infinite graphs that embed in the plane without accumulation points. We ask a similar question: *Which locally-finite infinite graphs embed in an annulus (the plane with a point removed) without accumulation points?* Boza, Gugúndez, Márquez, and Revuelta answer this question under the added condition that the rays of the obstruction are assigned particular ends of the annulus. The complete obstruction set for cubic graphs without this restriction is given in [4], see [3] for related material.

We close by noting that J. Cáceres [8] has results similar to those in this paper.

ACKNOWLEDGEMENTS: The first author thanks the University of Auckland, which hosted them during the time this research was conducted. The first three authors are thankful for support from the Marsden Fund (Grant Number UOA825) administered by the Royal Society of New Zealand. The fourth author is thankful for support from NASA's National Space Grant College and Fellowship Program (Grant Number NGT5-40110).

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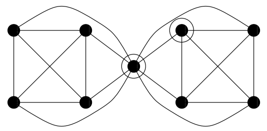
APPENDIX 1: Vertex Orbits of the 35 Minor-Minimal Non-Projective-Planar Graphs



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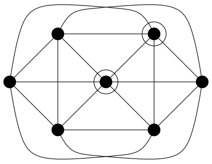

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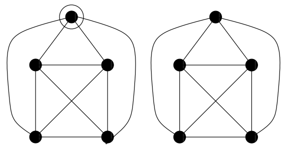

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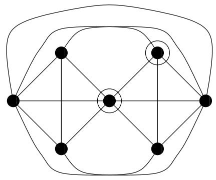

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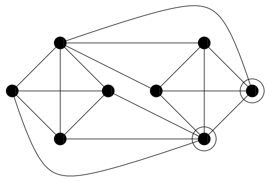

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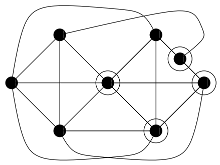

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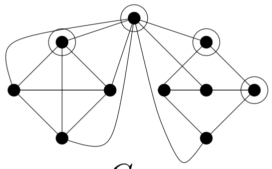

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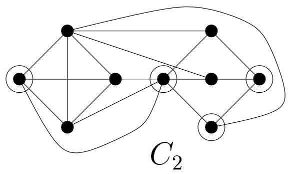

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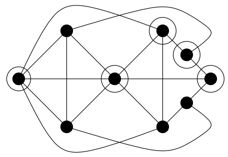

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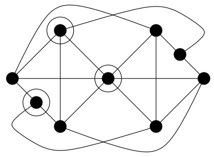

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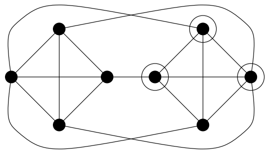

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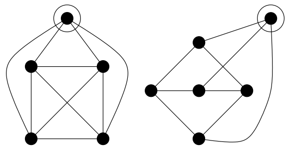

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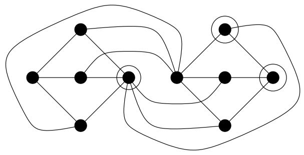

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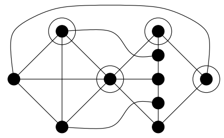

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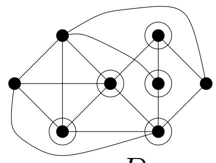

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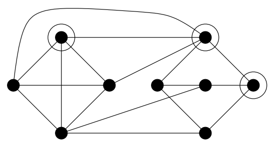

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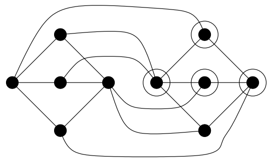

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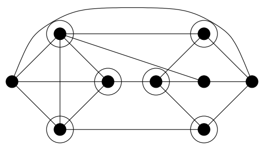

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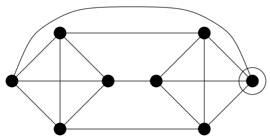

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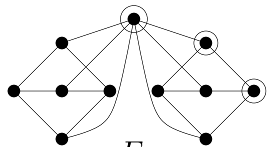

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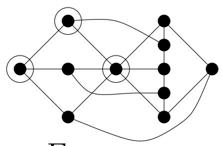

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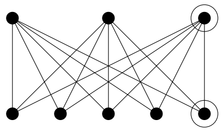

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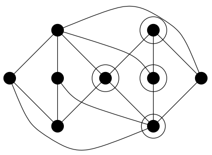

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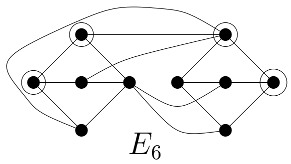

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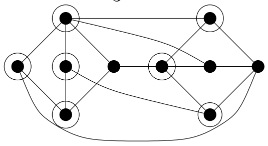

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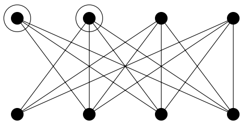

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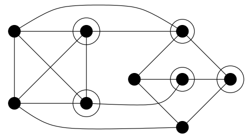

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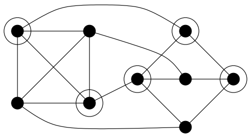
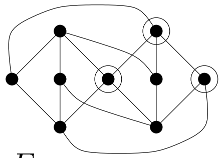
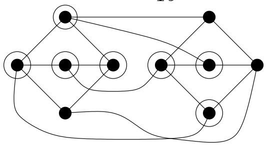


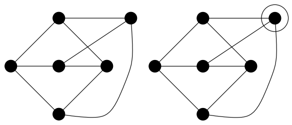
E_{20}



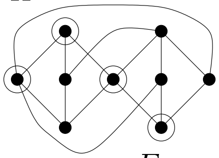
E_{22}



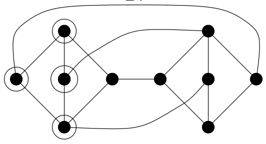
E_{27}



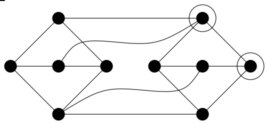
E_{42}



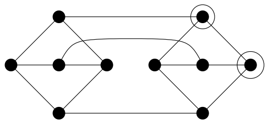
F_1



F_4



F_6



G_1

Appendix 2: Orbits of the 103 topological obstructions

	G	$\|\|V\|\|$	$\|\|E\|\|$	$\|\|V_c\|\|$	$\|\|E_c\|\|$		G	$\|\|V\|\|$	$\|\|E\|\|$	$\|\|V_c\|\|$	$\|\|E_c\|\|$
$\star$	A_1	2	2	2	2	$\star$	A_2	2	2	2	2
	A_3	4	5	1	0		A_4	3	3	1	0
$\star$	A_5	1	1	1	1	$\star$	B_1	2	3	2	3
	B_2	5	8	2	1	$\star$	B_3	2	2	2	2
	B_4	5	8	1	0		B_5	3	7	1	1
	B_6	5	9	0	0	$\star$	B_7	4	5	4	5
	B_8	3	4	1	1		B_9	5	10	0	0
	B_{10}	2	3	0	0		B_{11}	2	3	0	0
$\star$	C_1	4	4	4	4	$\star$	C_2	4	4	4	4
$\star$	C_3	5	7	5	7	$\star$	C_4	3	3	3	3
	C_5	6	7	2	1		C_6	4	5	1	0
$\star$	C_7	3	6	3	6		C_8	5	5	1	0
	C_9	5	5	0	0		C_{10}	5	6	1	0
$\star$	C_{11}	2	2	2	2	$\star$	D_1	3	2	3	2
$\star$	D_2	4	5	4	5	$\star$	D_3	5	7	5	7
$\star$	D_4	3	4	3	4		D_5	7	11	2	1
	D_6	5	9	1	0		D_7	9	17	2	1
	D_8	6	7	1	0	$\star$	D_9	4	4	4	4
	D_{10}	7	11	1	0		D_{11}	5	11	0	0
$\star$	D_{12}	6	9	6	9		D_{13}	6	7	2	1
	D_{14}	10	18	0	0		D_{15}	6	10	0	0
	D_{16}	8	11	0	0	$\star$	D_{17}	1	2	1	2
	D_{18}	4	5	0	0		D_{19}	4	5	0	0
$\star$	E_1	3	2	3	2	$\star$	E_2	3	2	3	2
$\star$	E_3	2	1	2	1		E_4	5	5	2	1
$\star$	E_5	4	4	4	4	$\star$	E_6	4	4	4	4

	G	$\|\|V\|\|$	$\|\|E\|\|$	$\|\|V_c\|\|$	$\|\|E_c\|\|$		G	$\|\|V\|\|$	$\|\|E\|\|$	$\|\|V_c\|\|$	$\|\|E_c\|\|$
	E_7	5	5	1	0		E_8	4	6	1	1
	E_9	5	5	0	0		E_{10}	6	8	0	0
$\star$	E_{11}	7	9	7	9		E_{12}	7	10	2	1
	E_{13}	7	7	2	1		E_{14}	8	11	0	0
	E_{15}	6	7	1	0		E_{16}	5	5	1	0
	E_{17}	5	5	0	0	$\star$	E_{18}	2	2	2	2
$\star$	E_{19}	5	7	5	7	$\star$	E_{20}	5	6	5	6
	E_{21}	7	9	2	1	$\star$	E_{22}	3	2	3	2
	E_{23}	5	8	0	0		E_{24}	7	10	0	0
	E_{25}	8	11	2	1		E_{26}	3	4	1	1
$\star$	E_{27}	7	9	7	9		E_{28}	9	13	0	0
	E_{29}	8	10	0	0		E_{30}	9	12	0	0
	E_{31}	5	5	0	0		E_{32}	11	18	0	0
	E_{33}	6	7	0	0		E_{34}	5	6	0	0
	E_{35}	6	9	0	0		E_{36}	5	5	0	0
	E_{37}	7	8	0	0		E_{38}	4	6	0	0
	E_{39}	9	12	0	0		E_{40}	3	3	1	0
	E_{41}	3	3	0	0	$\star$	E_{42}	1	1	1	1
$\star$	F_1	4	5	4	5		F_2	8	10	2	1
	F_3	6	8	1	0	$\star$	F_4	4	6	4	6
	F_5	4	6	1	1	$\star$	F_6	2	2	2	2
	F_7	4	4	1	1		F_8	4	4	0	0
	F_9	6	9	0	0		F_{10}	8	11	0	0
	F_{11}	2	3	0	0		F_{12}	2	3	0	0
	F_{13}	2	2	0	0		F_{14}	2	3	0	0
$\star$	G_1	2	2	2	2		**Totals**	**488**	**644**	**160**	**155**

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