Metrisability of Manifolds in Terms of Function Spaces

David Gauld and Frédéric Mynard

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1 Metrisability of manifolds and topological properties of function spaces

It is an obvious and well known fact that several topological properties that are different in general may collapse in the presence of additional properties. These additional properties may be algebraic (e.g., a topological group is metrisable if and only if it is first-countable) or purely topological. For instance, a large collection of topological properties which are different in general turns out to be all equivalent to metrisability for topological manifolds([1]) [6]. Another class of important topological objects in which several different topological properties may collapse is that of function spaces. Let $C_p(X)$ and $C_k(X)$ denote the set of real-valued continuous functions on a topological space X endowed with the topology of pointwise convergence and with the compact-open topology respectively. In general, Fréchetness implies sequentiality and sequentiality implies k-ness, but none of these implications can be reversed. However, these three properties coincide for function spaces like $C_p(X)$ and $C_k(X)$ [16].

In this paper, we show that (not surprisingly) even more properties collapse for function spaces over a topological manifold. Often more surprising are the new criteria of metrisability of a manifold that we derive, in terms of the topological properties of the function spaces over this manifold. Before we state these new criteria, we need to define the notions involved.

Definition 1. *A topological space X is*

1. metaLindelöf *if each open cover has a point-countable open refinement;*
2. Cečh-complete *if it is a G_δ-subset of a compact space;*
3. pseudocomplete *provided that it has a sequence $\langle \mathcal{B}_n \rangle$ of* π-bases *($\mathcal{B} \subset 2^X$ is a π-base if every non-empty open subset of X contains some member of $\mathcal{B}$) such that if $B_n \in \mathcal{B}_n$ and $\overline{B_{n+1}} \subset B_n$ for each n, then $\bigcap_{n\in\omega} B_n \neq \varnothing$;*
4. cosmic *([8, page 259]) if it has a countable* network, *i.e., a countable collection $\mathcal{N}$ such that if $x \in U$ with U open then $x \in N \subset U$ for some $N \in \mathcal{N}$;*
5. *a* σ-space *if it has a σ-discrete (that is, a countable union of discrete families) network;*
6. *a* (strong) Σ-space *if there exists a σ-locally finite (i.e., countable union of locally finite families) family $\mathcal{F}$ and a cover $\mathcal{C}$ by closed countably compact (compact) sets such that whenever $C \in \mathcal{C}$ and U is an open set that contains C, there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$;*

[1] i.e., a connected, Hausdorff space which is locally homeomorphic to euclidian space.

7. *a* q-space *if each point admits a sequence of neighbourhoods* Q_n *such that* $x_n \in Q_n$ *implies that* $\langle x_n \rangle$ *has cluster points;*

8. *of* point-countable type *if each point admits a sequence of neighbourhoods* Q_n *such that every filter that meshes every* Q_n *has cluster points.*

9. Fréchet *if whenever* $x \in \overline{A}$*, there exists a sequence* $\langle x_n \rangle$ *in* A *that converges to* x*;*

10. sequential *if sequentially closed and closed sets coincide;*

11. *a* k-space *if* $A \subset X$ *is closed whenever* $A \cap K$ *is closed in* K *for every compact subset* K *of* X*;*

12. ω-tight *or has* countable tightness *if whenever* $x \in \overline{A}$*, there exists a countable subset* B *of* A *such that* $x \in \overline{B}$*;*

13. ω-fan-tight *or has* countable fan tightness *([2]) if whenever* $x \in \bigcap_{n\in\omega} \overline{A_n}$*, there exists finite sets* $B_n \in A_n$ *such that* $x \in \overline{\bigcup_{n\in\omega} B_n}$*;*

14. *an* $\aleph_0$-space *([7, page 493]) provided that it has a countable* k-network*, i.e. a countable collection* $\mathcal{N}$ *such that if* $K \subset U$ *with* K *compact and* U *open then* $K \subset N \subset U$ *for some* $N \in \mathcal{N}$*;*

15. *a* $\aleph$-space *([7, page 493]) provided that it has a* σ*-locally finite* k-network*;*

16. analytic *if it is a continuous image of a Polish space (equivalently of the irrationals with their usual topology);*

17. hemicompact *[1, page 486] if there is a sequence* $\langle K_n \rangle$ *of compact subsets such that each compact subset of* X *is contained in some* K_n*;*

18. Hurewicz *(originally called* E^* *in [10, page 195]) if for each sequence* $\langle \mathcal{U}_n \rangle$ *of open covers there is a sequence* $\langle \mathcal{V}_n \rangle$ *such that* $\cup_{n\in\omega}\mathcal{V}_n$ *covers* X *and* $\mathcal{V}_n$ *is a finite subfamily of* $\mathcal{U}_n$ *for each* $n \in \omega$*.*

Even if we primarily focus on spaces of real-valued continuous functions, we also consider conditions on hyperspace topologies. Let $\mathcal{C}(X)$ denote the set of all closed subsets of X. This set is classically identified with the set of continuous functions from X to the Sierpiński topology \$ (i.e., topology on $\{0,1\}$ with $\varnothing$, $\{0,1\}$ and $\{0\}$ as open sets), by identifying closed sets with their characteristic functions. Thus, if $\mathcal{C}$ is endowed with the *cocompact topology*, that admits the sets $\{F \in \mathcal{C}(X) : F \cap K = \varnothing\}$ for K ranging over every compact subsets of X as a subbase, we denote by $C_k(X, \$)$ the resulting topological space. It is proved in [14] that $\mathcal{C}(X)$ endowed with the upper Kuratowski convergence is first-countable if and only if it is sequential if and only if it is countably tight if and only if X is hereditarily Lindelöf. But hereditary Lindelöfness is known to be equivalent to metrisability for a manifold [6]. On the other hand, it is well-known (e.g. [4]) that the upper Kuratowski convergence and the cocompact topology coincide for Hausdorff locally compact topological spaces X, in particular for manifolds. This proves equivalences 1 to 4 in Theorem 2 below and gives the first three criteria of metrisability of a manifold in terms of topological properties of function spaces.

Theorem 2. *Let* M *be a manifold. The following are equivalent:*

1. M *is metrisable;*

2. $C_k(M,\$)$ *is first-countable;*

3. $C_k(M,\$)$ *is sequential;*

4. $C_k(M,\$)$ *is* ω*-tight;*

5. $C_k(M)$ *is Polish;*

6. $C_k(M)$ *is a pseudocomplete* σ*-space (equivalently is completely metrisable);*

7. $C_k(M)$ *is second countable;*

8. $C_k(M)$ *is a q-space (equivalently is metrisable, equivalently contains a dense subset of point-countable type);*

9. $C_k(M)$ *is a k-space (equivalently is Fréchet);*

10. $C_k(M)$ *has countable tightness;*

11. $C_p(M)$ *has countable fan tightness;*

12. $C_p(M)$ *has countable tightness;*

13. $C_k(M)$ *is a* $\aleph_0$*-space;*

14. $C_k(M)$ *is cosmic;*

15. $C_p(M)$ *is cosmic;*

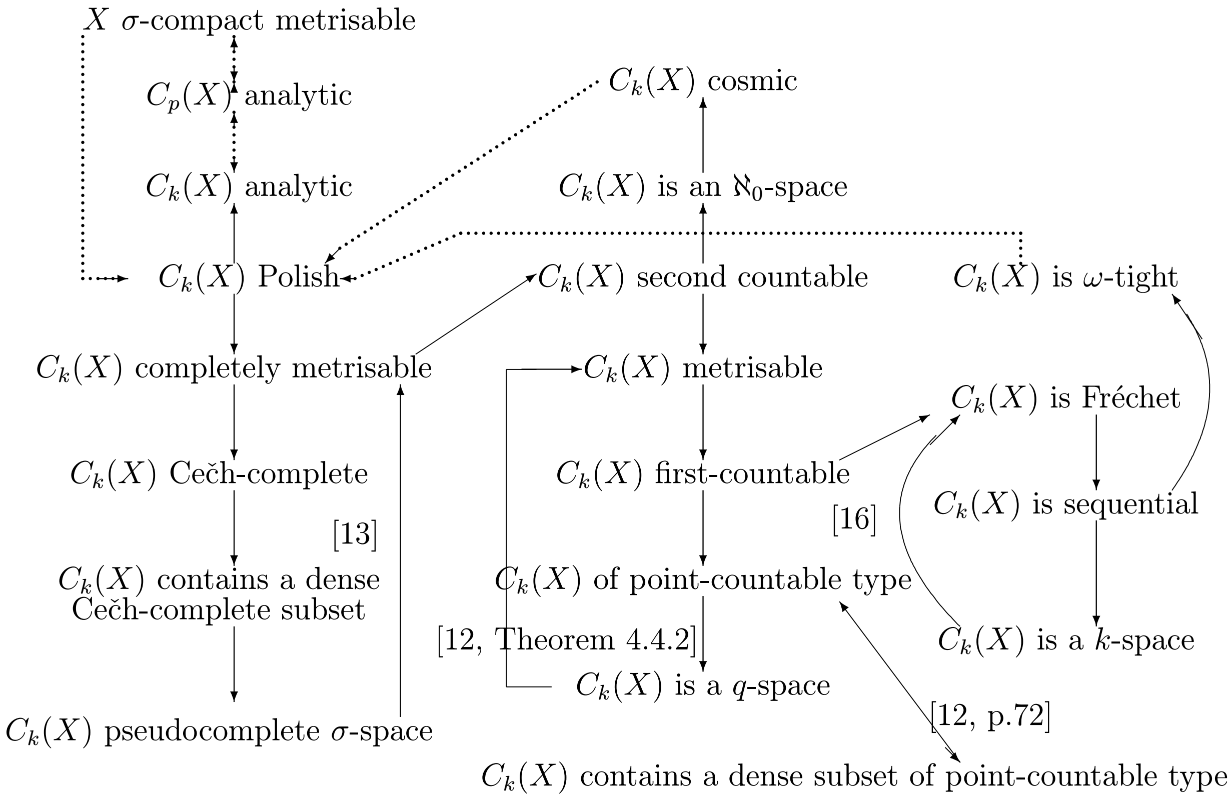
16. $C_k(M)$ *is analytic;*

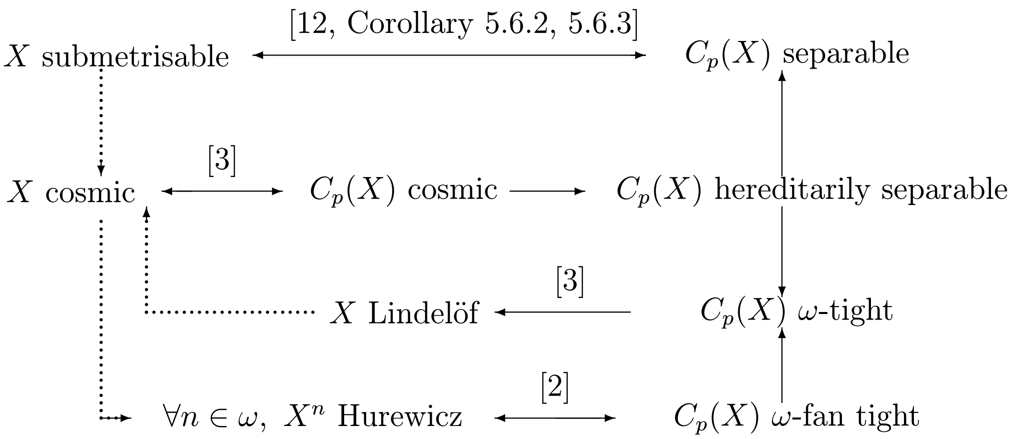
17. $C_p(M)$ *is analytic;*

18. $C_p(M)$ *is hereditarily separable;*

19. $C_p(M)$ *(equivalently* $C_k(X)$*) is separable.*

The following diagrams outline the connections that are true for general function spaces. In the next section, the dotted arrows are shown to be true when X is a manifold.





To prove the dotted implications in the first diagram, we use:

1. [12, Corollary 5.2.5]. $C_k(X)$ is Polish if and only if X is a hemicompact cosmic k-space.
2. [12, Corollary 4.1.3]. $C_k(X)$ is cosmic if and only if X is an $\aleph_0$-space.
3. [12, Corollary 4.7.2]. $C_k(X)$ is ω-tight if and only if every open k-cover ([2]) has a countable k-subcover.

[2] A *k-cover* of X is a collection $\mathcal{S}$ of subsets of X such that each compact subset of X lies in some member of $\mathcal{S}$.

4. [12, Theorem 5.7.5]. If X is a q-space (in particular a manifold), then $C_k(X)$ is analytic if and only if $C_p(X)$ is analytic if and only if X is σ-compact metrisable.

To complete the proof of Theorem 2 (and in particular dotted arrows in the two above diagrams), it remains to show that the following are equivalent for a manifold M:

1. M is hemicompact and cosmic [equivalently $C_k(X)$ is Polish, as a manifold is a k-space];
2. M is metrisable and σ-compact;
3. M is metrisable;
4. every open k-cover of M has a countable k-subcover [equivalently, $C_k(X)$ is ω-tight];
5. M is an $\aleph_0$-space [equivalently $C_k(X)$ is cosmic];
6. M is cosmic [equivalently $C_p(X)$ is cosmic];
7. M^n is Hurewicz, for every $n \in \omega$ [equivalently $C_p(X)$ is ω-fan-tight];
8. M is Lindelöf;
9. M is submetrisable [equivalently, $C_p(X)$ is separable, equivalently, $C_k(X)$ is separable].

While proving these equivalences in the next section, we actually prove that even more topological properties that are different in general are all equivalent to metrisability for manifolds. Hence we obtain both internal and external (in terms of function spaces) new criteria for metrisability of manifolds.

2 Internal criteria of metrisability

Lemma 3. *For a topological space X the following three conditions are equivalent.*

(a) X is hemicompact;

(b) There is an increasing sequence $\langle K_n \rangle$ of compact subsets of X such that each compact subset of X is contained in some K_n;

(c) Every k-cover of X has a countable k-subcover.

Proof. (a)⇒(b). If $\langle C_n \rangle$ is a sequence of compacta such that each compact subset of X lies in some C_n, then setting $K_n = \cup_{m \leq n} C_n$ gives a sequence satisfying (b).

(b)⇒(c). Let $\mathcal{S}$ be a k-cover of X and suppose that $\langle K_n \rangle$ is a sequence given by (b). For each $n \in \omega$ choose $S_n \in \mathcal{S}$ such that $K_n \subset S_n$. Then $\{S_n \ / \ n \in \omega\}$ is a countable k-subcover of $\mathcal{S}$.

(c)⇒(a). Let $\mathcal{K}$ consist of all compact subsets of X. Then $\mathcal{K}$ is a k-cover of X so has a countable k-subcover, say $\{K_n \ / \ n \in \omega\}$. The sequence $\langle K_n \rangle$ satisfies the definition of hemicompactness. ∎

Lemma 4. *Suppose that $\mathcal{N}$ is a k-network on the locally compact, regular space X. Then $\mathring{\mathcal{N}} = \{\mathring{N} \ / \ N \in \mathcal{N}\}$ is also a k-network for X.*

Proof. Let $K \subset U \subset X$ with K compact and U open. Use local compactness and regularity to find a finite collection $\{C_1, \ldots, C_n\}$ of compact subsets of U whose interiors cover K. As $\cup_{i=1}^n C_i$ is a compact subset of U there is $N \in \mathcal{N}$ such that $\cup_{i=1}^n C_i \subset N \subset U$. Then $K \subset \mathring{N} \subset U$. ■

Lemma 5. *Suppose that $\mathcal{U}$ is a collection of open subsets of the locally hereditarily separable space X. If $\mathcal{U}$ is point-countable on some dense subset of X then $\mathcal{U}$ is point-countable.*

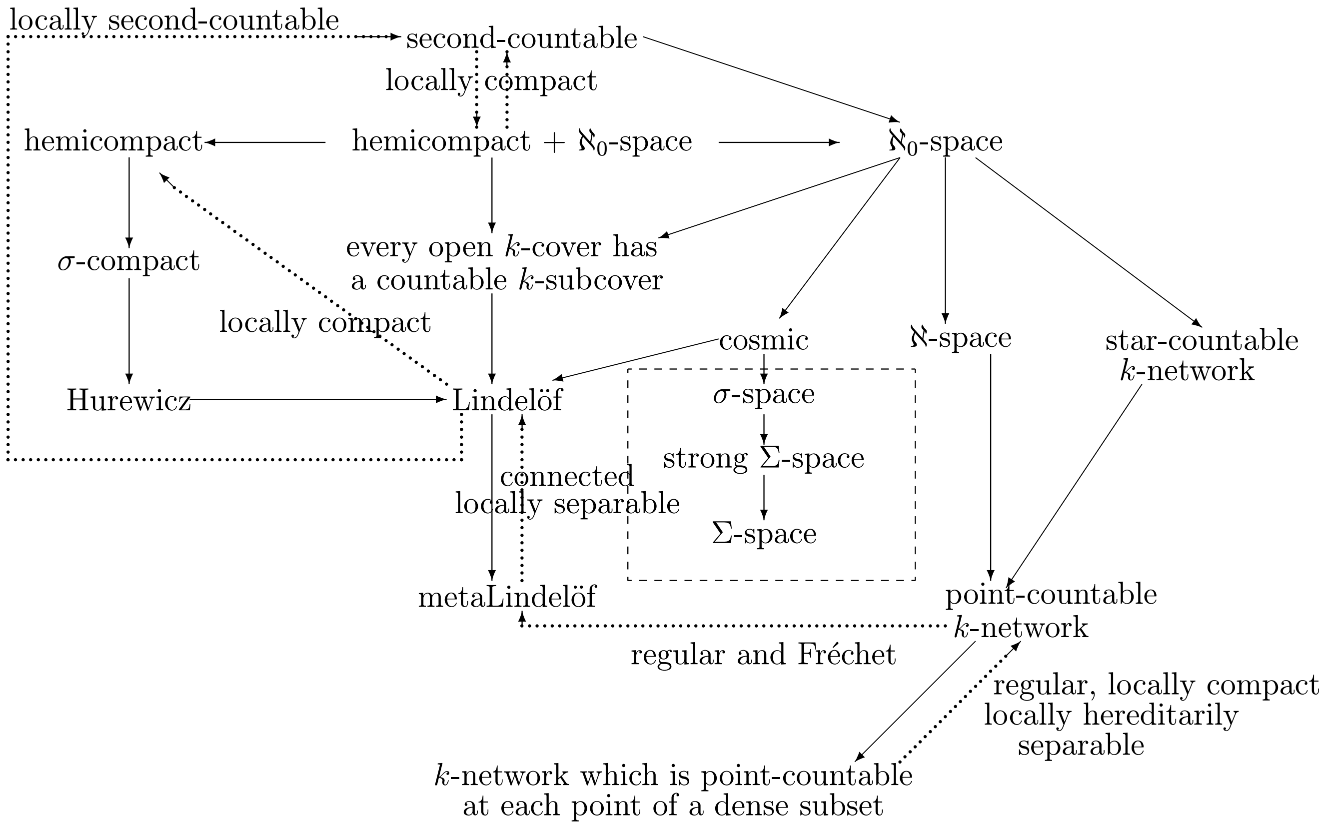
Proof. Suppose that $\mathcal{U}$ is point-countable on the dense subset $D \subset X$. Let $x \in X$ and let O be an open hereditarily separable neighbourhood of x. Let E be a countable dense subset of $D \cap O$; then E is also dense in O.

Choose $\varphi : \mathcal{U}_x \to E$, where $\mathcal{U}_x = \{U \in \mathcal{U} \ / \ x \in U\}$ so that for each $U \in \mathcal{U}_x$ we have $\varphi(U) \in E \cap O \cap U$: this is possible as $O \cap U$ is a non-empty open subset of O so that $E \cap O \cap U \neq \varnothing$. Note that for each $e \in E$ the set $\varphi^{-1}(e)$ is countable since $\mathcal{U}$ is point-countable at e. As E is also countable it follows that $\mathcal{U}_x = \cup_{e \in E} \varphi^{-1}(e)$ is countable. ■

Proposition 6. *Suppose that X is a locally hereditarily separable, locally compact, regular space which has a k-network which is point-countable on a dense subset of X. Then X has a point-countable k-network.*

Proof. let $\mathcal{N}$ be a k-network on X which is point-countable on a dense subset D of X. By Lemma 4, $\mathring{\mathcal{N}}$ is also a k-network on X. Clearly $\mathring{\mathcal{N}}$ is point-countable on D so by Lemma 5, $\mathring{\mathcal{N}}$ is point-countable. ■

Theorem 7. *All solid arrows in the diagram below represent general implications and the dotted arrows hold provided that X satisfies the extra conditions noted.*



Proof.

σ-compact implies Hurewicz.

We have $X = \cup_{n\in\omega} K_n$ for a sequence $\langle K_n \rangle$ of compact subsets of X. Suppose given a sequence $\langle \mathcal{U}_n \rangle$ of open covers of X. For each $n \in \omega$, $\mathcal{U}_n$ is an open cover of K_n so has a finite subcover, say $\mathcal{V}_n$. Then $\mathcal{V}_n$ satisfies the requirements.

Every open k-cover has a countable k-subcover implies Lindelöf.

Let $\mathcal{U}$ be an open cover of X and let $\hat{\mathcal{U}}$ consist of all open subsets of X which are finite unions of members of $\mathcal{U}$. Each compact subset of X is contained in a finite union of members of $\mathcal{U}$, hence in a single member of $\hat{\mathcal{U}}$. Thus $\hat{\mathcal{U}}$ is an open k-cover of X: let $\mathcal{V}$ be a countable k-subcover. Each member of $\mathcal{V}$ is a finite union of members of $\mathcal{U}$, so there is a countable subcollection $\mathcal{W} \subset \mathcal{U}$ such that each member of $\mathcal{V}$ is a finite union of members of $\mathcal{W}$. Then $\mathcal{W}$ is a countable subcover of $\mathcal{U}$.

$\aleph_0$-space implies that every open k-cover has a countable k-subcover.

Let $\mathcal{U}$ be an open k-cover of X and $\mathcal{N}$ be a countable k-network for X. For each $N \in \mathcal{N}$ for which there is $U \in \mathcal{U}$ with $N \subset U$ choose one such U; call it U_N. Then $\{U_N \ / \ N \in \mathcal{N}\}$ is a countable k-subcover of $\mathcal{U}$.

Cosmic implies Lindelöf.

Let $\mathcal{U}$ be an open cover of X and $\mathcal{N}$ be a countable network for X. For each $N \in \mathcal{N}$ for which there is $U \in \mathcal{U}$ with $N \subset U$ choose one such U; call it U_N. Then $\{U_N \ / \ N \in \mathcal{N}\}$ is a countable subcover of $\mathcal{U}$.

Lindelöf implies hemicompact under local compactness.

Use local compactness of X to cover X by open sets which are the interiors of compact sets. Because X is Lindelöf we need only countably many of these open sets to cover X. Thus we have compact sets $\{C_n \ / \ n \in \omega\}$ such that $X = \cup_{n\in\omega} \mathring{C}_n$. Let $K_n = \cup_{m\leq n} C_m$. It remains to show that each compact subset of X lies in some K_n. Given a compact subset $K \subset X$, as $\{\mathring{C}_n \ / \ n \in \omega\}$ is an open cover of K there is a finite subcover and if n is the largest index amongst the members of such a finite subcover then $K \subset K_n$.

Second-countable is equivalent to hemicompact $\aleph_0$-space under local compactness.

Assume X is second countable. To obtain a countable k-network, take the family of finite unions of elements of a countable base. On the hand, a second countable space is Lindelöf, hence hemicompact if X is moreover locally compact (by the above proof). To prove the converse (in fact the equivalence), we use a dual proof. By [12, Corollary 4.5.3], X is a hemicompact $\aleph_0$-space if and only if $C_k(X)$ is second-countable. Moreover, $C_k(X)$ coincides with $C_c(X)$, the set of real-valued continuous functions on X endowed with the continuous convergence, provided that X is locally compact [11, Theorem 3.2]. But $C_c(X)$ is second-countable if and only if X is second-countable [5, Theorem 1].

A regular Fréchet space with a point-countable k-network is metaLindelöf by [9, Proposition 8.6(b)]. By Proposition 6, a regular locally compact and locally hereditarily separable space with a k-network which is point countable at each point of a dense subset has a point-countable k-network.

The remaining implications in the diagram follow directly from the definitions. ∎

Corollary 8. *Let M be a manifold. The following are equivalent:*

1. *M is metrisable;*
2. *M is second-countable;*
3. *M is a hemicompact $\aleph_0$-space;*
4. *M is hemicompact;*
5. *M is an $\aleph_0$-space;*
6. *M is cosmic;*
7. *M is an $\aleph$-space;*
8. *M has a star-countable k-network;*
9. *M has a point-countable k-network;*
10. *M has a k-network which is point-countable at each point of a dense subset;*
11. *M is metaLindelöf;*
12. *M is Lindelöf;*
13. *M is Hurewicz;*
14. *M is σ-compact;*
15. *every open k-cover of M has a countable k-subcover.*

Proof. A manifold is regular, Fréchet, connected, locally compact, locally hereditarily separable and locally second-countable. Moreover, it is well-known (see for example [6, Theorem 2]) that a manifold is metrisable if and only if it is Lindelöf. ∎

In particular, the first eight properties quoted at the end of the first section are equivalent (including the seventh, because if a manifold M is Hurewicz, it is moreover second-countable, so that each finite power is also second-countable, hence Hurewicz).

Hence all the properties involved in the above diagram except those in the dotted box are criteria for metrisability of a manifold. What about these three additional properties?

Definition 9. *A topological space X is a* Moore space *if it is regular and has a development, i.e., a sequence $\langle \mathcal{U}_n \rangle$ of open covers such that for each $x \in X$ the collection $\{st(x, \mathcal{U}_n) : n \in \omega\}$ forms a neighbourhood basis at x. X has a (*regular*) G_δ*-diagonal *if its diagonal is a (regular) G_δ subset of $X \times X$ (H is a regular G_δ-set in Y if $H = \bigcap_n \overline{U_n}$, where each U_n is an open set containing H). A space X has a G^*_δ-diagonal if there exists a sequence $\langle \mathcal{G}_n \rangle$ of open covers such that for each $x \in X$, $\{x\} = \bigcap_n \overline{st(x, \mathcal{G}_n)}$.*

X is weakly normal *provided that for every pair A, B of disjoint closed subsets of X there is a continuous function f from X to a separable metric space such that $f(A) \cap f(B) = \varnothing$.*

It is well-known that each Moore space is a σ-space [7, Theorem 4.5] and that every σ-space has a G^*_δ-diagonal [7, Theorem 4.6]. On the other hand, by [7, Theorem 2.15], a locally compact and locally connected space (in particular a manifold) with a G^*_δ-diagonal is a Moore space. Hence Moore spaces, σ-spaces and spaces with a G^*_δ-diagonal coincide for manifolds. Moreover, there exists a non metrisable Moore manifold [15, Example 3.7]. Thus, none of the properties in the dotted box is a general criterion for metrisability of a manifold. However, it is known from [6] that "normal Moore space" and "weakly normal space with a G^*_δ-diagonal" are two such criteria.

Proposition 10. *The following are equivalent for a manifold M.*

1. *M is metrisable;*
2. *M is a (weakly) normal Moore space;*
3. *M is a (weakly) normal σ-space;*
4. *M is a (weakly) normal space with a G^*_δ-diagonal;*
5. *M has a regular G_δ-diagonal;*
6. *M is submetrisable.*

Proof. The equivalences between the four first points are obvious from the previous discussion while the equivalence between 1 and 5 follows from [7, Theorem 2.15,b)] that states that a locally compact locally connected space with a regular G_δ-diagonal is metrisable. Finally, a submetrisable space has a regular G_δ-diagonal [7, p. 430]. ∎

Question 11. *Now we can ask whether normality combined with the weaker property of (strong) Σ-space is also equivalent to metrisability for a manifold.*

3 Related notions and remarks.

Consider the following stronger variant of fan-tightness. A topological space X has *countable strong fan tightness* if whenever $x \in \bigcap_{n\in\omega} \overline{A_n}$, there exist $x_n \in A_n$ such that $x \in \overline{\{x_n : n \in \omega\}}$. In [17], it is shown that $C_p(X)$ has countable strong fan-tightness if and only if X^n has the following property (called C'') for every $n \in \omega$: for each sequence $\langle \mathcal{U}_n \rangle$ of open covers there is a sequence $V_n \in \langle \mathcal{U}_n \rangle$ such that $\cup_{n\in\omega} V_n$ covers X. As C'' is a slight strenghtening of the Hurewicz property which is equivalent to metrisability for a manifold, property C'' (and hence countable strong fan-tightness of $C_p(M)$) is a reasonable candidate to be another criterion of metrisability of a manifold M. However, taking $l = 1$ in Lemma 12 shows that such a space does not satisfy property C''. Hence, the function space $C_p(M)$ over a metrisable manifold M need not be (and actually is rarely) of countable strong fan-tightness .

Lemma 12. *Let (X, d) be a nontrivial, connected metric space and $l \in \mathbb{N}$. Then there is a sequence $\langle U_n \rangle$ of open covers such that for any sequence $\langle \mathcal{V}_n \rangle$, where $\mathcal{V}_n \subset \mathcal{U}_n$ and $\cup_{n\in\omega}\mathcal{V}_n$ covers X, there is $n \in \omega$ so that $|\mathcal{V}_n| > l$.*

Proof. Let (X, d) and $l \in \mathbb{N}$ be given. Choose two distinct points $a, b \in X$. For each $n \in \omega$ let $\mathcal{U}_n = \{B\left(x; \frac{d(a,b)}{2^{n+2}l}\right) / x \in X\}$, an open cover of X. Suppose that $U_{n,1}, \ldots, U_{n,l} \in \mathcal{U}_n$ for each $n \in \omega$: we show that $\{U_{n,j} / n \in \omega; j = 1, \ldots, l\}$ cannot cover X. Suppose to the contrary that $\{U_{n,j} / n \in \omega; j = 1, \ldots, l\}$ does cover X. As X is connected there must be $x_0, \ldots, x_k \in X$ such that $B\left(x_i; \frac{d(a,b)}{2^{n_i+2}l}\right) = U_{n_i,j_i}$ for some $n_i \in \omega$ and $j_i = 1, \ldots, l$ with $a \in U_{n_0,j_0}$, $b \in U_{n_k,j_k}$, the pairs (n_i, j_i) are distinct for $i = 0, \ldots, k$, and $U_{n_{i-1},j_{i-1}} \cap U_{n_i,j_i} \neq \varnothing$ for each $i = 1, \ldots, k$. Then

$$\begin{aligned} d(a,b) &\leq d(a,x_0) + \sum_{i=1}^{k} d(x_{i-1},x_i) + d(x_k,b) \\ &< \frac{d(a,b)}{2^{n_0+2}l} + \sum_{i=1}^{k}\left[\frac{d(a,b)}{2^{n_{i-1}+2}l} + \frac{d(a,b)}{2^{n_i+2}l}\right] + \frac{d(a,b)}{2^{n_k+2}l} \\ &= d(a,b)\sum_{i=0}^{k}\frac{1}{2^{n_i+1}l} \\ &\leq d(a,b)\sum_{i=0}^{k}\frac{1}{2^{i+1}} \\ &= d(a,b)\left(1-\frac{1}{2^{k+1}}\right) \\ &< d(a,b). \end{aligned}$$

This contradiction proves that if we select at most l members of each $\mathcal{U}_n$ then those sets selected from $\cup_{n\in\omega}\mathcal{U}_n$ collectively do not cover X. ∎

Remark. Note that, if in Theorem 7, we transform the condition that every open k-cover has a countable k-subcover by only requiring that every open k-cover has a countable subcover we obtain a condition equivalent to Lindelöfness.

We have some other like failures in the sense that we have conditions which superficially appear to be weaker than the standard condition, and hence may lead to a weaker criterion for metrisability of a manifold, but are not weaker at all. Of course in (1) below we could insist that the subcover be a k-cover, in (2) that the subcovers be lindelöf-subcovers and in (3) that the refinement be a k-cover but then the resulting conditions, while being stronger than the standard conditions in general, will be weaker than, for example, hemicompactness so not really of any interest in the context of manifolds. An *ideal open cover* is an open cover $\mathcal{U}$ such that any finite union of members of $\mathcal{U}$ is a member of $\mathcal{U}$ and any open subset of a member of $\mathcal{U}$ is also a member of $\mathcal{U}$.

Failure 13. *For any space X the following hold.*

(1) The following are equivalent:

(a) X is Lindelöf;

(b) every open k-cover has a countable subcover;

(c) every ideal open k-cover of X has a countable k-subcover.

(2) The following are equivalent:

(a) X is Lindelöf;

(b) every open lindelöf-cover has a finite subcover;

(c) every open lindelöf-cover has a countable subcover.

(3) X is metaLindelöf $\iff$ every open k-cover has a point-countable open refinement.

Proof of (1)(a)$\iff$(c). The implication (c)$\Rightarrow$(a) may be obtained from the proof of the non-ideal version of this implication in Theorem 7 above by taking $\hat{\mathcal{U}}$ to consist of all open subsets of finite unions of members of $\mathcal{U}$. Conversely suppose that $\mathcal{U}$ is an ideal open k-cover of X and let $\mathcal{V}$ be a countable subcover. Let $\hat{\mathcal{V}}$ be the set of all finite unions of members of $\mathcal{V}$. Then $\hat{\mathcal{V}}$ is still countable, is a subfamily of $\mathcal{U}$ and is a k-cover of X.

Proof of (3). $\Rightarrow$ is trivial, so we concentrate on $\Leftarrow$. Suppose that $\mathcal{U}$ is an open cover of X. Then $\hat{\mathcal{U}}$, which consists of the unions of all finite subfamilies of $\mathcal{U}$, is an open k-cover of X. Let $\mathcal{W}$ be a point-countable open refinement of $\hat{\mathcal{U}}$. For each $W \in \mathcal{W}$ choose $\widehat{U_W} \in \hat{\mathcal{U}}$ so that $W \subset \widehat{U_W}$ and then choose $U_{W,1}, \dots, U_{W,n_W} \in \mathcal{U}$ so that $\widehat{U_W} = U_{W,1} \cup \dots U_{W,n_W}$. Let $\mathcal{V} = \{U_{W,i} \cap W \ / \ W \in \mathcal{W} \text{ and } i = 1, \dots, n_W\}$. Then $\mathcal{V}$ is a point-countable open refinement of $\mathcal{U}$. ∎

The example below shows that for general topological spaces, 15$\not\Rightarrow$14 (hence 15$\not\Rightarrow$4), 15$\not\Rightarrow$6 (hence 15$\not\Rightarrow$5), 13$\not\Rightarrow$14 and 13$\not\Rightarrow$6 (hence 13$\not\Rightarrow$5) in Corollary 8.

Example 14. *Let X be ω_1 with the co-countable topology. Then X is Hurewicz and every open k-cover has a countable k-subcover but X is neither σ-compact nor cosmic.*

The only compact subsets of X are the finite subsets, so X is not σ-compact.

Let $\langle \mathcal{U}_n \rangle$ be a sequence of open covers of X. Choose any non-empty $U_0 \in \mathcal{U}_0$. As $X - U_0$ is countable, for each $n = 1, 2, \dots$ we can choose $U_n \in \mathcal{U}_n$ so that $\{U_n \ / \ n = 1, 2, \dots\}$ covers $X - U_0$. Now set $\mathcal{V}_n = \{U_n\}$ to get the sequence $\langle \mathcal{V}_n \rangle$ as in the definition of Hurewicz.

Let $\mathcal{U}$ be an open k-cover of X. For each finite subset $F \subset X$ choose non-empty $U_F \in \mathcal{U}$ and $\beta_F \in \omega_1$ such that $F \subset U_F$ and $[\beta_F, \omega_1) \subset U_F$; the latter follows from co-countability of U_F. Define inductively an increasing sequence $\langle \alpha_n \rangle$ as follows. Set $\alpha_0 = 0$. Given α_n, there are countably many finite subsets of $[0, \alpha_n)$ so $\{\alpha \ / \ \alpha > \beta_F \text{ for each finite } F \subset [0, \alpha_n)\}$ is non-empty; let α_{n+1} be the least member of this set or $\alpha_n + 1$, whichever is the greater. Let $\alpha = \lim_{n\to\infty} \alpha_n$. It is claimed that $\{U_F \ / \ F \text{ is a finite subset of } [0, \alpha)\}$ is a countable k-subcover of $\mathcal{U}$. Indeed, suppose that F is a finite subset of X and set $G = F \cap [0, \alpha)$. Then there is $n \in \omega$ such that $G \subset [0, \alpha_n)$. As $\beta_G < \alpha_{n+1} < \alpha$ it follows that $F \subset G \cup [\alpha, \omega_1) \subset U_G$.

X is not cosmic for if $\mathcal{N}$ is a network for X then for each $\alpha \in X$ there is $N_\alpha \in \mathcal{N}$ such that $\alpha \in N_\alpha \subset [\alpha, \omega_1)$. If $\alpha \neq \beta$ then $N_\alpha \neq N_\beta$. Thus $\mathcal{N}$ is uncountable. ∎

The next example shows that for general topological spaces 5$\not\Rightarrow$13 (hence 15$\not\Rightarrow$13, 12$\not\Rightarrow$13, 5$\not\Rightarrow$14, 5$\not\Rightarrow$4, 6$\not\Rightarrow$13, 6$\not\Rightarrow$14 and 6$\not\Rightarrow$4) in Corollary 8.

Example 15. *The space $\mathbb{P}$ of irrational numbers with the usual topology, which is homeomorphic to the product space ω^ω see [18, Theorem 3.11], is an $\aleph_0$-space but is not Hurewicz.*

$\mathbb{P}$ second countable, hence an $\aleph_0$-space.

To show that $\mathbb{P}$ is not Hurewicz we look at its homeomorph ω^ω. For each function $\sigma : \{0,1,\dots,n\} \to \omega$ set $U_\sigma = \{f \in \omega^\omega \ / \ f(i) = \sigma(i) \text{ for all } i \le n\}$. Let $\langle \mathcal{U}_n \rangle$ be a sequence of open covers of ω^ω defined by $\mathcal{U}_n = \{U_\sigma \ / \ \sigma = \{0,1,\dots,n\} \to \omega\}$. Suppose that $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ for each $n \in \omega$. We show that $\cup_{n\in\omega}\mathcal{V}_n$ does not cover ω^ω. Inductively construct a function $f : \omega \to \omega$. Let $f(0)$ be chosen so that $U_{f|\{0\}} \notin \mathcal{V}_0$. If $f(i)$ is defined for each $i \le n$ so that $U_{f|\{0,1,\dots,i\}} \notin \mathcal{V}_i$ then just select $f(n+1)$ so that $U_{f|\{0,1,\dots,n+1\}} \notin \mathcal{V}_{n+1}$. It follows that f does not lie in the union of all of the members of $\cup_{n\in\omega}\mathcal{V}_n$. ∎

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