Long initial value test problems from simulations of the Solar System

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Abstract

N-body simulations of the Solar System are a rich source of initial value problems for ordinary differential equations. The problems range from the simulation of two bodies requiring a few milliseconds of CPU time to large simulations of a hundred thousand bodies or more requiring CPU time counted in months. We describe five N-body models and use them with long intervals of integration to compare the integrators `DIVA`, `STEP`, `RKNINT` and `RKSUITE`.

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1 Introduction

Sets of test problems are important in the design, testing and use of integrators for initial value ordinary differential equations. They are used to decide between competing integration formulae and strategies, to augment analysis in selecting values for fine tuning constants, and to show a new integrator to advantage.

Simulations of the Solar System are a rich source of non-stiff test problems. These problems range from small simulations of two bodies over a short interval to long simulations of one hundred thousand or more bodies over a long interval. The well known sets of test problems in DETEST [8] and Hairer, Nørsett and Wanner [12], together with the lesser known set of Fine [9] contain simulations with short intervals of integration.

We describe five models for N-body simulations of the Solar System and use them with long intervals of integration to compare the integrators `DIVA` [14, 15], `STEP` [21], `RKNINT` [3] and `RKSUITE` [4]. The long intervals imply small tolerances and provide a demanding test of efficiency, error propagation and round-off error control. One problem has interpolatory

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g-stops, another is a mixed system of first and second order equations, and a third requires the solution of a small system of nonlinear equations on each evaluation of the acceleration.

`DIVA`, `STEP`, `RKNINT` and `RKSUITE` are general purpose integrators. All are variable-step and all control the estimated local error. `DIVA` is a variable-order Adams integrator for first order, second order and mixed-order problems. It uses PECE mode on the first step, PEC mode for the rest of the start phase, and PECE mode thereafter. The predictor and corrector are orders k and $k+1$ respectively, $1 \leq k \leq 19$, where k may differ from equation to equation. `DIVA` increases the tolerance if the round-off noise is deemed too large. It has an option for solving g-stops. `STEP` is a variable-order, Adams integrator for first order problems. It uses PECE mode with the predictor order k and the corrector order $k+1$, where $1 \leq k \leq 12$ and k is the same for all equations. `STEP` uses Møller's [16, 17] technique to reduce the round-off error when the tolerance is less than $100u$, where u is the unit round-off. `RKNINT` is an explicit Runge-Kutta Nyström (ERKN) integrator. It has a 4-6 ERKN pair and a 10-12 ERKN pair. We used the 10-12 pair because it is more efficient at severe tolerances. The pair uses 17 derivative evaluations per step. `RKSUITE` is an explicit Runge-Kutta integrator. It has three ERK pairs - we used the 7-8 pair because it is more efficient than the other pairs at severe tolerances. The pair uses 13 derivative evaluations per step.

The numerical comparisons were performed in double precision Fortran on a single processor of an SGI Origin 2000 with compiler options `-64`, `-mips4`, `-r10000` and `-O3`. The unit round-off for the Origin 2000 is 2.2×10^{-16}. The global error estimates were made at N evenly spaced values on the interval of integration, where N was either 100 or 1000. The reference solutions for the error estimates were calculated in quadruple precision with a tolerance at least 10^4 times smaller than that used for the double precision simulations. Quadruple precision on the Origin 2000 is approximately 20 times slower than double precision. This, together with the smaller tolerances, meant the reference solutions required at least 50 times as much CPU time as the double precision solutions. Despite this slowness, we used intervals up to one million years long.

The initial conditions and physical parameters for each problem are listed in the Appendix. To reduce the number of values listed, the initial conditions for a planet are the same for every problem the planet occurs in, and the barycentre of the Earth and Moon is used in place of the Earth.

2 Jovian problem

The Jovian planets Jupiter, Saturn, Uranus and Neptune are considerably heavier than the remaining five planets and their orbits collectively sweep out more of the Solar System. These planets and the Sun thus drive much of the dynamics of the Solar System and are included in many simulations, strong motivation for using a simulation of the five bodies as a test problem. We refer to this as the Jovian problem.

Let $\mathbf{r}_i$ denote the position of the i-th body in the Jovian problem, where the coordinate system is three-dimensional Cartesian with the origin at the barycentre of the Solar

System. The equations of motion for the Jovian problem are

$$\ddot{\mathbf{r}}_i(t) = \sum_{j=1, j\neq i}^{5} \frac{m_j(\mathbf{r}_j(t) - \mathbf{r}_i(t))}{||\mathbf{r}_j(t) - \mathbf{r}_i(t)||_2^3}, \quad i = 1, \ldots, 5, \tag{1}$$

where the dot operator denotes differentiation with respect to time t, $||\cdot||_2$ is the L_2 norm, and $\mu_j = Gm_j$, G being the gravitational constant and m_j the mass of the j-th body. The terms on the right of (1) model the Newtonian gravitational attraction between point-masses.

Except for the omission of Pluto, equations (1) are those for problem C5 in non-stiff `DETEST` [8]. C5 is an easy problem to solve because the interval of integration is short. C5 becomes a demanding problem when the interval of integration is long, such as 10^9 years, the length in some simulations (see for example [11]).

Since the origin of the Cartesian system is the barycentre, we must have $\sum_{i=1}^{5} \mu_i \mathbf{r}_i = 0$ and $\sum_{i=1}^{5} \mu_i \dot{\mathbf{r}}_i = 0$ which means the position and velocity of the Sun are known if the position and velocity of the Jovian planets is. This eliminates the equations of motion for the Sun, reducing the CPU required to evaluate the right side of (1) and possibly increasing the accuracy of the solution. The two vector identities for the barycentre can no longer be used to assess the accuracy of the solution.

The total energy $E(t)$ for the Jovian problem is

$$E(t) = \frac{1}{2}\left[\sum_{i=1}^{5}\left(m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i - \sum_{j=1, j\neq i}^{5} \frac{Gm_i m_j}{||\mathbf{r}_j - \mathbf{r}_i||_2}\right)\right]. \tag{2}$$

$E(t)$ is conserved, permitting an assessment of the accuracy of the numerical solution.

We integrated the Jovian problem for one million years using the four integrators with local error tolerances (`TOL`) of 10^{-13} and 10^{-14}. The motion of the Sun was found by integration. Figure 1 plots the L_2 norm of the global error and Table 1 lists information about the error growth and cost of integration. We have not distinguished between the evaluation of (1) written as a second order system and as a first order system because there is little difference in CPU time.

	`TOL`	No. of eval.	No. of steps	Av. h	CPU time	a	b	RMS	r
`DIVA`	10^{-13}	12,603,612	6,301,408	58.0 d	262 sec.	-12.4	1.66	0.164	0.9743
`DIVA`	10^{-14}	13,183,559	6,591,190	55.4 d	282 sec.	-13.1	1.80	0.074	0.9949
`STEP`	10^{-13}	23,563,296	11,734,218	31.1 d	401 sec.	-13.6	1.94	0.035	0.9991
`STEP`	10^{-14}	30,398,758	15,155,879	24.1 d	516 sec.	-15.5	2.04	0.120	0.9907
`RKNINT`	10^{-13}	20,042,551	1,767,174	208.7 d	264 sec.	-16.9	1.87	0.048	0.9982
`RKNINT`	10^{-14}	38,411,145	2,259,464	161.7 d	338 sec.	-15.9	1.62	0.066	0.9954
`RKSUITE`	10^{-13}	88,684,619	6,796,438	53.7 d	899 sec.	-15.2	2.00	0.015	0.9998
`RKSUITE`	10^{-14}	122,037,921	9,176,396	39.8 d	1234 sec.	-17.0	2.00	0.054	0.9980

Table 1: Jovian problem: `TOL` is the tolerance, a and b are the coefficients of the power law, RMS is the root mean square of the fit and r the coefficient of correlation.

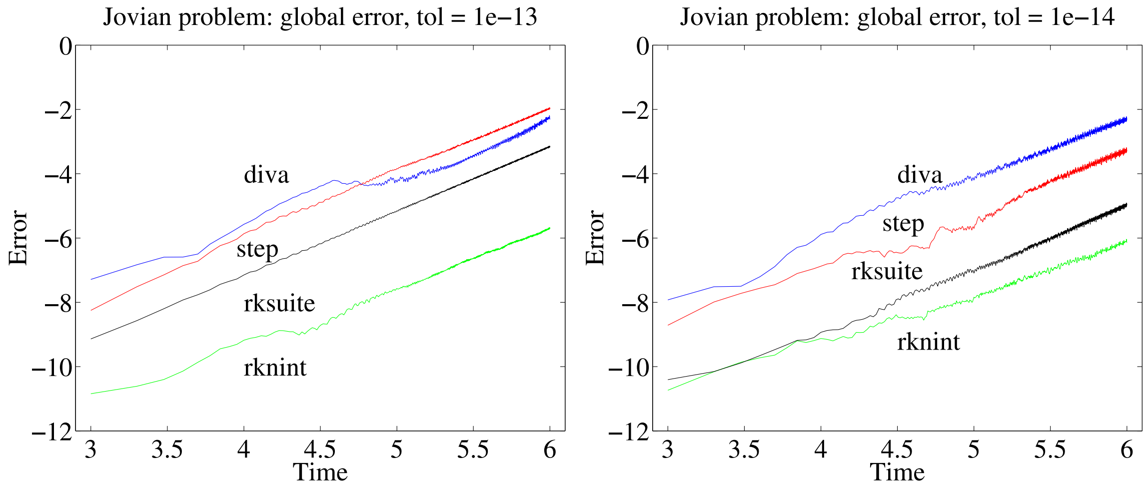

Figure 1: Jovian problem. Log-log graph (base 10) of the L_2 norm of the global error as a function of time t, where t is in years. Left - tolerance of 10^{-13}, right - tolerance of 10^{-14}.

We first comment on the results for $\texttt{TOL} = 10^{-13}$. `DIVA` used fewer derivative evaluations and less CPU time than `STEP`. Since the solutions for `DIVA` and `STEP` are of similar accuracy, the number of evaluations and CPU time imply `DIVA` was more efficient than `STEP`. `RKNINT` used a similar amount of CPU time as `DIVA` and produced a solution three orders of magnitude more accurate. Hence, if CPU time is used as the measure of work, `RKNINT` was more efficient than `DIVA`.

When we used `DIVA` with $\texttt{TOL} = 10^{-14}$, `DIVA` increased the tolerance to 3.29×10^{-14} at $t = 633.60$ years. This partly explains why the solution for `DIVA` with $\texttt{TOL} = 10^{-14}$ is only marginally more accurate than the solution with $\texttt{TOL} = 10^{-13}$. We could have forced `DIVA` to continue using $\texttt{TOL} = 10^{-14}$, but decided against this because we wanted to use the default settings of the integrators.

The solution for `RKNINT` with $\texttt{TOL} = 10^{-14}$ was also only marginally more accurate than for $\texttt{TOL} = 10^{-13}$. We did integrations with tolerances in the range $[10^{-13}, 10^{-10}]$ and found the global error was approximately proportional to the tolerance, evidence the round-off error in the solution with $\texttt{TOL} = 10^{-14}$ was significant.

The global error for `STEP` changed by an order of magnitude, suggesting the round-off error at $\texttt{TOL} = 10^{-14}$ was insignificant, possibly because `STEP` used Møller's technique to control the round with $\texttt{TOL} = 10^{-14}$ (but not with $\texttt{TOL} = 10^{-13}$ since $10^{-13} > 100u$). The solution for `RKSUITE` with $\texttt{TOL} = 10^{-14}$ was over an order of magnitude more accurate than the solution for $\texttt{TOL} = 10^{-13}$, again suggesting round-off error was insignificant.

We used linear least squares to fit the power law $10^a t^b$ to the the norm of the global error. The four right-most columns in Table 1 list a, b, the root mean square (RMS) and the coefficient of correlation r. The exponent b for `DIVA` and `RKSUITE` has a large variation between tolerances, indicating there was more than one contribution, including round-off error, to the global error. In contrast, there is only a small variation for `STEP` and no variation to two decimal places for `RKSUITE`, suggesting the global error for both integrators is dominated by truncation error.

The interval of integration for the simulation in [11] was 10^9 years. If we assume the power laws in Table 1 hold for $t > 10^6$ years and extrapolate to 10^9 years, the solution for `DIVA`, `STEP` and `RKSUITE` would have no digits of accuracy and that for `RKNINT` two digits.

Figure 2 are plots of the relative error in the energy, calculated as $(GE(t) - GE(0))/(GE(0))$ to avoid knowing G and μ separately. The error was calculated in quadruple precision to reduce the effects of round-off error. The plots confirm the relative accuracy of the solutions.

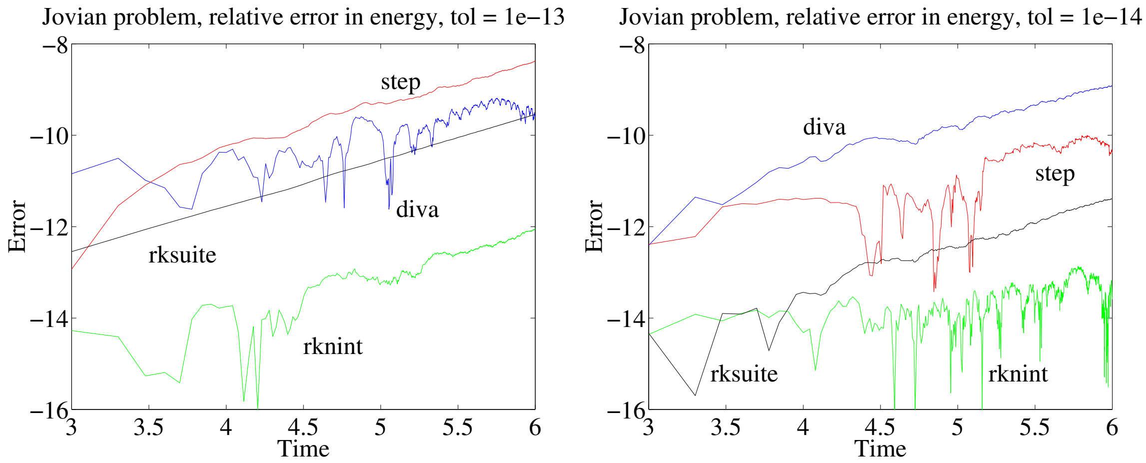

Figure 2: Jovian problem. Log-log graph (base 10) of the relative error in the energy as a function of time t, where t is in years. Left - tolerance of 10^{-13}, right - tolerance of 10^{-14}.

3 Plutonian problem

In 1965, Cohen and Hubbard [6] investigated how close Pluto came to Neptune by simulating the motion of the Sun, the Jovian planets and Pluto over 120,000 years, with all six bodies interacting with one another through Newtonian gravitational forces. They concluded that despite Pluto crossing Neptune's orbit twice a period, there was, to quote Cohen and Hubbard, *a remarkable libration of the orbit of Pluto relative to that for Neptune such that the distance between the bodies is never less than 18 a.u.*.[1]

The simulation was performed on the Naval Ordnance Research Calculator (NORC) using a high order Cowell method with a stepsize of 40 days. The NORC was commissioned in December 1954 and remained in service until 1968. It could perform 15000 operations per second and was for a time the most powerful computer in the world, once calculating π to 3089 digits in 13 minutes. The Cohen and Hubbard simulation took 80 hours. The simulation would take approximately one minute on a modern, fast workstation.

[1] a.u. denotes Astronomical Unit - one astronomical unit is approximately 149 million kilometres.

Longer simulations of this problem, which we call the Plutonian problem, have been performed, including a one million year simulation by Cohen, Hubbard and Oesterwinter [7] in 1973, a five million year simulation by Kinoshita and Nakai [13] in 1984, and a 200 million year simulation on the Digital Orrery [1] in 1986 by Applegate, Douglas, Gürsel, Sussman and Wisdom [2].

The equations of motion for the Plutonian problem are those for Jovian problem, except there is six bodies instead of five. The Plutonian problem is distinguished from the Jovian problem by the need to find the minimum of the distance between Pluto and Neptune. Cohen and Hubbard concluded the distance is almost periodic with a period of approximately 500 years, with three local minima and three local maxima per period. They also concluded the near periodicity and the number of local extrema per period holds for intervals far greater than 120,000 years. These conclusions were supported by the later simulations.

If $d(t)$ is the distance at time t, the local extrema for a simulation can be found by solving a sequence of interpolatory g-stops of the form $d'(t) = 0$ or, if local maxima are excluded, $d'(t) = 0$ subject to $d''(t) > 0$.

`DIVA` has an option for solving g-stops. We specified the g-stop $d'(t) = 0$ and integrated for 1.2 million years. `DIVA` evaluated $d'(t)$ 7,655,883 times, including once for each of the 7,562,202 steps to check for a solution of $d'(t) = 0$, and solved 14567 g-stops. `DIVA` took 392 seconds for the integration, and eight seconds for the evaluations of $d'(t)$ and the solution of the g-stops. `DIVA` found six local extrema each period.

4 Nine Planets problem

The next problem has the Sun and nine planets interacting through Newtonian gravitational forces. The equations of motion are those for the Jovian and Plutonian problems, except there is ten bodies instead of five.

We did an integration of 1000 years using the four integrators with `TOL` $= 10^{-13}$. Figure 3 and Table 2 summarise the results of the integration.

	`TOL`	No. of eval.	No. of steps	Av. h	CPU time	a	b	RMS	r
`DIVA`	10^{-13}	879,205	439,586	0.83 d	39 sec.	-11.8	2.13	0.168	0.9833
`DIVA`	10^{-14}	924,721	462,343	0.79 d	41 sec.	-12.6	2.04	0.299	0.9461
`STEP`	10^{-13}	1,198,585	598,065	0.61 d	49 sec.	-10.4	1.98	0.080	0.9956
`STEP`	10^{-14}	1,472,867	735,311	0.50 d	60 sec.	-10.4	1.61	0.069	0.9950
`RKNINT`	10^{-13}	1,940,840	114,167	3.20 d	48 sec.	-13.4	1.30	0.067	0.9929
`RKNINT`	10^{-14}	2,479,519	145,854	2.50 d	61 sec.	-14.2	1.73	0.083	0.9937
`RKSUITE`	10^{-13}	5,180,501	392,632	0.93 d	138 sec.	-13.5	1.89	0.063	0.9970
`RKSUITE`	10^{-14}	7,070,929	532,440	0.69 d	188 sec.	-14.0	2.00	0.075	0.9962

Table 2: Nine Planets problem: `TOL` is the tolerance, a and b are the coefficients of the power law, RMS is the root mean square of the fit and r the coefficient of correlation.

On planetary simulations, the stepsize for a variable-step integrator is driven by the shortest orbital period present. The shortest orbital period for the Jovian and Nine

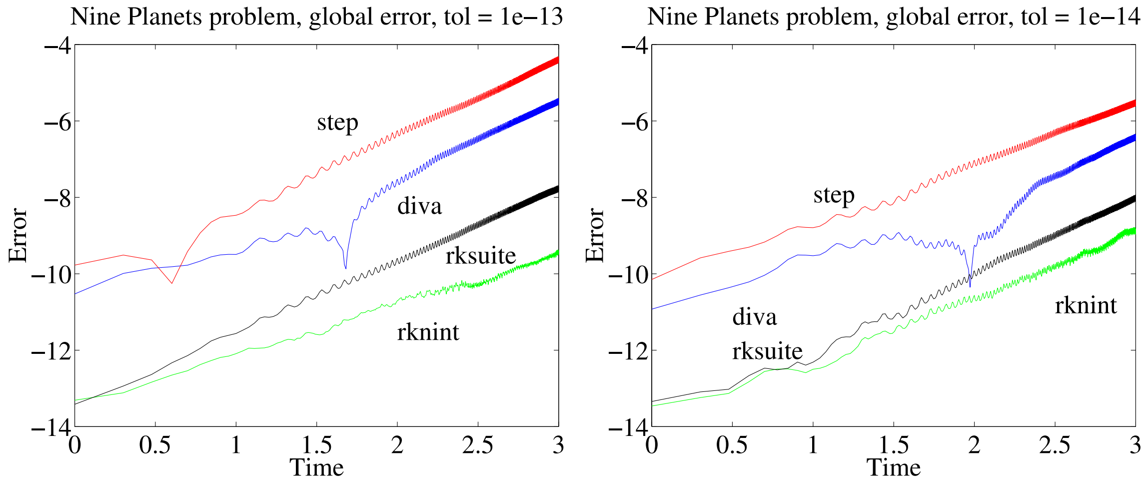

Figure 3: Nine planets problem. Log-log graph (base 10) of the L_2 norm of the global error as a function of time t, where t is in years. Left - tolerance of 10^{-13}, right - tolerance of 10^{-14}.

Planets problems are 4331 days (Jupiter) and 88 days (Mercury) respectively. Hence, the average stepsize for the Nine Planets problem should be about 50 times smaller than that for the Jovian problem. The average stepsize with $\texttt{TOL} = 10^{-13}$ was 70 times smaller for `DIVA`, 51 times smaller for `STEP`, 65 times smaller for `RKNINT` and 43 times smaller for `RKSUITE`, in reasonable agreement with expectations. The average stepsize for $\texttt{TOL} = 10^{-14}$ showed similar agreement.

The relative accuracy of the four integrators differs in two ways from that for the Jovian problem: `DIVA` is more accurate relative to `STEP` except in the first five years of the simulation, and `RKSUITE` is more accurate relative to `RKNINT`. These changes mirror, at least qualitatively, the changes in the CPU time and number of function evaluations. For example, on the Jovian problem with $\texttt{TOL} = 10^{-13}$, `DIVA` used 0.65 the CPU time of `STEP`, and 0.80 of the CPU time on the Nine Planets problem.

5 Spin Axis problem

Quinn, Tremaine and Duncan [20] performed a three million year integration of the planets and the Earth's spin axis. The equations of motion included corrections for the dominant parameterised post-Newtonian terms and the quadpole moment of the Earth-Moon system.

Quinn, Tremaine and Duncan used three-dimensional Cartesian coordinates with the origin at the centre of the Sun. The planets were ordered Mercury, Venus, Earth-Moon barycentre, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto, where the Earth-Moon barycentre is used in place of the Earth. The equations of motion for the i-th planet were

$$\ddot{\mathbf{r}}_i = -\frac{G(m_s+m_i)}{r_i^3}\mathbf{r}_i + \sum_{j=1,j\neq i} \frac{Gm_j}{||\mathbf{r}_j-\mathbf{r}_i||^3}(\mathbf{r}_j-\mathbf{r}_i) - \sum_{j=1,j\neq i}^{9} \frac{Gm_j\mathbf{r}_j}{r_j^3} +$$

$$\frac{Gm_s}{r_i^3}\mathbf{r}_i\left[2(\beta+\gamma)\frac{Gm_s}{c^2 r_i}-\gamma\frac{||\dot{\mathbf{r}}_i||_2^2}{c^2}\right]+ \tag{3}$$
$$(2+2\gamma)\frac{Gm_s}{c^2 r_i^3}(\dot{\mathbf{r}}_i\cdot\mathbf{r}_i)\dot{\mathbf{r}}_i-\frac{3}{4}\delta_{i3}\frac{Gm_s}{r_i^3}\mathbf{r}_i\left(\frac{R}{r_i}\right)^2 f\frac{m_3 m_l}{(m_3+m_l)^2}$$

where m_s, m_3 and m_l are the mass of the Sun, Earth and Moon respectively, m_j, $j = 1, 2, 4, \ldots, 9$ are the masses of the remaining planets, c is the speed of light, $r_k = ||\mathbf{r}_k||_2$, δ_{i3} is the Kronecker delta, and R and f are constants. The first three terms in (3) model point-mass interactions, the fourth and fifth the relativistic corrections and the sixth the quadpole moment of the Earth-Moon system.

The equations of motion for the Earth's spin axis $\widehat{\mathbf{a}}$ were

$$\dot{\widehat{\mathbf{a}}} = K(t)\left[\frac{2}{r_3^3}(\widehat{\mathbf{r}}_3\cdot\widehat{\mathbf{a}})(\widehat{\mathbf{r}}_3\times\widehat{\mathbf{a}})-B_l(t)(\widehat{\mathbf{b}}\cdot\widehat{\mathbf{a}})(\widehat{\mathbf{b}}\times\widehat{\mathbf{a}})\right]+\frac{3Gm_s}{2r_3^3c^2}(\mathbf{r}_3\times\dot{\mathbf{r}}_3)\times\widehat{\mathbf{a}}-\mathbf{c}\frac{d\epsilon}{dt}, \tag{4}$$

where $K(t) = K_0\,[1 + d_1(t - t_0)]$, $B_l(t) = B_{l,0}\,[1 + d_2(t - t_0)]$, $\mathbf{c} = [\widehat{\mathbf{b}} - (\widehat{\mathbf{a}}\cdot\widehat{\mathbf{b}})\widehat{\mathbf{a}}](\sin\epsilon)^{-1}$ and $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{r}}_3$ are unit vectors. The subscript 3 in (4) refers to the Earth-Moon barycentre and $\widehat{\mathbf{b}} = \widehat{\mathbf{r}}_3\times\widehat{\dot{\mathbf{r}}}_3$. The scalar ϵ is the obliquity of the ecliptic. The four terms in (4) model, respectively, torques due to the Sun and Moon, the geodetic precession and the effect of the Moon-Earth tides.

The present problem differs from the previous three problems in two important ways: the equations are mixed order and the right side of the second-order equations is of the form $f(t, y, y')$ and not $f(y)$.

Quinn, Tremaine and Duncan integrated equations (3) using the order 12 symmetric multistep method of Quinlan and Tremaine [19] and equations (4) using an order 13 Adams-Bashforth-Moulton method with one correction. The stepsize was 0.75 days for all equations. Techniques were used to control the round-off error. The simulation was run on a Silicon Graphics 4D-25 workstation and took about 65 days of CPU time.

We integrated the Spin Axis problem for 1000 years using `DIVA`, `STEP` and `RKSUITE` with `TOL` $= 10^{-13}$ and `TOL` $= 10^{-14}$, and using `RKSUITE` with `TOL` $= 10^{-10}$. The last integration was included to permit an easy comparison with `DIVA` and `STEP`. Figure 4 and Table 3 summarise the results of the integration.

	`TOL`	No. of eval.	No. of steps	Av. h	CPU time	a	b	RMS	r
`DIVA`	10^{-13}	845,946	422,953	0.86 d	68 sec.	-11.5	2.13	0.074	0.9963
`DIVA`	10^{-14}	892,631	446,314	0.82 d	71 sec.	-11.5	1.63	0.098	0.9888
`STEP`	10^{-13}	1,199,641	598,613	0.61 d	85 sec.	-10.5	2.01	0.076	0.9956
`STEP`	10^{-14}	1,474,773	736,273	0.50 d	105 sec.	-12.0	2.15	0.084	0.9953
`RKSUITE`	10^{-10}	2,184,595	163,614	2.23 d	128 sec.	-10.0	1.91	0.067	0.9963
`RKSUITE`	10^{-13}	5,170,849	392,088	0.93 d	304 sec.	-13.5	1.91	0.064	0.9965
`RKSUITE`	10^{-14}	7,056,690	531,365	0.69 d	414 sec.	-14.5	2.18	0.102	0.9932

Table 3: Spin Axis problem. `TOL` is the tolerance, a and b are the coefficients of the power law, RMS is the root mean square of the fit and r the coefficient of correlation.

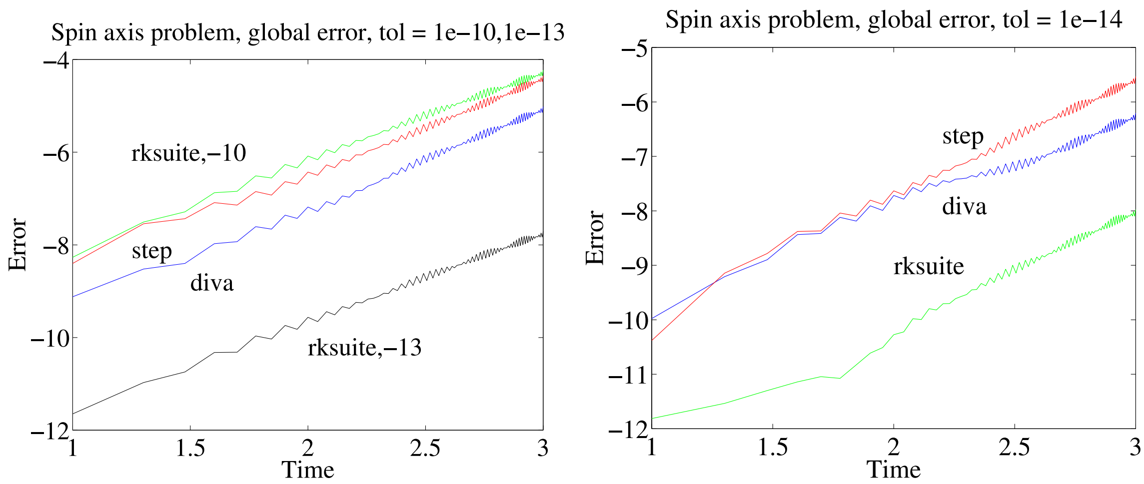

Figure 4: Spin Axis problem. Log-log graph (base 10) of the L_2 norm of the global error as a function of time t, where t is in years. Left - tolerance of 10^{-13}, right - tolerance of 10^{-14}. The plots labelled `rksuite,-10` and `rksuite,-13` are for `RKSUITE` with `TOL` $= 10^{-10}$ and 10^{-13} respectively.

The number of derivative evaluations for each integration is close to that for the Nine Planets problem, evidence the parameterised post-Newtonian terms and the quadpole of the Earth-Moon system do not drive the stepsize selection. The relative accuracy of the solutions is similar to that for the Nine Planets problem, except for `TOL` $= 10^{-14}$ where the solution for `DIVA` is less accurate relative to that for `STEP`. The global error for `RKSUITE` with `TOL` $= 10^{-10}$ is similar to that for `STEP` with `TOL` $= 10^{-13}$; the CPU time is 22 % larger and the number of function evaluations 82% larger. Except for `DIVA` with `TOL` $= 10^{-14}$, the exponent b for the power law fit shows less variation than for the previous problems. This, along with b being near two, suggests the global error is dominated by the truncation error. An extrapolation of the power law for `RKSUITE` gives a global error of about 10^{-2} after one million years.

6 DE102 problem

Newhall, Standish and Williams [18] used a detailed parameterised post-Newtonian model for the orbital motion of the Sun and planets. The equations of motion for the i-th planet, $i = 1, \ldots, 9$, were

$$\ddot{\mathbf{r}}_i = \left(\sum_{j=1, j\neq i}^{10} \rho_{ij} \frac{\mu_j(\mathbf{r}_j - \mathbf{r}_i)}{||\mathbf{r}_j - \mathbf{r}_i||_2^3} \right) + A_i, \quad i = 1, \ldots, 9, \tag{5}$$

where $j = 10$ refers to the Sun.

The factor ρ_{ij} is

$$1 - \frac{2(\beta+\gamma)}{c^2} \sum_{k=1,k\neq i}^{10} \frac{\mu_k}{||\mathbf{r}_i - \mathbf{r}_k||_2} - \frac{2\beta - 1}{c^2} \sum_{k=1,k\neq j}^{10} \frac{\mu_k}{||\mathbf{r}_j - \mathbf{r}_k||_2} + \gamma \frac{||\dot{\mathbf{r}}_i||_2^2}{c^2} + (1+\gamma)\frac{||\dot{\mathbf{r}}_i||_2^2}{c^2} -$$

$$\frac{2(1+\gamma)}{c^2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j - \frac{3}{2c^2}\left[\frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot \dot{\mathbf{r}}_j}{||\mathbf{r}_i - \mathbf{r}_j||_2}\right]^2 + \frac{1}{2c^2}(\mathbf{r}_j - \mathbf{r}_i)\cdot \ddot{\mathbf{r}}_j,$$

where $\ddot{\mathbf{r}}_j$ is the Newtonian acceleration of the j-th body by the remaining nine bodies.

The term A_i is

$$\frac{1}{c^2}\sum_{j=1,j\neq i}^{10} \frac{\mu_j}{||\mathbf{r}_j - \mathbf{r}_i||_2^3} \left\{[\mathbf{r}_i - \mathbf{r}_j]\cdot[(2+2\gamma)\dot{\mathbf{r}}_i - (1+2\gamma)\dot{\mathbf{r}}_j]\right\}(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + \frac{3+4\gamma}{2c^2}\sum_{j=1,j\neq i} \frac{\mu_j \ddot{\mathbf{r}}_j}{||\mathbf{r}_i - \mathbf{r}_j||_2}$$

The barycentre of the ten bodies was defined by the vector identities

$$\sum_{i=1}^{10} \mu_i^\star \mathbf{r}_i = 0, \quad \sum_{i=1}^{10} \mu_i^\star \dot{\mathbf{r}}_i = 0, \tag{6}$$

where

$$\mu_i^\star = \mu_i \left\{1 + \frac{1}{2c^2}||\dot{\mathbf{r}}_i||_2^2 - \frac{1}{2c^2}\sum_{j=1,j\neq i}^{10} \frac{\mu_j}{||\mathbf{r}_i - \mathbf{r}_j||_2}\right\}. \tag{7}$$

Since $\mu_j^\star$ depends on $\mathbf{r}_j$, the two identities (6) form a system of six nonlinear equations for the Sun's position and velocity.

We integrated the DE102 problem for 1000 years using `DIVA`, `STEP` and `RKSUITE`. Figure 5 and Table 4 summarise the results of the integrations. As in [18], we solved (6) and (7) using fixed-point iteration.

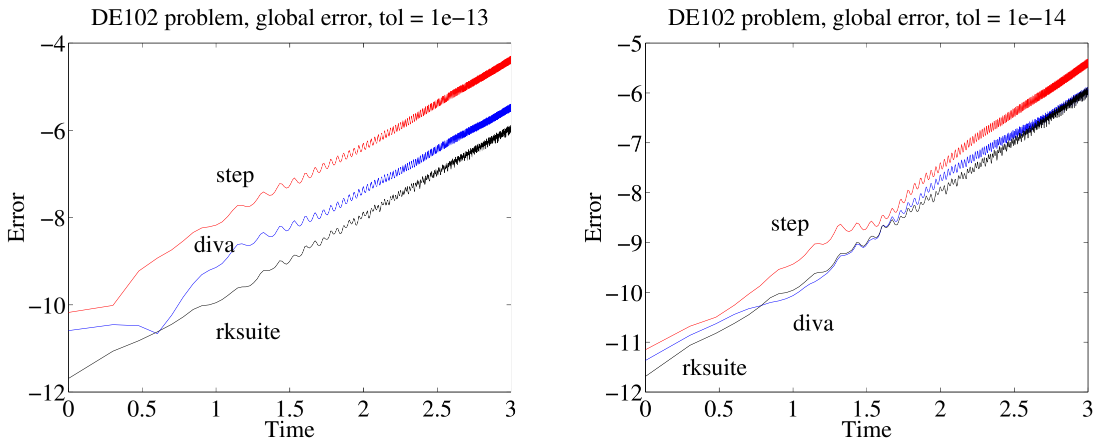

Figure 5: DE102 problem. Log-log graph (base 10) of the L_2 norm of the global error as a function of time (t), where t is in years. Left - tolerance of 10^{-13}, right - tolerance of 10^{-14}.

	TOL	No. of eval.	No. of steps	Av. h	CPU time	a	b	RMS	r
DIVA	10^{-13}	874,151	437,057	0.84 d	299 sec.	−11.1	1.86	0.077	0.9954
DIVA	10^{-14}	919,280	459,620	0.79 d	314 sec.	−11.6	1.90	0.091	0.9938
STEP	10^{-13}	1,196,640	597,129	0.61 d	404 sec.	−10.2	1.92	0.072	0.9962
STEP	10^{-14}	1,468,058	732,948	0.50 d	495 sec.	−11.6	2.06	0.081	0.9958
RKNINT	10^{-13}	5,179,968	392,627	0.93 d	1690 sec.	−11.9	1.98	0.057	0.9977
RKNINT	10^{-14}	7,057,458	531,389	0.69 d	2303 sec.	−11.9	1.97	0.057	0.9977

Table 4: DE102 problem: TOL is the tolerance, a and b are the coefficients of the power law, RMS is the root mean square of the fit and r the coefficient of correlation.

The number of derivative evaluations for each integrator is close to that for the Nine Planets problem. With TOL $= 10^{-13}$ the accuracy of the solutions rank as for the Spin Axis problem, but the difference in accuracy is less. With TOL $= 10^{-14}$, the solutions for the three integrators are of similar accuracy. RKSUITE used over four times as much CPU time as STEP and over five times as much CPU time as DIVA.

7 Discussion

We described five N-body models for simulations of the Solar System and used the models with long intervals of integration to compare the performance the multistep integrators DIVA and STEP, and the one-step integrators RKSUITE and RKNINT. The comparisons were done in double precision and most comparisons were for local error tolerances of 10^{-13} and 10^{-14}.

The solutions for DIVA were usually at least as accurate as the solutions for STEP. Since this accuracy was achieved using less CPU time and fewer derivative evaluations, we recommend, with one exception, DIVA in preference to STEP. The exception is when the round-off error, if not controlled, would contribute noticeably to the global error - STEP has better tolerance proportionality than DIVA. For problems of the form $\ddot{y} = f(y)$, the average stepsize for RKNINT was about three times larger than that for DIVA, and the solutions for RKNINT were two to three orders of magnitude more accurate than those for DIVA. These results, together with the lower overhead of RKNINT, meant RKSUITE required less CPU time than DIVA to achieve the same global error. This advantage decreases as the cost per component of evaluating the derivative increases, and for sufficiently high cost, DIVA would be more efficient than RKNINT. Nevertheless, RKNINT would be more efficient on many simulations, especially those containing massless particles since the cost per component for these simulations depends only weakly on the total number of particles.

Our testing showed the limitations of double precision on long integrations. For example, on the simulation of the Sun and Jovian planets, an extrapolation of our results suggests the global error at 10^9 years for the four integrators would be at least 10^{-2}. This estimate agrees well with that calculated for Störmer methods [10] and illustrates the need for fast extended precision.

Appendix

Table 5 lists the initial conditions for the Spin Axis problem. Row i, $i = 1, \ldots, 9$, is the initial position for the i-th body, with the third body the Earth-Moon system. The tenth row is the initial position for the Earth's spin axis and row $10 + i$, $i = 1, \ldots, 9$, is the initial velocity for the i-th body.

The initial conditions for the Jovian problem are found by taking the position and velocity of the Sun and Jovian planets listed in Table 5 and transforming to barycentric coordinates. The transformation is done by first using the expressions

$$\mathbf{r}_b = \frac{\sum_{j=1}^{5} \mu_j \mathbf{r}_j}{\sum_{j=1}^{5} \mu_j}, \quad \dot{\mathbf{r}}_b = \frac{\sum_{j=1}^{5} \mu_j \dot{\mathbf{r}}_j}{\sum_{j=1}^{5} \mu_j},$$

to find the position $\mathbf{r}_b$ and velocity $\dot{\mathbf{r}}_b$ of the barycentre and then subtracting $\mathbf{r}_b$ and $\dot{\mathbf{r}}_b$ from the positions and velocities listed in Table 5.

The initial conditions for the Plutonian and Nine Planets problems are found in the same way as for the Jovian problem, except the number of bodies is six and ten respectively. The initial conditions for the DE102 problem are also found in the same way, except $\mu_j^\star$ is used in place of μ_j.

	x	y	z
Mercury	0.343926450169642	0.0456154799533135	-0.0109240372119325
Venus	0.142965184343246	0.647005066033887	0.28248240066038
Earth	-0.136364695954795	0.893397922857	0.387458344639337
Mars	-1.36983397618342	0.843135248017904	0.42383290661143
Jupiter	3.34936422369601	-3.47376144901258	-1.5721496863938
Saturn	-8.97250506828211	2.27968200813286	1.33033860971146
Uranus	-1.00300399532732	17.3235084732718	7.60482504641591
Neptune	-29.1945807386112	-7.71928519199847	-2.42724537828877
Pluto	-26.2336507820155	20.5619754200559	14.4445571277807
Spin Axis	1.51920829e-07	-2.57060482e-06	0.999999999996684
Mercury	-0.00846653204500986	0.0256146053072818	0.0145868453362396
Venus	-0.0198938122793425	0.00311311946611859	0.0026594458057458
Earth	-0.0173199988485296	-0.00224430473176756	-0.000973361115758044
Mars	-0.00738456383127117	-0.0094773586392742	-0.00415165513666213
Jupiter	0.00558564314958231	0.00496226113722251	0.00199227692673937
Saturn	-0.00185825195699671	-0.00498385858141744	-0.00198025741280725
Uranus	-0.00395525416301772	-0.000375913785112941	-0.000108849991287794
Neptune	0.000820748057818085	-0.00277209825958023	-0.00115611603043592
Pluto	-0.00131588869828116	-0.00262012820549352	-0.000427083355026298

Table 5: The initial conditions for the Spin Axis problem.

Table (6) lists $\mu = Gm$ for the planets, where the third planet is the Earth-Moon system, and μ_s is μ for the Sun. We used $\mu_s = (0.01720209895)^2$ AU^3 d^{-2}, where AU and d denote Astronomical Unit and day respectively. Table 7 lists the value of the parameters for the Spin Axis problem.

Mercury	Venus	Earth-Moon	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
$\frac{\mu_s}{6023600}$	$\frac{\mu_s}{408523.5}$	$\frac{\mu_s}{328900.53}$	$\frac{\mu_s}{3098710}$	$\frac{\mu_s}{1047.355}$	$\frac{\mu_s}{3498.5}$	$\frac{\mu_s}{22869}$	$\frac{\mu_s}{19314}$	$\frac{\mu_s}{3000000}$

Table 6: The μ for the planets.

$c = 173.144633484$ AU d^{-1}	$f = 0.9473$	$K_0 = 2.306559456 \times 10^{-7}$ d^{-1}
$R = 0.0025696$ AU	$\beta = 1$	$d_1 = -6.39235695008 \times 10^{-8}$ d^{-1}
$d_2 = -7.9488 \times 10^{-13}$ d^{-1}	$\gamma = 1$	$\epsilon = 23.43929 - 3.60288 \times 10^{-14}(t - t_0)$
$m_3/m_l = 81.3007$	$B_{l,0} = 2.161225$	

Table 7: The parameters for the Spin Axis problem.

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