ON THE ORIENTABLE GENUS OF THE CARTESIAN PRODUCT OF A COMPLETE REGULAR TRIPARTITE GRAPH WITH A EVEN CYCLE

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ABSTRACT. We apply the technique of patchwork embeddings to find orientable genus embeddings of the Cartesian product of a complete regular tripartite graph with a even cycle. In particular, the orientable genus of $K_{m,m,m} \times C_{2n}$ is determined for $m \geq 1$ and for all $n \geq 3$ and $n = 1$. For $n = 2$ both lower and upper bounds are given. We see that the resulting embeddings may have a mixture of triangular and quadrilateral faces, in contrast to previous applications of patchwork method.

1. INTRODUCTION

In [1, 2, 3], Pisanski develops the theory of "patchworks" that can be used to derive, for example, exact values for the genus of the Cartesian product of regular bipartite graphs. The resulting embeddings are quadrangulations. The purpose of this paper is to show these techniques can be extended to other families of Cartesian products where the resulting embeddings may have a mixture of triangular and quadrilateral faces. In particular, we show that for the Cartesian product of the complete regular tripartite graph $K_{m,m,m}$ with the even cycle C_{2n}

$$\gamma(K_{m,m,m} \times C_{2n}) = 1 + m(m-1)n, \quad m \geq 1, n \geq 3.$$

The orientable genus of $K_{m,m,m}$ was shown to be $(m-1)(m-2)/2$ by Ringel and Youngs [4] and independently by White [5]. (For $m = 3$, the genus embedding is in the torus – see Figure 1.) There is an embedding of $K_{m,m}$ in the surface of genus $(m-1)(m-2)/2$ with m faces such that every face is a Hamiltonian cycle of the $2m$ vertices (see, for example [6]). Placing a new vertex into every face that is adjacent to

The work of the first author is supported by the Marsden fund grant number UOA825 administered by the Royal Society of New Zealand.

The work of the second author is supported in part by "Ministrstvo za znanost in tehnologijo Republike Slovenije", proj. no. J1-8901 and J1-8549. The paper was written while the second author was visiting the University of Auckland, New Zealand

Submitted for publication, 1 September 2001.

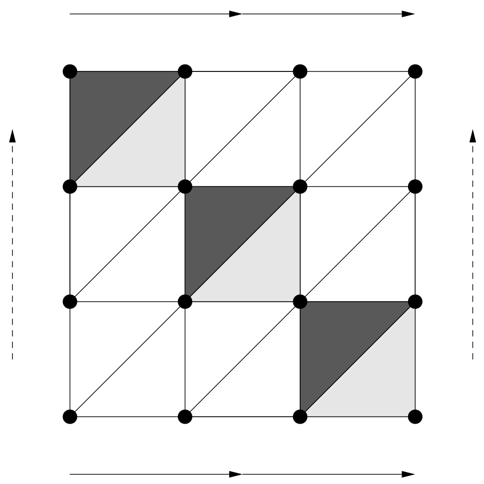

FIGURE 1. Case $m = 3$. Triangular embedding of $K_{m,m,m}$ in torus with two patchworks indicated.

the vertices on the boundary gives a required triangulation of $K_{m,m,m}$ in the surface of genus $(m-1)(m-2)/2$. A *patchwork* in an embedded graph is a 2-factor in which the connected 2-regular subgraphs are all facial boundaries. By the construction of the triangulation of $K_{m,m,m}$ it is immediate that there are $2m$ disjoint patchworks in this embedding. (For $m = 3$ the embedding has 6 disjoint patchworks, two of which are indicated in Figure 1.)

2. Main Result

Theorem 1. *The genus of $K_{m,m,m} \times C_{2n}$ for $m \geq 1, n \geq 3$ is given by*

$$\gamma(K_{m,m,m} \times C_{2n}) = 1 + m(m-1)n.$$

Proof. We first prove $\gamma(K_{m,m,m} \times C_{2n}) \leq 1 + m(m-1)n$. For $m = 1$, we have $K_{m,m,m} = C_3 \times C_{2n}$ which is obviously toroidal, and hence the result holds.

Now assume $m \geq 2$. We start with $2n$ copies of the above mentioned triangulation of $K_{m,m,m}$ in a surface S_g of genus $g = (m-1)(m-2)/2$. Since C_{2n} is a bipartite 2-regular then by the patchwork methods of [1, 2, 3], $K_{m,m,m} \times C_{2n}$ has an embedding in the orientable surface of genus $1 + m(m-1)n$. The two patchworks used in this method may be constructed (for instance) by taking alternating edges of any Petrie walk of the above mentioned embedding of $K_{m,m}$ in the surface of genus $(m-1)(m-2)/2$, then augmenting the edges to appropriate triangles of $K_{m,m,m}$ in the same surface. We double-check the genus formula by the following argument. Note that:

(1) there are $2n$ copies of S_g, arranged in a circle, each triangulated by a copy of $K_{m,m,m}$,
(2) there are m tubes $(C_3 \times K_2)$ between any two consecutive S_g, giving a total of $2nm$ tubes, and
(3) of these tubes, $2n-1$ are needed to connect the $2n$ copies S_g to a single surface Σ_0,

Hence the final surface Σ is homeomorphic to a sphere with $2ng + 2mn - (2n-1) = 1 + m(m-1)n$ handles attached. The embedding consists of $4m(m-1)n$ triangles remaining in the original surfaces S_g and $6mn$ quadrilaterals along the $2mn$ tubes. There are $2m+2$ faces incident with any vertex: $2m-2$ triangles and 4 quadrilaterals. The result that $\gamma(K_{m,m,m} \times C_{2n}) \leq 1 + m(m-1)n$ follows.

We now show that $\gamma(K_{m,m,m} \times C_{2n}) \geq 1 + m(m-1)n$.

Take an embedding of a graph with vertices $x_1, x_2, \ldots, x_v$ and a total of f faces. Let f_k denote the total number of faces of size k and let $a_k(x)$ denote the number of faces of size k incident with a given vertex x. Clearly:

$$\deg(x) = a_3(x) + a_4(x) + \cdots, \tag{1}$$

$$kf_k = a_k(x_1) + a_k(x_2) + \cdots + a_k(x_v),$$

and

$$f = f_3 + f_4 + \cdots .$$

For a vertex x define its *face contribution* to be

$$\phi(x) = a_3(x)/3 + a_4(x)/4 + \cdots .$$

Let ϕ_0 denote the average face contribution $(\phi(x_1) + \phi(x_2) + \cdots + \phi(x_v))/v$. Evidently, $f = \phi(x_1) + \phi(x_2) + \cdots + \phi(x_v)$. If a graph has v vertices, e edges then the genus of this embedding can be expressed as: $\gamma = 1 + e/2 - v(1+\phi_0)/2$. Therefore minimizing γ is equivalent to maximizing ϕ_0.

Now let's return to graph $K_{m,m,m} \times C_{2n}$; here, $v = 6mn, e = 6m(m+1)n$. Hence $\gamma(K_{m,m,m} \times C_{2n}) \geq 1 + m(m-1)n$ is equivalent to saying that for any embedding of $K_{m,m,m} \times C_{2n}$ we have $\phi_0 \leq (2m+1)/3$. If we can show this inequality not only for the average face contribution but for the maximal face contribution we are done.

Let $t = a_3(x)$ be the number of triangles incident with a vertex x. Since $\deg(x) = 2m+2$, it follows by Equatio (1) above that $\phi(x) \leq (m+1)/2 + t/12$. Since adjacent vertices in different copies of $K_{m,m,m}$ do not belong to a common triangle, then $0 \leq t \leq 2m$. The case $t = 2m$ is impossible to attain in an embedding in a surface since the triangles would "close-up" and the rotation at that vertex would consist of more than one cycle. If $t \leq 2m-2$ then $\phi(x) \leq (2m+1)/3$ where equality is attained only if $t = 2m - 2$ and the remaining four faces are quadrilaterals. This solution is indeed possible by our 2-patchwork construction in the first half of the proof.

In the remaining case ($t = 2m - 1$) we have $2m - 1$ triangular faces and 3 other faces incident with x. The triangular faces are necessarily consecutive in the rotation around x, since two of the neighbors of x are not in triangles with x. There are 4 sub-cases, concerning the number of quadrilateral faces $q = a_4(x)$. We may have $0 \leq q \leq 3$. By an arithmetical argument we rule out the cases $q = 0$ and $q = 1$. Case $q = 3$ is impossible, since $n > 2$ and one face has two edges projecting to C_{2n}. This leaves us with $q = 2$ and the remaining face either pentagonal (i.e. $a_5(x) = 1$) or hexagonal (i.e. $a_6(x) = 1$). Indeed, if the remaining face has size greater than 6, the value $(2m + 1)/3$ cannot be attained. The value $a_6(x) = 1$ gives us exactly $\phi(x) = (2m+1)/3$. The only way that $a_5(x) = 1$ could occur is to have a consecutive sequence of $2m - 1$ triangles ended on each side by a quadrilateral and the pentagonal face at x has both edges, say xy and xz projecting on C_{2n}. (See Figure 2.) But this is impossible, since the shortest path from y to z not using edge xy and/or xz has length 4. $\square$

3. Small cases

In Theorem 1, $n \geq 3$. The cases $n = 1$ and $n = 2$ turn out to be non-trivial. For $n = 1$ exact results are given; for $n = 2$ we present close bounds.

Theorem 2. *The genus of $K_{m,m,m} \times C_2, m \geq 1$ is given by the formula:*

$$\gamma(K_{m,m,m} \times C_2) = \gamma(K_{m,m,m} \times K_2) = 1 - 2m + m^2 = (m - 1)^2$$

Proof. It is easy to see that the two graphs have the same genus embedding and hence consider K_2 instead of C_2. The proof is simpler but analogous to the proof of Theorem 1. In the construction we only need one patchwork. The surface is composed of two surfaces S_g joined by m tubes, hence, it has genus $(m - 1)^2$. The converse is easy since each vertex must necessarily contribute only $2m - 1$ triangles, and 2 additional quadrilaterals is the best one can hope for. $\square$

Theorem 3. *In general the genus of $K_{m,m,m} \times C_4$ is bounded as follows:*

$$\lceil 2m^2 - 5m/2 + 1 \rceil \leq \gamma(K_{m,m,m} \times C_4) \leq 1 + 2m(m - 1) = 2m^2 - 2m + 1.$$

In particular, $\gamma(K_{1,1,1} \times C_4) = 1$, $\gamma(K_{2,2,2} \times C_4) = 5$ and $\gamma(K_{3,3,3} \times C_4) = 12$.

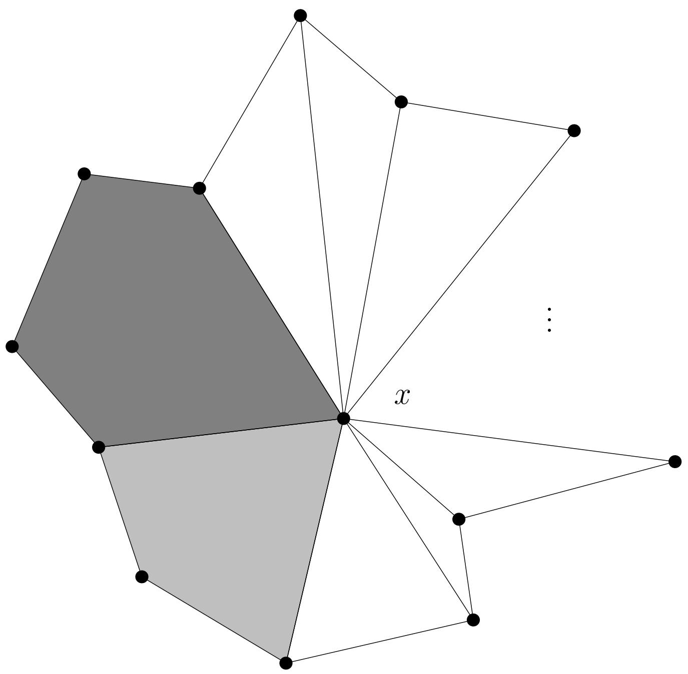

FIGURE 2. A consecutive sequence of $2m-1$ triangles ended on each side by a (shaded) quadrilateral face and a (shaded) pentagonal face at x.

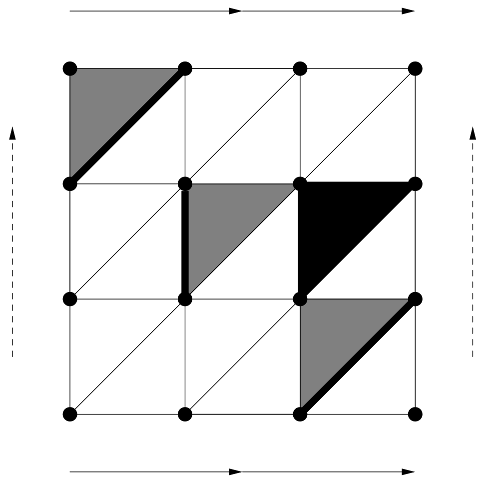

FIGURE 3. Embedding $K_{3,3,3} \times K_2$ and $K_{3,3,3} \times K_2 \times K_2$.

Proof. The upper bound $1 + 2m(m-1)$ is obtained from construction of Theorem 1. The lower bound also follows from the argument in the proof of Theorem 1. Namely, here we cannot rule out the possibility that $\phi_0 = (m+1)/2 + (2m-1)/12 = (8m+5)/12$ that would arise if $2m-1$ triangles and 3 quadrilaterals are incident with each vertex. For $m = 1$ the two bounds coincide. For $m = 2$ the genus is between 4 and 5 and one can easily check that no genus 4 orientable embedding exists. For $m = 3$ the lower bound is $\lceil 11.5 \rceil = 12$. In order to lower the upper bound to 12 we may use the fact that $K_{m,m,m} \times C_4$ is isomorphic to $K_{m,m,m} \times K_2 \times K_2$. We start with the genus embedding of $K_{m,m,m} \times K_2$ described in Theorem 2. It contains a patchwork consisting of 2 triangles and 3 quadrilaterals. (See Figure 3. The three triangles indicate the patchwork that was used for embedding $K_{3,3,3} \times K_2$. The three thick edges mark the 3 selected quadrilaterals and the black triangle comes in two copies to complete the new patchwork of the embedded $K_{3,3,3} \times K_2$.) Using this patchwork one can produce an embedding of $K_{m,m,m} \times K_2 \times K_2$ that has 56 triangular and 30 quadrilateral faces and is therefore an embedding on the surface of genus 12. The same idea could be explored for more general values of m. It would slightly improve the upper bound at least for m that is divisible by 3. $\square$

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