LEMNISCATES AND THE SPECTRUM OF THE PERTURBED SHIFT

V.L.Oleinik[1], A.P.Kalupin[2]

[1] *On leave from the Department of Physics, St.Petersburg State University, Ul'yanovskaya 1, Petrodvorets, St.Petersburg, 198504, Russia.*
[2] *Department of Physics, St.Petersburg State University, Ul'yanovskaya 1, Petrodvorets, St.Petersburg, 198504, Russia.*

Department of Mathematics report series 463 May 2001
The University of Auckland, New Zealand.

Abstract

The spectrum of the perturbed shift operator T, $T: f(n) \to f(n+1) + a(n)f(n)$, in $\ell^2(\mathbf{Z})$ is considered for $a(n)$ taking a finite set of values. It is proven that if all values of the function $a(n)$ have uniform frequencies on $\mathbf{Z}$ then the essential part of the spectrum is continuous and fills a lemniscate.

1 Introduction

In this paper we determine conditions which should be imposed on a function $a : \mathbf{Z} \to \mathbf{C}$ in such a way that the spectrum of the perturbed shift operator

$$T : f(n) \to f(n+1) + a(n)f(n),$$

acting on the space of sequences $\ell^2(\mathbf{Z})$, fills a generalized lemniscate. The case of periodic function $a(n)$ was considered in [9] and [8]. Here we deal with the general case. In particular, we allow deformations of the periodic function $a(n)$ with rare layout along the axis $\mathbf{Z}$.

1.1 Lemniscate

The essential part of the spectrum of the operator considered here lies on a lemniscate.

Definition 1.(see [4]) *Let $\{z_s\}_{s=1}^N \subset \mathbf{C}$ be a finite set of distinct complex points and let $\{\alpha_s\}_{s=1}^N$ be a set of positive numbers such that $\sum_{s=1}^N \alpha_s = 1$. For a given $r > 0$ the level curve l_r of the function*

$$p(z) := \prod_{s=1}^{N} |z - z_s|^{\alpha_s}, \tag{1.1}$$

is said to be a generalized lemniscate, that is

$$l_r := \{z \in \mathbf{C} : p(z) = r\}.$$

The points z_s are foci of the lemniscate l_r.

In what follows we will drop the word "generalized" and put $l := l_1$.

Some properties of the lemniscate can be deduced from the maximum modulus principal for subharmonic function:
1) the lemniscate separates each point z_k from infinity;
2) no point of the lemniscate l_r can lie interior to a Jordan curve consisting wholly of the points of l_r;
3) each such Jordan curve must contain inside at least one point z_k;
4) the lemniscate l_r consists of a finite number M of bounded Jordan curves, $1 \leq M \leq N$.

For rational α_s we obtain the classical definition of lemniscate (see [10], [7]). Thus, the lemniscate with equation $|z^2 - 1| = r^2$ is called the Cassini oval, or, in the case $r = 1$ the Bernoulli lemniscate (which looks like the "infinity"-sign ∞).

The lemniscate actually can be a rather intricate curve. In 1897 D. Hilbert had showed (see [5]) that for any bounded simply connected domain G and for every $\varepsilon > 0$ there exists a connected lemniscate $l_r \subset G$ such that the boundary ∂G lies in the ε-neighborhood of l_r, and l_r lies in the ε-neighborhood of ∂G:

$$\partial G \subset \{z \in \mathbf{C} : \operatorname{dist}(z, l_r) < \varepsilon\}, \;\; l_r \subset \{z \in \mathbf{C} : \operatorname{dist}(z, \partial G) < \varepsilon\}.$$

Hilbert meant a "classical" lemniscate with rational powers α. The extension of Hilbert's theorem to the general case of unbounded multiply connected domains with a compact compliment one can find in [10].

1.2 Shift operator.

Let $f(n)$ be a sequence, $f : \mathbf{Z} \to \mathbf{C}$, and S the shift operator, i.e.,

$$(Sf)(n) = f(n+1), \; n \in \mathbf{Z}.$$

Then S is a unitary operator on the space

$$\ell^2(\mathbf{Z}) = \left\{ f : |f|^2 = \sum_{n=-\infty}^{+\infty} |f(n)|^2 < \infty \right\},$$

and its spectrum $\sigma(S)$ coincides with the unit circle, which is the simplest lemniscate, $\sigma(S) = \{|z| = 1\}$. Further, let a be a multiplication operator on a function $a(n) : \mathbf{Z} \to \mathbf{C}$, and $T := S + a$ the perturbed shift, i.e.,

$$(Tf)(n) = f(n+1) + a(n)f(n), \;\; n \in \mathbf{Z}.$$

In the paper [9] (see also [8]) it is shown that for a periodic function $a(n)$ $(a(n+m) = a(n), \;\; n \in \mathbf{Z})$ the spectrum $\sigma(T)$ fills a lemniscate and the set of foci of the lemniscate is exactly the set of values of the function $a(n)$. Moreover, all powers α_s in (1.1) are rational numbers and $p(z)$ is a fractional power of the modulus of a polynomial $\mathcal{P}(z)$. The set of roots of the polynomial is a range of values of the function $a(n)$, that is

$$p(z) = |\mathcal{P}(z)|^{1/m}.$$

We will consider the similar case when the function a has a finite range of values $\{a_1, a_2, ..., a_N\}$ and every value a_s is met in the sequence $\{a(n)\}_{n=-\infty}^{+\infty}$ with a uniform "frequency" $\alpha_s \geq 0$, $\sum_{s=1}^{N} \alpha_s = 1$. Here we allow rare allocation of some values of the function a, that is $\alpha_s = 0$.

2 Main theorem

Let $a(n)$ be a function on $\mathbf{Z}$ with a finite range of values, i.e.,

$$a : \mathbf{Z} \to \{a_1, a_2, \ldots, a_N\} \subset \mathbf{C}.$$

Denote by $m_s(n, j)$ the number of integer t from the interval $[n, n+j]$ such that $a(t) = a_s$, i.e.,

$$m_s(n, j) := \#\{t \in [n, n+j] : a(t) = a_s\}; \; s = 1, 2, \ldots, N; \; n \in \mathbf{Z}; \; j = 0, 1, 2, \ldots.$$

It is clear that

$$\sum_{s=1}^{N} m_s(n,j) = j+1. \tag{2.1}$$

To each number a_s we couple non-negative number $\alpha_s \geq 0$ which has a meaning of frequency of occurrence of the number a_s in the sequence $\{a(n)\}_{n=-\infty}^{+\infty}$. More precisely, we suppose that

$$\sum_{s=1}^{N} \alpha_s = 1,$$

and

$$m_s(n,j) = \alpha_s j + \beta_s(n,j);\;\; j \geq 0;\;\; 1 \leq s \leq N; \tag{2.2}$$

where $\beta_s(n,j) = o(j)$, and $\beta_s(n-j,j) = o(j)$ as $j \to \infty$.

The small functions $\beta_s(n,j)$ can have both positive and negative values. However, if $\alpha_s = 0$ then $\beta_s(n,j) = m_s(n,j) \geq 0$. Furthermore, we have

$$0 \leq m_s(n,j) \leq j+1,$$

thus

$$-\alpha_s j \leq \beta_s(n,j) \leq j+1,$$

and in particular,

$$\max_{s,n} |\beta_s(n,j)| \leq j+1.$$

Finally, it follows from (2.1) that $\sum_{s=1}^{N} \beta_s(n,j) = 1$.

Definition 2. *A function $a(n) : \mathbf{Z} \to \{a_1, a_2, \ldots, a_N\}$ such that conditions (2.2) are fulfilled is said to be of class*
1) $\mathcal{A}$, if quotients $\beta_s(n,j)/j$ tend to zero uniformly with respect to n as j goes to infinity, i.e.

$$\lim_{j\to\infty} (\frac{1}{j} \max_{s,n} |\beta_s(n,j)|) = 0; \tag{2.3}$$

2) $\mathcal{B}$, if $\beta_s(n,j)$ are uniformly bounded, i.e.

$$D := \max_{s,n,j} |\beta_s(n,j)| < \infty. \tag{2.4}$$

Note that according to (2.4) any number a_s with $\alpha_s = 0$ appears in the sequence $\{a(n)\}_{n=-\infty}^{+\infty}$ not more then D times. It is clear that $\mathcal{B} \subset \mathcal{A}$. All the functions a considered in [9] and [8] belong to the class $\mathcal{B}$.

For a given function a of the class $\mathcal{A}$ we put

$$p(z) = p_a(z) := \prod_{s:\ \alpha_s \neq 0} |z - a_s|^{\alpha_s}. \tag{2.5}$$

Denote by $\tau(a)$ the union of the lemniscate $l = \{z : p(z) = 1\}$ and the set of those points a_s which are external to l, i.e.

$$\tau(a) = \{z \in \mathbf{C} : p(z) = 1\} \bigcup \{a_s : p(a_s) > 1\}.$$

Now we are able to formulate our main result.

Theorem. *Let $a \in \mathcal{A}$. Then:*
1) the open set $\mathbf{C} \setminus \tau(a)$ is contained in the resolvent set of the operator T;
2) if $a \in \mathcal{B}$, then

i) the spectrum $\sigma(T)$ of the operator T coincides with the closed set $\tau(a)$, i.e. $\sigma(T) = \tau(a)$; moreover, there exist positive numbers δ, C_1 and C_2 such that the estimate

$$\frac{C_1}{|p(z)-1|} \leq \|(T-z)^{-1}\|, \;\; z \in U_\delta \setminus l,$$

holds in the neighborhood $U_\delta := \{z \in \mathbf{C} : 1-\delta \leq p(z) \leq 1+\delta\}$ of the lemniscate l

ii) if no point a_s lies on l then we have

$$\|(T-z)^{-1}\| \leq \frac{C_2}{|p(z)-1|}, \;\; z \in U_\delta \setminus l.$$

3 Proof of the theorem

In the first part of the proof we will represent the resolvent of the operator T via generalized Jacobi interpolation series [10] (see also [2], [6]). Then estimates of the resolvent norm in a neighborhood of the corresponding lemniscate will be proved.

3.1 Jacobi series for the resolvent

In order to describe the spectrum $\sigma(T)$ of the operator T we construct the resolvent $(T-z)^{-1}$. To do this on the Hilbert space $\ell^2(\mathbf{Z})$ we need to consider the next nonhomogeneous equation:

$$(T-z)f = g, \;\; g \in \ell^2(\mathbf{Z}). \tag{3.1}$$

We are looking for the values of the parameter z such that for any $g \in \ell^2(\mathbf{Z})$ there exists a solution $f \in \ell^2(\mathbf{Z})$. In a more detailed appearance, the equation (3.1) looks as follows:

$$f(n+1) + (a(n)-z)f(n) = g(n), \;\; n \in \mathbf{Z}. \tag{3.2}$$

So, we deal with a linear nonhomogeneous difference equation of the first order [1].

Let us consider the next polynomial:

$$P(n,j;z) := \prod_{k=0}^{j}(z - a(n+k)); \;\; n \in \mathbf{Z}; \;\; j = 0,1,2,\ldots. \tag{3.3}$$

Then

$$P(n-j,j-1;z) = \prod_{k=1}^{j}(z - a(n-k)); \;\; n \in \mathbf{Z}; \;\; j = 1,2,3,\ldots, \tag{3.4}$$

and

$$P(n,j+1;z) = P(n,j;z)(z-a(n+j+1)),$$

$$P(n,j;z) = (z-a(n))P(n+1,j-1;z),$$

$$|P(n,j;z)| = p(z)^j B(n,j;z),$$

where

$$B(n,j;z) := \prod_{s=1}^{N} |z-a_s|^{\beta_s(n,j)}; \quad n \in \mathbf{Z}; \quad j = 0,1,2,\ldots. \tag{3.5}$$

For a given $z \in \mathbf{C} \setminus \tau(a)$ and for any finite sequence $g = g(n)$ we define a sequence $V_g(n,z)$:

$$V_g(n,z) := -\sum_{j=0}^{\infty} \frac{g(n+j)}{P(n,j;z)}, \quad \text{if } z : p(z) > 1, \ z \notin \{a_s\}_{s=1}^{N}, \tag{3.6}$$

i.e., the point z belongs to the exterior of the lemniscate l and coincides with neither of the values of the function $a(n)$;

$$V_g(n,z) := g(n-1) + \sum_{j=1}^{\infty} g(n-j-1)P(n-j,j-1;z), \quad \text{if } z : p(z) < 1, \tag{3.7}$$

i.e., the point z belongs to the interior of the lemniscate l.

We will show that the sequence V_g is a solution to the nonhomogeneous equation (3.1) and $V_g \in \ell^2(\mathbf{Z})$ for $g \in \ell^2(\mathbf{Z})$. Convergence of series (3.6) and (3.7) for any $g \in \ell^2(\mathbf{Z})$ follows from the next lemma:

Lemma 1. *For any number $z \in \mathbf{C} \setminus \tau(a)$ there exists a constant $C(z)$ such that for every finite sequence $g = g(n)$ the following inequality holds:*

$$\|V_g(z)\| \le C(z)\|g\|.$$

Proof. Let us consider points z, distinguished by their layout relative to the lemniscate.

Case 1. The point z belongs to the exterior of the lemniscate l and coincides with neither of values of the function $a(n)$, that is $z : p(z) > 1, \ z \notin \{a_s\}_{s=1}^{N}$.

Let the indicated point z be fixed. Let us estimate $|V_g(n)|$. Due to the definition of $p(z)$ in (2.5), and taking into account (2.2), (3.3), (3.5), by the Cauchy inequality we obtain:

$$|V_g(n)| \le \sum_{j=0}^{\infty} \frac{|g(n+j)|}{|P(n,j;z)|} = \sum_{j=0}^{\infty} \frac{|g(n+j)|}{p(z)^j B(n,j;z)}$$

$$\le \left(\sum_{j=0}^{\infty} \frac{1}{p(z)^j B(n,j;z)^2} \right)^{1/2} \left(\sum_{j=0}^{\infty} \frac{|g(n+j)|^2}{p(z)^j} \right)^{1/2}. \tag{3.8}$$

To estimate the first factor, we put:

$$K = K(z) := \max_s(\max\{|z-a_s|, |z-a_s|^{-1}\}).$$

It is clear that $K > 1$, and since $z \notin \{a_s\}_{s=1}^N$ we have $K < \infty$. It follows from (2.3) that for every $\varepsilon > 0$ there exists a number J such that for any $j > J$ uniformly with respect to s and n the following inequality holds

$$|\beta_s(n,j)| < \varepsilon j.$$

Choosing such J we get the next finite number

$$C_1(z) := \max_n \sum_{j=0}^{J} \frac{1}{p(z)^j B(n,j;z)^2}.$$

Since

$$\frac{1}{|z-a_s|^{2\beta_s(n,j)}} \le K^{2\varepsilon j},$$

we can estimate the first factor in the last part of (3.8) in the following way

$$\sum_{j=0}^{\infty} \frac{1}{p(z)^j B(n,j;z)^2} \le C_1(z) + \sum_{j=J+1}^{\infty} \frac{K^{2\varepsilon j N}}{p(z)^j} = C_1(z) + \sum_{j=J+1}^{\infty} \left(\frac{K^{2\varepsilon N}}{p(z)}\right)^j.$$

We know that $p(z) > 1$ and we first choose $\varepsilon = \varepsilon(z) < 1$ such that

$$K^{2\varepsilon N} = (1-\varepsilon)p(z).$$

Then we can find an appropriate number $J = J(z)$. So, we get the estimations

$$\sum_{j=0}^{\infty} \frac{1}{p(z)^j B(n,j;z)^2} \le C_1(z) + \frac{1}{\varepsilon(z)},$$

and

$$|V_g(n)|^2 \le C_2(z) \sum_{j=0}^{\infty} \frac{|g(n+j)|^2}{p(z)^j},$$

with

$$C_2(z) := C_1(z) + \frac{1}{\varepsilon(z)}.$$

Further:

$$\sum_{n=-\infty}^{+\infty} |V_g(n)|^2 \le C_2(z) \sum_{n=-\infty}^{+\infty} \sum_{j=0}^{\infty} \frac{|g(n+j)|^2}{p(z)^j} = C_2(z) \sum_{j=0}^{\infty} \frac{1}{p(z)^j} \sum_{n=-\infty}^{+\infty} |g(n+j)|^2$$

$$= C_2(z)\|g\|^2 \frac{1}{1-1/p(z)} = C_2(z)\frac{p(z)}{p(z)-1}\|g\|^2.$$

Finally putting

$$C(z)^2 = C_2(z)\frac{p(z)}{p(z)-1}$$

we obtain the required estimation.

Case 2. The point z belongs to the interior of the lemniscates l and coincides with neither of its foci, i.e. $z : p(z) < 1, \ \ z \notin \{a_s : \alpha_s > 0\}_{s=1}^N$.

Let the indicated point z be fixed. Using the inequality $(a+b)^2 \le 2(a^2+b^2)$ we obtain:

$$|V_g(n)|^2 \le 2|g(n-1)|^2 + 2\left(\sum_{j=1}^{\infty}|g(n-j-1)||P(n-j,j-1;z)|\right)^2. \tag{3.9}$$

Taking into account (2.2) and (3.4), we can present $|P(n-j,j-1;z)|$ in (3.9) as a product of terms $|z-a_s|$ where $s = 1,2,\dots,N$, of two types: the first type contains foci of the lemniscate, i.e. such a_s that $\alpha_s > 0$, and the second one contains a_s with $\alpha_s = 0$. So, we need only upper estimations of the factors $|z-a_s|$ with $\alpha_s = 0$. By

$$|P(n-j,j-1;z)| = p(z)^{j-1}B(n-j,j-1;z),$$

where $B(n-j,j-1;z)$ is the product of factors of two types mentioned above, we obtain:

$$\sum_{j=1}^{\infty}|g(n-j-1)||P(n-j,j-1;z)| = \sum_{j=1}^{\infty}|g(n-j-1)|p(z)^{j-1}B(n-j,j-1;z)$$

$$\le \left(\sum_{j=1}^{\infty}p(z)^{j-1}B(n-j,j-1;z)^2\right)^{1/2}\left(\sum_{j=1}^{\infty}|g(n-j-1)|^2p(z)^{j-1}\right)^{1/2}.$$

Define $Q(z) := \max_k |z-a_k|$. Then, for any $s = 1,2,\dots,N$ the following estimation holds $|z-a_s| \le Q(z)$. Further, considering only foci we put $q(z) := \min_{k:\alpha_k>0}|z-a_k|$. Then for any s with $\alpha_s > 0$ the estimation is valid:

$$|z-a_s| \ge q(z) > 0.$$

We designate the greatest of values $Q(z)$ and $1/q(z)$ as L:

$$L = L(z) := \max\{Q(z), 1/q(z)\}.$$

Then, for all s, n, j it is true that

$$|z-a_s|^{\beta_s(n-j,j-1)} \le L^{|\beta_s(n-j,j-1)|}.$$

Now by (2.3) for a fixed $\varepsilon > 0$ we choose J such that for any $j > J$ we have

$$|\beta_s(n-j,j-1)| < \varepsilon j.$$

Thus

$$|z-a_s|^{2\beta_s(n-j,j-1)} \le L^{2\varepsilon j}.$$

Therefore

$$\sum_{j=1}^{\infty} p(z)^{j-1} B(n-j,j-1;z)^2 \le C_3(z) + \sum_{j=J+1}^{\infty} p(z)^{j-1} L^{2\varepsilon jN},$$

with

$$C_3(z) := \max_n \sum_{j=1}^{J} p(z)^{j-1} B(n-j,j-1;z)^2 < \infty.$$

In the case $0 < p(z) < 1$ let us fix $\varepsilon = \varepsilon(z) < 1$ such that the following condition would be fulfilled

$$p(z)L^{2\varepsilon N} = 1-\varepsilon.$$

Then we can find an appropriate number $J = J(z)$. So, we obtain:

$$\sum_{j=J+1}^{\infty} p(z)^{j-1} L^{2\varepsilon jN} = \sum_{j=J+1}^{\infty} \frac{p(z)^{j-1}(1-\varepsilon)^j}{p(z)^j} = \frac{1}{p(z)} \sum_{j=J+1}^{\infty} (1-\varepsilon)^j \le \frac{1}{p(z)\varepsilon(z)}$$

and

$$|V_g(n)|^2 \le 2|g(n-1)|^2 + 2C_4(z) \sum_{j=1}^{\infty} |g(n-j-1)|^2 p(z)^{j-1},$$

with

$$C_4(z) := C_3(z) + \frac{1}{p(z)\varepsilon(z)}.$$

Finally

$$\frac{1}{2}\sum_{n=-\infty}^{+\infty} |V_g(n)|^2 \le \sum_{n=-\infty}^{+\infty} |g(n-1)|^2 + C_4(z) \sum_{n=-\infty}^{+\infty} \sum_{j=1}^{\infty} |g(n-j-1)|^2 p(z)^{j-1}$$

$$= \|g\|^2 + C_4(z) \sum_{j=1}^{\infty} p(z)^{j-1} \sum_{n=-\infty}^{+\infty} |g(n-j-1)|^2 = \|g\|^2 + \frac{C_4(z)\|g\|^2}{1-p(z)}.$$

So, we obtain the required estimation with

$$C(z)^2 := 2 + 2C_4(z)/(1-p(z)).$$

Case 3. The point $z = z_0$ coincides with a focus of the lemniscate, i.e. there exists a number t with $a_t = z_0$ and $\alpha_t > 0$.

We will show the existence of such number $M \ge 1$ that the function $a(n)$ takes the value z_0 on the interval $[n, n+M]$. Therefore, $P(n, M; z_0) = 0$ for all $n \in \mathbf{Z}$.

Really, let the product $P(n, j;\ z_0)$ be nonzero. Then

$$m_t(n,j) = \alpha_t j + \beta_t(n,j) = 0. \tag{3.10}$$

The relation (3.10) can be fulfilled only for the interval $[n, n+j]$ of finite length since this is equivalent to the equality:

$$\frac{\beta_t(n,j)}{j} = -\alpha_t.$$

The ratio in the left part of the equality tends to zero uniformly with respect to n as $j \to \infty$. Therefore, there exists a number M such that for any $j \geq M$ and $n \in \mathbf{Z}$ the relations

$$\frac{|\beta_t(n,j)|}{j} < \alpha_t; \quad m_t(n,j) \geq 1; \quad P(n,j;z_0) = 0;$$

are fulfilled. The existence of the above mentioned number M is proved.

Define

$$R := \max\{1, \max_{n,j} |P(n-j, j-1; z_0)|\}.$$

Then

$$|V_g(n)| \leq |g(n-1)| + \sum_{j=1}^{M} |g(n-j-1)||P(n-j, j-1; z_0)|$$

$$\leq |g(n-1)| + R\sum_{j=1}^{M} |g(n-j-1)| \leq R\sum_{j=0}^{M} |g(n-j-1)|$$

$$\leq R(M+1)^{1/2} \left(\sum_{j=0}^{M} |g(n-j-1)|^2 \right)^{1/2}.$$

Further

$$\sum_{n=-\infty}^{+\infty} |V_g(n)|^2 \leq R^2(M+1) \sum_{n=-\infty}^{+\infty} \sum_{j=0}^{M} |g(n-j-1)|^2 = R^2(M+1) \sum_{j=0}^{M} \sum_{n=-\infty}^{+\infty} |g(n-j-1)|^2$$

$$= R^2(M+1)^2 \|g\|^2.$$

Supposing

$$C(z) := R(M+1),$$

we again obtain the required estimation. •

Extending the map $g \to V_g$ on the space $\ell^2(\mathbf{Z})$ by continuity, we obtain the following statement.

Lemma 2. *Let $z \in \mathbf{C} \setminus \tau(a)$. Then for any $g \in \ell^2(\mathbf{Z})$ the sequence $f(n) := V_g(n) \in \ell^2(\mathbf{Z})$, and f is a solution to the equation (3.2).*

Proof. Let us check the second part of the assertion of the lemma by means of direct substitution of $f(n)$ into the equation.

Case 1. The point z belongs to the exterior of the lemniscate l and coincides with neither of values of the function $a(n)$, i.e. $z : p(z) > 1, z \notin \{a(s)\}_{s=1}^{N}$. In this case we have

$$f(n) = -\sum_{j=0}^{\infty} \frac{g(n+j)}{P(n,j;z)}; \quad f(n+1) = -\sum_{j=0}^{\infty} \frac{g(n+j+1)}{P(n+1,j;z)}.$$

Taking (3.4) into account , we obtain:

$$(a(n)-z)f(n) = (z-a(n))(\frac{g(n)}{P(n,0;z)} + \sum_{j=1}^{\infty} \frac{g(n+j)}{P(n,j;z)}) =$$

$$= g(n) + \sum_{j=1}^{\infty} (z-a(n)) \frac{g(n+j)}{(z-a(n))P(n+1,j-1;z)} = g(n) + \sum_{j=0}^{\infty} \frac{g(n+j+1)}{P(n+1,j;z)} =$$

$$= g(n) - f(n+1),$$

i.e. $f(n)$ is a solution of the equation (3.2).

Case 2. The point z belongs to the interior of the lemniscate l and can coincide with a focus. It was shown in the proof of Lemma 1 that if $z \in \{a_s\}$ then the formula (3.6) contains only a finite number of items, so, $f \in \ell^2(\mathbf{Z})$. As above we have

$$f(n) = g(n-1) + \sum_{j=1}^{\infty} g(n-j-1)P(n-j, j-1; z),$$

$$f(n+1) = g(n) + \sum_{j=1}^{\infty} g(n-j)P(n-j+1, j-1; z).$$

Taking into account (3.4), we get

$$(a(n)-z)f(n) = (z-a(n))g(n-1) - \sum_{j=1}^{\infty} g(n-j-1)(z-a(n))P(n-j,j-1;z) =$$

$$= -\sum_{j=1}^{\infty} g(n-j)P(n-j+1, j-1; z) = g(n) - f(n+1),$$

which means that $f(n)$ is a solution of the equation (3.2). •

¿From Lemmas 1 and 2 we obtain representation for the resolvent on the open set $\mathbf{C} \setminus \tau(a)$.

Corollary. *The resolvent of the operator T on $\ell^2(\mathbf{Z})$ can be written as follows*

$$((T-z)^{-1}g)(n) = -\sum_{j=0}^{\infty} \frac{g(n+j)}{P(n,j;z)}, \tag{3.11}$$

if the point z belongs to the exterior of the lemniscate l and does not coincide with a value of the function $a(n)$, i.e. $z : p(z) > 1, z \notin \{a_s\}_{s=1}^N$, and

$$((T-z)^{-1}g)(n) = g(n-1) + \sum_{j=1}^{\infty} g(n-j-1)P(n-j, j-1; z), \tag{3.12}$$

if z belongs to the interior of the lemniscate l, i.e. $z : p(z) < 1$.

The points a_s from the exterior of the lemniscate l are singular points of the resolvent. Thus, the part 1) of Theorem is proved.

Remarks. 1) It is easy to show that if $p(a_{s_0}) > 1$ and

a) $N_0 := \inf\{n : a(n) = a_{s_0}\} > -\infty$ then the number $z = a_{s_0}$ is an eigenvalue of the operator T, and the dimension of the corresponding invariant subspace coincides with the number of different values n for which $a(n) = a_{s_0}$ (see [8]). Thus the operator T has a Jordan one-side bounded box at the point a_{s_0};

b) $N_0 = -\infty$ then the corresponding Jordan box should be two-side unbounded.

2) Let no point a_s lies on l. Then we can choose a positive number δ such that the neighborhood U_δ is free of a_s. Let $a \in \mathcal{B}$ then there exist constants b_1 and b_2 such that for any n, j and $z \in U_{\delta/2}$ we have

$$b_1 \le |B(n,j;z)| \le b_2.$$

Thus, there exists a constant C such that

$$\|(T-z)^{-1}\| \le \frac{C}{|p(z)-1|}, \quad z \in U_{\delta/2} \setminus l.$$

This proves the statement ii) of Theorem.

3.2 A lower estimate of the resolvent near the lemniscate

In this section we will get a lower estimate only in the case $p(z) > 1$. The opposite case $p(z) < 1$ can be considered in a similar way.

As far as $p(z) > 1$, the formula (3.11) is valid for the resolvent. Let us apply this to the function

$$g(n) = \chi(n) := \begin{cases} 1, \ n = 0, \\ 0, \ n \ne 0. \end{cases}$$

We obtain:

1) if $n \ge 1$ then $f(n) \equiv 0$;

2) if $n \le 0$ then

$$f(n) = ((T-z)^{-1}\chi)(n) = -\sum_{j=0}^{\infty} \frac{\chi(n+j)}{P(n,j;z)} = -\frac{1}{P(n,|n|;z)}.$$

Therefore, we have:

$$|f(n)| = \frac{1}{|P(n,|n|;z)|} = \frac{1}{p(z)^{|n|} B(n,|n|;z)}, \quad n \le 0.$$

We need an upper estimate for $B(n,|n|;z)$ on the set $U_{\delta/2}$, where U_δ is free of a_s. Note that $\beta_s(n,|n|;z) = m_s(n,|n|) \ge 0$. Denote

$$A_1 := \max_{z \in U_{\delta/2}} \max_s |z - a_s|.$$

The sets $U_{\delta/2}$ and $\{a_s\}_{s=1}^N$ are bounded, therefore $A_1 < \infty$. In particular,

$$\max_{s:\ \alpha_s} |z - a_s| \le A_1.$$

On the other hand closed sets $U_{\delta/2}$ and the set of foci of the lemniscate are disjoint. Thus

$$A_2 := \min_{z \in U_{\delta/2}} \max_{s:\ \alpha_s > 0} |z - a_s| > 0,$$

and by (2.4) we have the next estimate

$$0 \le \max_{z \in U_{\delta/2}} B(n, |n|; z) \le A^{DN},$$

with

$$A := \max\{A_1, A_2^{-1}, 1\}.$$

Thus, taking into account that $\|\chi\| = 1$, we get the necessary estimation of the resolvent norm for $z \in U_{\delta/2} \setminus l$:

$$\|(T - z)^{-1}\ |^2 \ge \|(T - z)^{-1}\chi\|^2 = \sum_{k=0}^{\infty} |f(-k)|^2$$

$$\ge A^{-2DN} \sum_{k=0}^{\infty} \frac{1}{p(z)^{2k}} = \frac{p(z)^2}{A^{2DN}(p(z)^2 - 1)} \ge \frac{1}{(2 + \delta)A^{2DN}(p(z) - 1)},$$

Thus, it is proved that the set $\mathbf{C} \setminus \tau(a)$ is the resolvent set of the operator T, and its spectrum $\sigma(T)$ coincides with the set $\tau(T)$. The latter completes the proof of Theorem.•

Remark. The proof of our main theorem depends crucially on the product rule $|zw| = |z||w|$ which holds for all $z, w \in \mathbf{C}$. One can expect that once this condition is fulfilled, we may consider the shift on generalized complex numbers: the quaternions $\mathbf{H}$ and the Cayley numbers $\mathbf{K}$ (see [3], [11]). Note that for these numbers the commutative law of multiplication does no hold and for the Cayley numbers even the associative law of multiplication is lost. However every non-zero element of $\mathbf{H}$ or $\mathbf{K}$ has an inverse. The case of periodic function $a(n)$ see in [9].

4 Acknowledgements

This research was carried out at the University of Auckland, New Zealand, and the first author (V.O.) is grateful to the Department of Mathematics for the excellent working conditions. The authors sincerely thank B. S. Pavlov and M. M. Faddeev for many useful discussions.

References

[1] R.P. Agarwal, Difference Equations and Inequalities: Theory, Methods, and Applications, Marcel Dekker, 1992.

[2] J.H. Curtiss, On the Jacobi series, Trans. of the AMS, v. 49, 3, 1941, pp. 467-501.

[3] H.-D. Ebbinghaus et al., Numbers, Springer-Verlag, 1991.

[4] H. Grunsky, Lectures on Theory of Functions in Multiply Connected Domains, Gttingen ; Zrich : Vandenhoeck und Ruprecht, 1978

[5] D. Hilbert, Über die Entwicklung einer beliebigen analytischen Funktion einer Variabeln in eine unendliche nach ganzen rationalen Funktionen fortschreitende Reihe, *Göttinger Nachrichten*, 1897, pp. 63-70.

[6] P.Ja. Kiselev, 1) The coefficients of Jacobi series, (Ukrainian), Dopovidi Akad. Nauk Ukrain. RSR ser. A, v. 1970, 1970, pp. 978-980, 1052; 2) The Jacobi series, (Russian), Ukrain. Mat. Z., v. 23, 1971, pp. 399-405.

[7] A.I. Markushevich, Theory of Functions of a Complex Variable, v.1, Prentice-Hall, 1965.

[8] V.L. Oleinik, The tight-binding approximation method on the lemniscate, (Russian), Zap.Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), v. 270, 2000, pp. 258-276, 368.

[9] V.L. Oleinik and B.S. Pavlov, Hilbert theorem on lemniscate and the spectrum of the perturbed shift, Department of Mathematics, University of Auckland,New Zealand, report series 417, May 1999, pp. 1-11.

[10] J.L. Walsh, Interpolation and approximation rational functions in the complex domain, 2nd ed., American Mathematical Society, 1956.

[11] J.P. Ward, Quaternions and Cayley Numbers, Kluwer Academic Publishers, 1997.