The Journey of the Four Colour Theorem Through Time*

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Abstract

This is a historical survey of the Four Colour Theorem and a discussion of the philosophical implications of its proof. The problem, first stated as far back as 1850s, still causes controversy today. Its computer-aided proof has forced mathematicians to question the notions of proofs and mathematical truth.

1 Introduction

The Four Colour Theorem is one of the simplest mathematical problems to state and understand. It says that any map is four-colorable. In other words, if given a map, one can colour its regions using at most four colours, such that no two adjacent regions are colored by the same colour. As simple as this result may seem, it took mathematicians over 100 years to prove, and some may still not agree that it has indeed been proved. This article discusses the historical aspects of the problem and the philosophical impacts that it had in the field of mathematics.

Firstly, we will look into how the problem was first stated and how it was received by specialists. We will see the famous minds that were boggled and the not so famous minds that were fascinated. Several "proofs" were given along the course of the problem's history, which will be discussed in Section 2.

Secondly, in Section 3 we look at the philosophical aspects of the Four Colour Theorem. What are "proofs" and what do we expect from a proof? This is touched upon in Section 3.1. Finally, in 3.2, the controversy created by the recent proofs of the theorem is presented.

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2 History of the Four Colour Problem

2.1 Early beginnings

The Four Colour Conjecture was first stated by Guthrie in 1850. He was studying law at University College London and he was taught, among others, by De Morgan. Guthrie had noticed that four colours seemed sufficient to colour most maps, but he wondered if that was true for all maps. This led him to try and prove the conjecture or at least find out why it might be true. De Morgan could not come up with any answers and passed the problem on to Hamilton, whose reply was "I am not going to attempt your quaternion of colour very soon". The problem did not seem to pick Hamilton's attention for some time and interest in it almost seemed to die down. In 1860, Peirce tried to prove the conjecture too, but failed. Cayley heard of the problem in 1878 and he published a paper entitled "*On the coloring of maps*" in 1879 in which he explains why it is difficult to produce a proof for the Four Colour Problem.

2.2 The temperature is rising

In 1879, the mystery seemed to have been solved. Kempe submitted a proof of the Four Colour *Theorem* to the American Journal of Mathematics. His proof was very welcomed within the mathematical community and he was made a Fellow of the Royal Society, and knighted in 1912. Kempe introduced a new technique now referred to as "Kempe's Chains". His method involves creating "chains of two colours" within a graph, in order to predict possible combinations of such colourings. However useful this method still proves to be today, his actual proof of the Four Colour Theorem itself is flawed, as established by Tait in 1880. In fact, Tait went so far as to come up with his own proof, which once again proved to be wrong. He based this proof on the false assumption that every three-connected planar graph is Hamiltonian. So the Four Colour *Theorem* became the Four Colour *Problem* for the second time in history.

2.3 Hot from the Oven

However, this was only the beginning. In 1890, Heawood published "Map colouring theorem" which is a proof of the Five Colour Theorem (i.e., that any map can be coloured with at most five colours) using Kempe's Chains. In 1891, Petersen realised that Tait's methods show that the Four Colour Problem can be adapted to a conjecture on colouring edges. He went on to show that the Four Colour Problem is equivalent to the conjecture that any planar cubic graph contains a Hamiltonian circuit (i.e., a closed walk containing each vertex exactly once) or by a collection of mutually disjoint subcircuits of even length. In 1904, Weinecke published a paper on avoidable sets (i.e., a set of elements which can be "left out" when testing a graph for specific properties). In 1912, Veblen wrote one too, generalizing Heawood's work and Birkoff introducing theories on reducibility and chromatic polynomials. Franklin used unavoidable sets and went further to prove in 1922 that a map with at most 25 regions is four-colourable. He was followed by Reynolds who showed any graph with 28 regions is four-colourable in 1926. It is interesting to note that the first book on graph theory appeared in 1936, published by König. The saga of regions continued with 32 regions in 1938, proved by Franklin, 36 regions in 1940, proved by Winn, then Ore and Stemple proved that any graph with 40 regions is four-colourable in 1970. In 1976, Mayer showed the same result for a map with 90 regions.

In the meantime, Hadwiger's conjecture was born in 1943 - the Four Colour Problem being a special case of this conjecture. Dykin and Uspensky published a small book on elementary graph colouring exercises in 1952 and Ore wrote what was to become a famous book entitled "The Four Color Problem" in 1967. Finally, in 1976, Heesch introduced the method of discharging.

2.4 Eureka ... or is it?

By now, the ideas about discharging methods and configuration sets were introduced and studied by several people. These methods constituted the basis for what was to leave the mathematical community in shock: the world's first computer-aided proof. The year of 1976 saw Appel and Haken march onto the center of the mathematical stage, announcing their proof of the Four Colour Theorem. This new version of the proof engaged a computer of the time in 1200 hours of computation which is mostly used up by verifications of various configurations. By 1977, details of the proof appeared in articles and the controversy began. The mathematical community watches in semi-desperation and semi-ecstacy as this tiny problem, which an average child has the capacity to understand, puts the very core of what mathematics stands for - the ultimate, unquestionable truth - on shaky ground. Can we and should we accept this kind of proof?

Due to this proof's practical impossibility of human verification, many have decided to come up with their own, new proof. Allaire produced such a proof in late 1977. The computer was again used in case testing, but this time the discharging method was more powerful than Appel and Haken's one, taking less computer time to execute.

Furthermore, in 1996, Robertson, Sanders, Seymour and Thomas came up with yet another version of a computer-aided proof, using more powerful discharging methods again and cutting the number of cases being tested to from 1500 to 633. The team is confident in their result but at the same time somewhat surprised at the way in which it was finally obtained "It is amazing that such a simply stated result resisted a proof for one and a quarter centuries, and even today it is not yet fully understood", says Thomas [20].

2.5 What now?

So what happens now? On one hand we have *three* slightly different proofs which confirm the result. On the other hand no one was yet able to check "manually" that these proofs are correct. In fact, it is highly improbable that anyone will ever be able to check them in this way. Worse even, some predict that one will never be able to come up with a proof for the Four Colour Theorem without requiring the help of machines. If so, what does that mean for mathematics and its credibility? Horgan [12] says that a "proof which offers a probability of truth - and not certainty - is an oxymoron". Is he right in thinking that the "death of proofs" as we know them is getting closer and closer?

3 Philosophical Implications of the Four Colour Theorem

3.1 The birth of proofs

The Greeks and in particular Pythagoras, introduced the key concept which tortured, fascinated and become a true addiction for mathematicians for the last 2000 years - the concept of proof. Pythagoras knew in 500 BC that his theorem (concerning right angled triangles) was to remain true for eternity. Are the three teams of mathematicians who provided computer-aided proofs for the Four Colour Theorem also confident that the result will remain valid for eternity?

Up until recently, it was understood that an acceptable definition for the concept of *proof* is "a finite sequence of formulae of some given system, where each of the sequence is either an axiom of the system or a formula derived by a rule of the system from some of the preceding formula" [4]. However, due to practicality, this definition is not followed closely, as many proofs are in fact too long to be expressed in terms of axioms (i.e., formalised). There are two other question marks about this definition. What is the aim of a proof? When is a proof not really a proof?

Some mathematicians feel the need to go beyond the formal definition of proofs. Tmockzo [5] characterises proofs to be convincing, formalisable and surveyable. In short, it is expected that proofs are *convincing* enough to persuade even the skeptics of the accuracy of the claimed results. They should be *formalisable* at least in theory, i.e., the conclusion should be reached starting from axiomatic systems as demanded by the formal definition of proofs. It is the *surveyable* aspect that causes the most controversy in the case of the Four Colour Theorem. Surveyability is best illustrated by the "falling-tree" conundrum: Has a tree fallen if no one can hear it? Has a theorem been proved if no one can read its proof?

3.1.1 Mathematical science versus Natural Science

The other approach is to look closely at what it is that differentiates pure mathematical science from natural sciences. Philosophers have been concerned with this vital difference for as long as mathematics itself can remember. Kant and Plato explained it by using the “a priori - a posteriori” dichotomy. On one hand we have physical experiments and personal experience, often referred to as “common knowledge”, which may not always turn out to be how we perceive it. This is considered to be *posteriori*. On the other hand, there is pure mathematical knowledge, derived from an axiomatic system whose truth value is guaranteed to transcend time itself is - this is thought of as a *priori*. This kind of “absolute” truth is what mathematics has been thriving on for the last 2000 years. Throughout its journey, the field adopted various other modern terms for “priori” such as innate, formal, certain, analytic and conversely for “posteriori”: learned, empirical, dubitable, synthetic. However, Kant and Plato’s ideology remains firmly ingrained in many mathematical circles today, regardless of its exact linguistic form.

As clear cut as this definition might seem, there are simple examples, which puzzle even the most convinced believers. Take the concept of a prime number, i.e., a number whose only divisors are one and itself. Many prime numbers can be calculated by hand, simply by trying out various possible divisors. Clearly, this exercise only works for reasonably small numbers. In spite of that, mathematicians were interested to obtain large prime numbers too and they employed computers to calculate them. Does this mean that some primes are priori (those founds by hand) and some posteriori (those found by a machine)?

A further distinction between mathematics and natural sciences is illustrated by the contrasting concepts of “repeating a proof” and “repeating an experiment”. According to Wittgenstein, a proof is “the picture of an experiment, as it practically always turns out”. This draws upon the issue of surveyability or “übersehbar”, as Wittgenstein [10] himself would have defined it. He goes on to explain that an experiment (for example in physics) does not guarantee that the original result will always be obtained, if the experiment were to be repeated; whereas proofs do. Proofs are not temporally relative, they are eternal truths in all possible worlds.

3.1.2 The end product

What do proofs prove? Theorems! Swart [6], argues that there is not one type of theorem but four:

- theorems whose proofs can be done in one’s head;
- theorems which require pen and paper for proving;
- theorems which not only require pens and paper, but also a great amount of effort and time;

- theorems which can only be proved with the help of a computer.

The question is which of these types of theorems would Kant and Plato accept as valid a priori mathematical truths? Where does one draw the line between a priori and a posteriori?

On a different tangent, some people argue that a proof does not only prove a certain result, but in fact it provides insight as to why the result is true. In a way, putting our minds to rest that the proof for Pythagoras's Theorem is true constitutes only half the benefits. The fact that it explains why this is the case is what makes it a "beautiful" and a "proper" proof. In other words, it provides the "a-ha" feeling which is the core motivation for many mathematicians.

3.2 The controversy of the proof

3.2.1 Eureka!

The long awaited proof for the Four Colour Theorem did come eventually, but not in the form that the mathematical community expected it to. The day Wiles presented his outline of the proof of Fermat's Last Theorem the audience rejoiced in his achievement and clapped ecstatically. In contrast, those who witnessed the most daring and unusual proof in mathematics, produced by Appel and Haken, fell into a deadly silence. No one clapped. No one rejoiced. Even today, almost a quarter of a century afterwards, many mathematicians who accept the proof, will not admit that the Four Colour Theorem is "just as proved as" Fermat's Last Theorem. Suddenly, the most precise concept in mathematics, the ultimate kind of truth, the notion of proof has been transformed into an adjective with comparative and superlative forms. Why has this happened?

3.2.2 The skeptics

The issue of *surveyability* is burning more and more holes in the credibility of the proof. If proofs are to be verified, then it automatically seems to follow that a person - as opposed to a machine - has to complete this task. This has not been achieved, due to the length of the proof. In principle, we do know what the computer is doing, but no one can say they have personally done that, or even ever hope to be able to do that.

Halmos [18] explains that a proof done with the use of the computer has the same amount of credibility as one done by a "reputable fortune teller", as these machines are similar to "oracles" since they have certain physical properties that we have yet to understand. Similarly, Deligne [12], from the Institute for Advanced Study (a 1978 Fields Medalist) shares the same point of view: "I don't believe in a proof done on a computer. In a way, I am very egocentric. I believe in a proof if I understand it, if it's clear. A computer will also make mistakes, but they are much more difficult to find".

Halmos has another complaint: "we don't learn anything from the proof". The proof gives no indications as to why it is that we only need four colours to colour maps; why not thirteen? What is so special about the number four? As good as an "oracle" gets, it still can't provide an explanation for that one! Stewart [22] agrees with Halmos and Hersch joins them in their disappointment about there being no "enlightenment" brought out through the proof. Some go as far to claim that mathematics is not about finding answers, but about the methods used to obtain them. Cohen [18] explains that "admitting these computer shenanigans" would lead to an "intellectually unfulfilled" result. Computer checked cases do not belong to mathematical science because one cannot accept a "quasi-proof, hidden in a black box", according to Bonsall [22].

Tmockzo [5] draws upon the distinction between a priori and posteriori truths. To use a computer in establishing a mathematical truth is to transform proofs into experiments. He claims that the Four Colour Theorem has been "confirmed" though a theoretical physics experiment, but not proved in a formal way. Hence, this result is posteriori because it manipulates machines as well as logical operators. The trouble with these experiments is that although we have an approximate idea of what the computer is testing, no one can be one hundred percent sure of the methods used. This means that the nature of the result is of the type "Simon says" where mathematicians are invited to "have faith" and "believe" whatever some Simon god-like creature claims. But this is mathematics and not religion!

While Tmockzo does not completely deny the validity of the results, he underlines the fact that the theorem is not "as proved as" other theorems (those whose proofs do not involve computers). He calls the result "a new kind of knowledge". Swart [6] reconciles this divergence between conventional and unconventional theorems by introducing a new idiom: "agrograms". The latter terminology is changed as follows: we start at conjectures when a new problem is introduced; move up to agrograms - truth statements which are verified as best possible, and then finish with theorems, the absolute rulers of the mathematical market. While agrograms are still considered mathematical truths, they do not possess the authority that theorems do; in fact, they represent the fuzzy boundary between a priori and posteriori.

3.2.3 The believers

In spite of all this, there is also plenty of support for the Four Colour Theorem proof. Unconventional types of proofs are being welcomed in the mathematical community more and more. In 1988, Lam [8] and his colleagues at the Concordia University used the computer to examine various cases as part of a geometry proof. In 1981, Lanford from the Institut des Hautes Etudes Scientifiques, Bures-Sur-Yvette proved the Feigenbaum conjecture, also with the help of computers. Meyer and Schmidt from the University of Cincinnati employed computer algebra to get further insight into the n-body problem. So why is it that some are prepared to accept all these proofs as valid mathematical truths?

Firstly, the complaint that computers may have bugs or produce errors can be applied in the same way to humans. People are also likely to make mistakes [9]. In fact, Kempe's proof itself is a perfect example that humans are only human. His error was detected years after the proof was widely recognized and accepted. Furthermore, even though computer errors might be harder to detect, humans are more likely to make mistakes in their proofs - computers follow a pre-designed rigid program and are not distracted by moods, stress and other outside factors.

Secondly, the sheer length of some proofs is beyond the scope of human computation, but perfectly acceptable by machine standards. In other words, the kinds of results we are interested in obtaining involve complex and lengthy computations, which might be understood by humans, but by no means executable without the use of computers. The Four Colour Theorem is one of those results. What initially took 1200 hours of computer work, would be practically impossible for one single mathematician to do; which explains why the proof is not surveyable. And even for those results which can be done by humans, the speed of a computer outweighs that of the human [9]. Why should you walk to work, when you can drive in?

Thirdly, "the idea that you don't use computers is going to be increasingly foreign to the next generation", according to Taylor [12]. It is a matter of acceptance and familiarity. Many fresh mathematicians are educated "with computers"; they are taught to use them as any other tool, such as a pencil or paper. For those who did not start their mathematical career during the computer revolution, these powerful machines are more of a threat than a help. Their instinct tells them to question the validity of the results obtained from such machines. Hersch [18] has some bad news for these researchers: "those who will try to reject computer proofs will have as much success as King Canate"- by royal order he commanded the tide to turn back.

Fourthly, we have another perspective altogether: "that mathematics reduces in principle to formal proofs is a shaky idea", according to Thurston [12]. "In practice, mathematicians prove theorems in a social context. It is a socially conditioned body of knowledge and techniques." De Millo, Lipton and Perlis [3] agree. Proofs in practice are completely different from what we would like them to be in theory. They start out by being messages on paper, or even napkins. Then they are developed into proper arguments and circulated around the mathematical community. As a result of this journey, multiple versions of the resulting theorem are stated. Consequently, the "baby theorems" get used here and there by supporters among researchers. Finally, they start to form links with previous results in various other areas of mathematics. And there you have it, a "grown-up" theorem in all its rights. In other words, it is not a formal process but a social ritual of acceptance which allows theorems to achieve a certain amount of credibility. Arguing on priori versus posteriori terms is not a suitable approach in this case. It all comes down to the opinions and beliefs of those in the mathematical community: will we believe in a computer-aided result or not?

One mathematician goes so far to criticize the existing proof of the Four

Colour Theorem that he asserts:

> Since the problem has been taken care of by a totally inappropriate means, no first-rate mathematician would now work any more on it, because he would not be the first one to do it, and therefore a decent proof might be delayed indefinitely. ...So we have done something very, very bad, and things like that should not be committed again. [22]

In response to this view, Archdeacon of Vermont University says: “there were many ugly paintings of gardens, but that didn't stop Van Gogh from painting his sunflowers” [23]. People are looking for better things in every aspect of life, and similarly, mathematicians are looking for better, more elegant, shorter and more beautiful proofs. This is the purpose of Erdős's idea of a book with ideal proofs, entitled “*Proofs from The Book*” [21].

Casti [22] writes that the Four Colour Theorem is one of the most interesting problems in mathematics because it initiated a completely new field in mathematics - that of probabilistic mathematics - and also because it forced researchers to look back and question the notion of proof. Perhaps this is the foremost reason for which the Four Colour Theorem will be remembered for.

4 Conclusion

This article discusses the Four Colour Theorem from a historical and philosophical point of view. We have seen that a problem as simple to state as this one can bring with it much more complicated and controversial issues for the entire field of mathematics.

Three different computer-aided proofs are supporting the validity of the theorem. However, no one has been able to verify any of these proofs in their entirity as yet - in other words, question marks are still lingering with respect to the surveyability of the proofs. On one hand, we have the *skeptics*, who argue against such computer-aided proofs and, on the other hand, there are the *believers,* who invite us to accept the immense aid that machines can provide to the field of mathematics as a whole. The discussion is still open and the two camps are both passionate about their arguments.

The question which follows from all this is: what does it mean for the field of mathematics if the idea of proof was to be adjusted to include computer-aided proofs?

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References

[1] Davis, Philip J. (1972). *Fidelity in Mathematical Discourse: Is One and One Really Two?. New Directions in the Philosophy of Mathematics.* Princeton University Press. Princeton. New York.

[2] Appel, K. and Haken, W. (1978). *The Four-Color Problem. Mathematics Today - Twelve Informal Essays.* Springer Verlag. New York.

[3] De Millo, R., Lipton, R. and Perlis, A. (1979). Social Processes and Proofs of Theorems and Programs. *New Directions in the Philosophy of Mathematics.* Princeton University Press. Princeton. New York.

[4] Lakatos, I. (1979). What does a Mathematical Proof Prove?. *New Directions in the Philosophy of Mathematics.* Princeton University Press. Princeton. New York.

[5] Tmoczko, Thomas. (1979). The Four-Color Problem and Its Philosophical Significance. *New Directions in the Philosophy of Mathematics.* Princeton University Press. Princeton. New York.

[6] Swart, E.R. (1980). The Philosophical Implications of the four-color theorem. *American Math. Monthly*, 87, 697-707.

[7] Saaty, L. Thomas & Kainen, Paul C. (1986). *The Four-Color Problem - Assaults and Conquest.* Dover Publications Inc., New York.

[8] Cipra, A. (1989). Do Mathematicians still do Math?. *Science*, 244, 769-770.

[9] Epstein, R.L. and Carnielli, W.A. (1989). *Computability - Computable Functions, Logic, and the Foundations of Mathematics.* Wadsworth & Brooks/Cole Advanced Books & Software. Pacific Grove. California.

[10] Stillwell, S. (1992). Empirical Enquiry and Proof. *Proof and Knowledge in Mathematics.* Routledge. London.

[11] Holton, D.A. & Sheehan, J. (1993). *The Petersen Graph.* Cambridge University Press, Great Britain.

[12] Horgan, J. (1993). The death of proof. *Scientific American*, 269, 74-103.

[13] Andrews, G. (1994). The death of proof? Semi-rigorous mathematics? You've got to be kidding. *The Mathematical Intelligencer*, 16 (4), 16-18.

[14] Mackenzie, D. (1995). The automatization of proof: a historical and sociological exploration. *IEEE Annals of the History of Computing*, 17, 7-29.

[15] Gardner, M. (1996). *The Universe in a Handkerchief: Lewis Carroll's Mathematical Recreations, Games, Puzzles and Word Plays.* New York: Copernicus.

[16] Eccles, P.J. (1997). *An Introduction to Mathematical Reasoning: numbers, sets, and functions.* Cambridge University Press. Cambridge.

[17] Dales, H.G. and Oliveri, G. (1998). *Truth in Mathematics.* Claredon Press. Oxford.

[18] Hersch, Reuben. (1998). *What is Mathematics Really?*. Vintage. London.

[19] Petersen, I. (1998). *The Mathematical Tourist.* WH Freeman and Company. New York.

[20] Thomas, R. (1998). An update on the Four-Color Theorem. *Notices of the AMS*, volume 47, number 7, pages 848-859.

[21] Aigner, M. and Ziegler, G.M. (1999). *Proofs from The Book.* Berlin. Germany.

[22] Casti, J. (to appear in July 2001). *Mathematical Mountaintops.* Oxford University Press. Oxford.

[23] Archdeacon, Dan. University of Vermont, personal communication.