On π-Consistent Social Choice Functions

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Abstract: In this paper we introduce the concept of $\mathcal{F}$-consistency of a social choice function relative to the given class $\mathcal{F}$ of social choice functions. This refines the concept of consistency, introduced by Koray [2], and allows to discover a number of classes $\mathcal{F}$ for which there exist $\mathcal{F}$-consistent social choice functions which are neither dictatorial nor antidictatorial. Furthermore, under certain mild conditions on $\mathcal{F}$ all $\mathcal{F}$-consistent social choice functions are described.

1. Introduction

It is not uncommon that a society of voters $\mathcal{N}$ facing a choice between the alternatives of a set A has also to decide on the procedure which is to be employed to make this choice. Suppose that only the social choice functions (SCFs) from a class $\mathcal{F}$ are socially acceptable (for example, the Paretian omes). Suppose also that the information about voters' preferences is publicly known. Then the voters are able to compare the outcomes, which will result from the implementation of each of the available SCFs, and their preferences on the set $\mathcal{F}$ are therefore defined. Thus, any preference profile on A leads to a dual preference profile that the same voters have on $\mathcal{F}$. It is now possible to employ any SCF from $\mathcal{F}$ to choose the SCF, which will be employed to make a choice from A. It would be consistent to use for the main choice any SCF that chooses itself at the first stage of the procedure. Koray [2] investigated such self-selective SCFs with $\mathcal{F}$ being the class of all possible SCFs on A - he called such SCFs consistent. Under these assumptions he proved that all neutral and unanimous consistent SCFs are the dictatorial ones.

In this paper we investigate the concept of consistency in the most general situation. We do not assume neither neutrality nor unanimity and allow for a large variety of classes $\mathcal{F}$, e.g., $\mathcal{F}$ may be the class of all Paretian SCFs. In these circumstances we characterize all $\mathcal{F}$-consistent SCFs and show that, when $\mathcal{F}$ is not the whole set of all possible SCFs, interesting non-dictatorial $\mathcal{F}$-consistent SCFs do exist. For example, when $\mathcal{F}$ consists of all Paretian SCFs, the SCF, which chooses the worst Pareto-optimal alternative for the ith voter, is $\mathcal{F}$-consistent.

2. Social Choice Functions and Correspondences

Let A be a finite set of m alternatives and $\mathcal{N}$ be a finite set of n agents who are to make a choice of one best alternative from A. To this end, the agents submit strict linear orders $R_1, \ldots, R_n$ on A as their preferences over the given set of alternatives. The n-tuple $R = (R_1, \ldots, R_n)$ is called the n-profile or, simply, the profile. The set of all strict linear orders on A is denoted as $\mathcal{L}(A)$ and the set of all n-profiles is denoted as $\mathcal{L}(A)^n$.

Definition 1 *A social choice function (SCF) F on A is a mapping*

$$F\colon \mathcal{L}(A)^n \to A. \tag{1}$$

A social choice correspondence (SCC) π on A is a mapping

$$\pi\colon \mathcal{L}(A)^n \to \mathcal{P}(A),$$

where $\mathcal{P}(A)$ is the power set of A.

In this section we will pay attention to the following question: how, given an SCF (or an SCC) on A, to define an SCF (or an SCC) on any set B which is equinumerous to A.

Let A and B be two equinumerous finite sets and let $\nu\colon A \to B$ be a bijection. Then, for all $n \geq 1$, we can define a bijection

$$\bar{\nu}\colon \mathcal{L}(A)^n \to \mathcal{L}(B)^n$$

as follows. Suppose that we have a relation Q on A. Then we can define a relation $Q^{\bar{\nu}}$ on B in the following way:

$$bQ^{\bar{\nu}}b' \stackrel{\text{def}}{=} \nu^{-1}(b)Q\nu^{-1}(b'). \tag{2}$$

If $R = (R_1, \ldots, R_n) \in \mathcal{L}(A)^n$ is a profile on A, then we define a profile $R^{\bar{\nu}}$ on B by setting $R^{\bar{\nu}} = (R_1^{\bar{\nu}}, \ldots, R_n^{\bar{\nu}})$.

Proposition 1 *Let A, B, C be three finite equinumerous sets and $\mu: A \to B$ and $\nu: B \to C$ be bijections. Then $\overline{\mu \circ \nu} = \bar{\mu} \circ \bar{\nu}$.*

Proof: It suffices to prove the statement for $n = 1$, i.e. for linear orders. Let Q be a linear order on A. Then for any $c, c' \in C$

$$cQ^{\overline{\mu \circ \nu}}c' = (\mu \circ \nu)^{-1}(c)Q(\mu \circ \nu)^{-1}(c') = (\nu^{-1} \circ \mu^{-1})(c)Q(\nu^{-1} \circ \mu^{-1})(c')) =$$

$$\nu^{-1}(\mu^{-1}(c))Q\nu^{-1}(\mu^{-1}(c')) = \mu^{-1}(c)Q^{\bar{\nu}}\mu^{-1}(c') = c\,(Q^{\bar{\nu}})^{\bar{\mu}}\,c' = cQ^{\bar{\mu} \circ \bar{\nu}}c'. \ \square$$

Suppose now that F is an arbitrary SCF on a set X. Let A be a set equinumerous to X, and let $\nu: A \to X$ be a bijection. Let $R \in \mathcal{L}(A)^n$ be a profile on A. We define

$$F^{\nu}(R) \stackrel{\text{def}}{=} \nu^{-1}(F(R^{\bar{\nu}})). \tag{3}$$

This could be illustrated by the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\bar{\nu}} & R^{\bar{\nu}} \\ {\scriptstyle F^{\nu}}\downarrow & & \downarrow{\scriptstyle F} \\ A & \xrightarrow{\nu} & X \end{array}$$

Clearly, F^{ν} is an SCF on A. Therefore, having an SCF on X we can induce an SCF on A for each bijection $\nu: A \to X$. Thus, if X was of cardinality m, then we can get $m!$ different SCFs on A. Of course, in some cases, for different bijections μ and ν the SCFs F^{μ} and F^{ν} may well coincide.

Proposition 2 *Let A, B, C be three finite equinumerous sets and $\mu: A \to B$ and $\nu: B \to C$ be bijections. Let F be an SCF on C. Then*

$$F^{\nu \circ \mu}(R) = \mu^{-1}F^{\nu}(R^{\bar{\mu}}), \tag{4}$$

i.e. the following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\bar{\mu}"] \arrow[d, "F^{\nu\circ\mu}"'] & R^{\bar{\mu}} \arrow[d, "F^{\nu}"] \\ A \arrow[r, "\mu"] & B \end{tikzcd}$$

Proof: We have

$$F^{\nu\circ\mu}(R) = (\nu\circ\mu)^{-1}F(R^{\overline{\nu\circ\mu}}) = \mu^{-1}\circ\nu^{-1}F((R^{\bar{\mu}})^{\bar{\nu}}) = \mu^{-1}F^{\nu}(R^{\bar{\mu}}).\Box$$

Suppose now that P is an arbitrary SCC on X. Let A be a set equinumerous to X, and let $\nu: A \to X$ be a bijection. Let $R \in \mathcal{L}(A)^n$ be a profile on A. We define

$$P^{\nu}(R) \stackrel{\text{def}}{=} \nu^{-1}(P(R^{\bar{\nu}})). \tag{5}$$

This could be illustrated by the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\bar{\nu}"] \arrow[d, "P^{\nu}"'] & R^{\bar{\nu}} \arrow[d, "P"] \\ \mathcal{P}(A) \arrow[r, "\nu"] & \mathcal{P}(X) \end{tikzcd}$$

Proposition 3 *Let A, B, C be three finite equinumerous sets and $\mu: A \to B$ and $\nu: B \to C$ be bijections. Let P be a SCC on C. Then*

$$P^{\nu\circ\mu}(R) = \mu^{-1}P^{\nu}(R^{\bar{\mu}}). \tag{6}$$

The proof of this following proposition is similar to that of Proposition 2.

3. Universal Social Choice Functions and Correspondences

The common sense says that a social choice function must be applicable to a large variety of the situations in which choice is to be made. In particular, we want SCFs and SCCs to be applicable for any finite set of agents $\mathcal{N}$ and for any finite set of alternatives A. To this end, we need a generic set of alternatives, and in this capacity we take the subset $I_m = \{1, 2, \ldots, m\}$ of the set of all positive integers $\mathbb{N}$. We define

Definition 2 *A social choice function (SCF) F is a mapping*

$$F\colon \bigcup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n \to \mathbb{N} \tag{7}$$

such that $F(R) \in I_m$ for each $R \in \mathcal{L}(I_m)^n$. A social choice correspondence (SCC) π is a mapping

$$\pi\colon \bigcup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n \to \mathcal{P}(\mathbb{N}), \tag{8}$$

where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$, such that $F(R) \subseteq I_m$ for each $R \in \mathcal{L}(I_m)^n$.

In the sequel, if we do not specifically mention the set of alternatives, on which the SCF is defined, it means that we have in mind a universal SCF of the latter definition.

Now, using the machinery defined in the previous section, we may extend any of the universal SCFs and SCCs to SCFs and SCCs on A. Let F be an arbitrary SCF and let the cardinality of A be m. Suppose that $\nu\colon A \to I_m$ is a bijection. Then a pair (A, ν) is called an indexed (numbered) set. Let $R \in \mathcal{L}(A)^n$ be a profile on A. We define

$$F^{\nu}(R) \stackrel{\text{def}}{=} \nu^{-1} F(R^{\bar{\nu}}). \tag{9}$$

By S_m we denote the symmetric group of all permutations on I_m.

Definition 3 *An SCF F is said to be neutral at a profile $R \in \mathcal{L}(I_m)^n$ if for every permutation $\sigma\colon I_m \to I_m$ from S_m we have*

$$F(R^{\bar{\sigma}}) = \sigma F(R). \tag{10}$$

An SCF F is said to be neutral for m alternatives if it is neutral at any profile of $R \in \mathcal{L}(I_m)^n$. An SCF F is said to be neutral if it is neutral at any profile.

The symmetric group S_m acts on $\mathcal{L}(I_m)^n$ by means of $\sigma \cdot R \stackrel{df}{=} R^{\bar{\sigma}}$. Let us denote by $<R>$ the orbit in $\mathcal{L}(I_m)^n$ to which R belongs, i.e. $<R> = \{\sigma \cdot R \mid \sigma \in S_m\}$.

Lemma 1 *If an SCF F is neutral at a profile R, then it is neutral at any profile of the orbit $<R>$.*

Proof: Let us assume that F is neutral at a profile R. Suppose $\nu \in S_m$ is fixed and $\mu \in S_m$ is arbitrary. Let $\sigma = \mu \circ \nu$. Then, since F is neutral at R,

$$F(R^{\overline{\mu \circ \nu}}) = F(R^{\bar{\sigma}}) = \sigma F(R) = (\mu \circ \nu)F(R),$$

or due to Proposition 1 and neutrality

$$F((R^{\bar{\nu}})^{\bar{\mu}}) = \mu(\nu F(R)) = \mu F(R^{\bar{\nu}}).$$

This means that F is neutral at $R^{\bar{\nu}}$. But $R^{\bar{\nu}}$ is an arbitrary profile of $<R>$. $\square$

Proposition 4 *Let R be a profile on A and $\eta\colon A \to I_m$ be a bijection. Then the following statements are equivalent:*

1. *F is neutral at $<R^{\bar{\eta}}>$;*
2. *$F^{\eta}(R)$ does not depend on η, i.e., for any other bijection $\mu\colon A \to I_m$ we have $F^{\eta}(R) = F^{\mu}(R)$.*

Proof: Let $\sigma = \mu \circ \eta^{-1}$. The neutrality condition written for σ and the profile $R^{\bar{\eta}}$ would be

$$F\left((R^{\bar{\eta}})^{\bar{\sigma}}\right) = \sigma F\left(R^{\bar{\eta}}\right)$$

which is by Proposition 1 can be written as

$$F\left(R^{\bar{\mu}}\right) = (\mu \circ \eta^{-1}) F\left(R^{\bar{\eta}}\right).$$

This, in turn, is equivalent to

$$\mu^{-1} F\left(R^{\bar{\mu}}\right) = \eta^{-1} F\left(R^{\bar{\eta}}\right),$$

or $F^{\mu}(R) = F^{\eta}(R)$. $\square$

Definition 4 *If the conditions of Proposition 4 hold, we will say (with some abuse of the terminology) that F is neutral at R and denote as $F(R)$ the common value of $F^{\eta}(R)$ since it does not really depend on the bijection η.*

Proposition 5 *Let R be a profile on A and $\mu: A \to B$ be a bijection. Let F be an SCF on A which is neutral at R. Then F is also neutral at $R^{\bar{\mu}}$ and*

$$F(R) = \mu^{-1} F(R^{\bar{\mu}}). \tag{11}$$

Proof: Let $\nu: B \to I_m$ be any bijection and $\eta = \nu \circ \mu$. Then by (4) of Proposition 2

$$F^{\eta}(R) = F^{\nu \circ \mu}(R) = \mu^{-1} F^{\nu}(R^{\bar{\mu}}).$$

Since the left-hand-side does not depend on η, the right-hand-side does not depend on ν, which proves the statement. $\square$

Definition 5 *An SCC P is said to be neutral at a profile $R \in \mathcal{L}(I_m)^n$ if for every permutation $\sigma: I_m \to I_m$ from S_m we have*

$$P(R^{\bar{\sigma}}) = \sigma(P(R)). \tag{12}$$

An SCC is said to be neutral if it is neutral at every profile.

Everything that was said in relation to the neutrality of SCFs can be repeated word by word in relation to SCCs.

4. Two Main Concepts.

These two main concepts to be defined in this section are π-completeness and independence of π-irrelevant alternatives for π being any SCC.

Definition 6 *Let π be an SCC. We say that elements of $\pi(R)$ are the π-optimal alternatives of R. We say that an SCF F is π-ian if for every profile R the alternative $F(R)$ is π-optimal.*

Example 1 *If $\pi(R)$ is the set of all Pareto optimal alternatives of the profile R, then the π-ian SCFs are exactly the Paretian ones.*

Definition 7 *Let $\mathcal{F}$ be a set of SCFs and let π be an SCC. We say that $\mathcal{F}$ is a π-complete set of SCFs if for every profile R and every element $a \in \pi(R)$ there exists an SCF $F \in \mathcal{F}$ such that $F(R) = a$.*

Example 2 *Let $\pi(R)$ be the set of all first preferences of the profile R, then any set of SCFs containing all dictatorial SCFs is π-complete.*

Let $R \in \mathcal{L}(I_m)^n$ be a profile which represents preferences of agents from $\mathcal{N}$. Suppose that we have also a finite set of SCFs $\mathcal{F}$. Then the agents can compare also these SCFs in a way that the i-th agent prefers an SCF F to an SCF G, if $F(R)R_iG(R)$, or she is indifferent, if $F(R) = G(R)$. We defined therefore a complete preorder P_i^* on $\mathcal{F}$. By breaking ties and introducing strict linear orders on indifference classes we may obtain a strict linear order R_i^*. Of course, this can be done in many different ways. Any profile $R^* = (R_1^*, \ldots, R_n^*)$, so obtained, will be called a profile dual to R on the set of SCFs $\mathcal{F}$.

Definition 8 *Let $\mathcal{F}$ be a set of SCFs. An SCF F is said to be $\mathcal{F}$-consistent at a profile $R \in \mathcal{L}(I_m)^n$ if, for any finite set of SCFs $\mathcal{F}' \subseteq \mathcal{F}$, there exists at least one dual profile R^* on $\mathcal{G} = \{F\} \cup \mathcal{F}'$ such that for every bijection $\nu: \mathcal{G} \to I_k$, where k is the cardinality of $\mathcal{G}$, the SCF F^ν, being applied to R^*, chooses F, i.e., $F^\nu(R^*) = F$ for all ν. An SCF F is said to be $\mathcal{F}$-consistent if it is $\mathcal{F}$-consistent at every profile.*

Let π be an SCC and R be a profile. By $\pi^-(R)$ we will denote the set of all alternatives which are not π-optimal relative to R. The following key lemma relates the condition of $\mathcal{F}$-consistency with the more familiar conceptual framework.

Lemma 2 *Let π be an SCC and let $R \in \mathcal{L}(I_m)^n$ be a profile. Let F be an SCF which is $\mathcal{F}$-consistent at R for some π-complete set $\mathcal{F}$ of SCFs. Let Q be a subset of I_m of cardinality $m-k$ such that $\pi^-(R) \subseteq Q \subseteq I_m$. Let $A = I_m \setminus Q$. Then for every bijection $\nu: A \to I_k$*

$$(F(R) \in A) \Longrightarrow (F(R) = F^\nu(R|_A)). \tag{13}$$

In particular, F is neutral at $R|_A$.

Proof: Suppose $F(R) \in A$. Note that all elements in A are π-optimal, hence there exists a subset $\mathcal{G} \subseteq \mathcal{F}$ of cardinality k such that $F \in \mathcal{G}$ and for every $a \in A$ there exists an SCF $G \in \mathcal{G}$ such that $G(R) = a$. Let $\mu: \mathcal{G} \to A$ be a bijection such that $\mu(G) = G(R)$. Let $\sigma = \nu \circ \mu$.

Let us denote $S = R|_A$ to be the restriction of R onto A. Then, using the mapping μ^{-1} we can induce a profile $S^{\overline{\mu^{-1}}}$ on $\mathcal{G}$. Note that $S^{\overline{\mu^{-1}}}$ coincides with the unique dual profile S^* on $\mathcal{G}$. Thus, by $\mathcal{F}$-consistency of F, we have

$$F^\sigma(S^{\overline{\mu^{-1}}}) = F^\sigma(S^*) = F.$$

Having the definition of μ in mind, by Proposition 1 and by (4) of Proposition 2, we obtain

$$F(R) = \mu(F) = \mu F^{\sigma}(S^{\overline{\mu^{-1}}}) = F^{\nu}(S).$$

Since the right-hand-side does not depend on ν, by Proposition 4 we have the neutrality of F at S. $\square$

We will call the condition (13) the Independence of π-Irrelevant Alternatives (π-IIA). Note that, although we proved that F is neutral at $R|_A$, it does not mean that F is neutral because the profile $R|_A$ is not arbitrary. For example, if π is the SCC from Example 1, then all alternatives of $R|_A$ are Pareto optimal, which is not the case for an arbitrary profile.

Corollary 1 *Let $R \in \mathcal{L}(I_m)^n$ be a profile and F be an π-ian SCF satisfying π-IIA. Then*

$$F(R) = F^{\nu}(R|_{\pi(R)}).$$

Proof: Since F is π-ian, $F(R) \notin \pi^{-}(R)$. By π-IIA

$$F(R) = F(R|_{I_m \setminus \pi^{-}(R)}) = F(R|_{\pi(R)}). \ \square$$

Corollary 2 *Let F be a π-ian SCF satisfying π-IIA. Then F is neutral at every profile $R \in \mathcal{L}(I_m)^n$ for which all alternatives are π-optimal, i.e., $\pi(R) = I_m$.* $\square$

Let R be a profile on a finite set A and let F be an SCF. In the sequel, if $F^{\nu}(R)$ does not depend on ν, and no confusion can emerge, then this ν will be omitted and we will write simply $F(R)$.

Definition 9 *We say that an SCC π is hereditary if for every profile R and for every subset $X \subseteq \pi(R)$ it is true that $\pi(R|_X) = X$.*

Example 3 *Let $a \in I_m$, R be a profile, and $U_i(a)$ be the upper contour set of a relative to R_i. An element a is said to be q-Pareto optimal if*

$$card\left(\bigcap_{i=1}^{n} U_i(a)\right) \leq q.$$

Let $P_q(R)$ be the set of all q-Pareto optimal elements of R. Then for every $q \geq 0$ the SCC P_q is hereditary.

Definition 10 *An SCC π is said to be tops-inclusive if*

1. *for every profile R the set $\pi(R)$ contains all first preferences of agents;*

2. *if $\pi(R)$ strictly contains $P_1(R)$, i.e. the set of all Pareto optimal elements of R, for at least one profile R, then $\pi(R)$ contains all second preferences of agents as well.*

For the rest of the article we will fix a neutral, hereditary and tops-inclusive SCC π, for example, any of the given in Examples 1,2 or 3.

Let F be an SCF satisfying π-IIA and let R be a profile. Then for $X \subseteq \pi(R)$ (using the convention above) we define

$$c_R(X) \stackrel{\text{def}}{=} F(R|_X);$$

and for every $x, y \in \pi(R)$

$$x \succ_R y \stackrel{\text{def}}{\Leftrightarrow} c_R(\{x,y\}) = x.$$

By doing this, we attach to every SCF F and every profile R a binary relation $\succ_R$ on $\pi(R)$. These binary relations satisfy the following conformity condition: if a profile R' is a restriction of R, then $\succ_{R'}$ is a restriction of $\succ_R$.

Lemma 3 *Let F be an SCF satisfying π-IIA. Then for every profile R the restriction of the binary relation $\succ_R$ to $\pi(R)$ is a linear order on $\pi(R)$.*

Proof: Suppose $x \succ_R y$ and $y \succ_R z$, where $x, y, z \in \pi(R)$ are distinct. Then $x, y, z \in \pi(R|_{\{x,y,z\}})$ since π is hereditary. Let us prove that $c_R(\{x,y,z\}) = x$. Indeed, if $c_R(\{x,y,z\}) = z$, then π-IIA implies $c_R(\{y,z\}) = z$ which contradicts to $y \succ_R z$. If $c_R(\{x,y,z\}) = y$, then π-IIA implies $c_R(\{x,y\}) = y$ which contradicts to $x \succ_R y$. $\square$

The following proposition reveals the mechanism behind an π-ian SCF satisfying π-IIA. It can be viewed as an extention of Corollary 1.

Proposition 6 *Let π be hereditary and let F be a π-ian SCF satisfying π-IIA and R be a profile. Let us enumerate elements of $\pi(R)$ so that $\pi(R) = \{b_1, \ldots, b_r\}$ and*

$$b_1 \succ_R b_2 \succ_R \ldots \succ_R b_r.$$

Then $F(R) = c_R(\{b_1, \ldots, b_r\}) = b_1$.

Proof: The equality $F(R) = c_R(\{b_1, \ldots, b_r\})$ is implied by Corollary 1. Let us prove $c_R(\{b_1, \ldots, b_k\}) = b_1$ by induction on k. If $k = 2$, then $c_R(\{b_1, b_2\}) = b_1$ is equivalent to $b_1 \succ_R b_2$. Suppose that $c_R(\{b_1, \ldots, b_k\}) = b_1$, let us consider $b_1, \ldots, b_{k+1}$. If $c_R(\{b_1, \ldots, b_{k+1}\}) = b_{k+1}$, then π-IIA implies $b_{k+1} \succ_R b_k$, the contradiction. Then $c_R(\{b_1, \ldots, b_{k+1}\}) \in \{b_1, \ldots, b_k\}$. Then by π-IIA

$$c_R(\{b_1, \ldots, b_{k+1}\}) = c_R(\{b_1, \ldots, b_k\}) = b_1.$$

The proposition is proved. □

We will assume that $\mathcal{N} = \{1, 2, \ldots, n\}$ denoting the i-th voter as i. It will not lead to a confusion.

Definition 11 *Let F be an SCF. We say that a coalition $\mathcal{D} \subseteq \mathcal{N}$ is π-decisive for F and a pair (a, b) of distinct alternatives $a, b \in I_m$, if for an arbitrary profile R, such that $a, b \in \pi(R)$, aR_ib for $i \in \mathcal{D}$, and bR_ja for $j \in \mathcal{N} \setminus \mathcal{D}$, imply $a \succ_R b$. We say that $\mathcal{D}$ is π-decisive for F, if it is π-decisive on every pair of π-optimal distinct alternatives.*

Lemma 4 *Let F be an SCF satisfying π-IIA and let $\mathcal{D}$ be a coalition. Suppose that there exists a profile R, such that for some $a, b \in \pi(R)$, aR_ib for $i \in \mathcal{D}$, and bR_ja for $j \in \mathcal{N} \setminus \mathcal{D}$, and $a \succ_R b$. Then $\mathcal{D}$ is π-decisive for F and a pair (a, b). If the coalition $\mathcal{D}$ is proper, i.e $\emptyset \neq \mathcal{D} \neq \mathcal{N}$, then the reverse is also true.*

Proof: Suppose that there exists a profile R, such that $a, b \in \pi(R)$, aR_ib for $i \in \mathcal{D}$, and bR_ja for $j \in \mathcal{N} \setminus \mathcal{D}$, and $a \succ_R b$. Let R' be any profile with $a, b \in \pi(R')$ such that $a, b \in \pi(R')$, aR'_ib for $i \in \mathcal{D}$, and bR'_ja for $j \in N \setminus \mathcal{D}$. Then $R'|_{\{a,b\}} = R|_{\{a,b\}}$, whence by π-IIA $F(R'|_{\{a,b\}}) = F(R|_{\{a,b\}}) = a$, and $a \succ_{R'} b$.

Suppose now that a proper coalition $\mathcal{D}$ is π-decisive for F and a pair (a, b). Let us consider any profile R of the following type:

$$\begin{array}{lll} a \succ b \succ \ldots & : & \text{agents from } \mathcal{D}, \\ b \succ a \succ \ldots & : & \text{agent from } \mathcal{N} \setminus \mathcal{D}. \end{array}$$

Then $a, b \in \pi(R)$, since π is tops-inclusive, and hence $a \succ_R b$ by π-decisiveness of $\mathcal{D}$. Therefore the required profile exists. □

Lemma 5 *Let F be an SCF satisfying π-IIA. Then a coalition $\mathcal{D}$ is π-decisive for F if and only if it is π-decisive for F and a pair (a,b) for some distinct alternatives $a,b \in I_m$.*

Proof: Suppose $\mathcal{D}$ is π-decisive for F and a pair (a,b), of distinct alternatives $a,b \in I_m$. First, we suppose that there exists a profile R, such that $a,b \in \pi(R)$, aR_ib for $i \in \mathcal{D}$, and bR_ja for $j \in \mathcal{N} \setminus \mathcal{D}$, and $a \succ_R b$. By the definition the latter means that $a = F(R|_{\{a,b\}})$. Let us denote $R|_{\{a,b\}} = P$. By Lemma 2 F is neutral at P.

Let us consider any profile R' such that such that $c,d \in \pi(R')$, cR'_id for $i \in \mathcal{D}$, and dR'_jc for $j \in \mathcal{N} \setminus \mathcal{D}$. Let us denote $R'|_{\{c,d\}} = Q$. Consider the mapping $\mu\colon \{a,b\} \to \{c,d\}$ such that $\mu(a) = c$ and $\mu(b) = d$. Then by Proposition 5

$$a = F(P) = \mu^{-1}F(Q),$$

whence $F(Q) = c$. The latter means $c \succ_{R'} d$ and by Lemma 4 $\mathcal{D}$ is π-decisive for (c,d).

Let us consider the remaining case, when no profiles exist such that $a,b \in \pi(R)$, aR_ib for $i \in \mathcal{D}$, and bR_ja for $j \in \mathcal{N} \setminus \mathcal{D}$, then no profile Q can exist such that $c,d \in \pi(Q)$, cQ_id for $i \in \mathcal{D}$, and dQ_jc for $j \in \mathcal{N} \setminus \mathcal{D}$ because π is neutral. Thus, in both cases, $\mathcal{D}$ is also π-decisive for F. □

Corollary 3 *Let F be an SCF satisfying π-IIA. Let $\mathcal{D}$ be a proper subset of $\mathcal{N}$. Then either $\mathcal{D}$ is π-decisive or its complement $\mathcal{N} \setminus \mathcal{D}$ is π-decisive.*

Proof: Suppose that a coalition $\mathcal{D}$ is π-decisive for F and a pair (a,b). Then $\mathcal{D}$ is decisive by Lemma 5. If $\mathcal{D}$ is not π-decisive for F and a pair (a,b), then there exists a profile R such that $a,b \in \pi(R)$, and aR_ib for $i \in \mathcal{D}$, and bR_ja for $j \in \mathcal{N} \setminus \mathcal{D}$, but $b \succ_R a$. But now by Lemmata 4 and 5 $\mathcal{N} \setminus \mathcal{D}$ is π-decisive. □

4. Main Results

In what follows, we follow the ideas of the original proof of Arrow's Impossibility Theorem [1]. The difference is that we have transitivity only on a variable set of alternatives which depend on the profile.

The following Lemmata on the structure of the set of π-decisive subsets of $\mathcal{N}$ will be proved in the assumption that F is a SCF which satisfies π-IIA, where π is a neutral, hereditary, and tops-inclusive SCC.

Lemma 6 *If a π-decisive set $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, different from $\mathcal{N}$, is a disjoint union ($\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$) of two nonempty subsets $\mathcal{D}_1$ and $\mathcal{D}_2$, then either $\mathcal{D}_1$ or $\mathcal{D}_2$ is π-decisive as well.*

Proof: Let $\mathcal{N} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{M}$, where $\mathcal{M} = \mathcal{N} \setminus \mathcal{D} \neq \emptyset$. Consider any profile R such that for some $a, b, c \in I_m$:

$$\begin{array}{lcl} a \succ b \succ c \succ \ldots & : & \text{agents from } \mathcal{D}_1, \\ b \succ c \succ a \succ \ldots & : & \text{agents from } \mathcal{D}_2, \\ c \succ a \succ b \succ \ldots & : & \text{agents from } \mathcal{M}. \end{array}$$

Then $a, b, c \in \pi(R)$ as π is tops-inclusive. Then $b \succ_R c$ as $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is π-decisive. If $b \succ_R a$ then $\mathcal{D}_2$ is π-decisive and the result is proved. If not, then $a \succ_R b$. Since by Lemma 3 the relation $\succ_R$ is transitive on $\pi(R)$, $a \succ_R b$ and $b \succ_R c$ imply $a \succ_R c$, which means that in this case $\mathcal{D}_1$ is π-decisive. $\square$

Lemma 7 *There exists a singleton $v \in \mathcal{N}$ such that $\{v\}$ is π-decisive.*

Proof: Let $\mathcal{N}' = \mathcal{N} \setminus \{u\}$, where $u \in \mathcal{N}$ is arbitrary. Then by Corollary 3 either $\{u\}$ or $\mathcal{N}'$ is decisive. In the first case we are done. In the second, we may repeatedly apply Lemma 6 to $\mathcal{N}'$ and then to its decisive subsets until a decisive singleton is obtained. $\square$

Lemma 8 *Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ be three nonempty disjoint subsets of $\mathcal{N}$ such that $N = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. Then all three subsets cannot be simultaneously π-decisive.*

Proof: If it were possible, then consider the following profile R:

$$\begin{array}{lcl} a \succ b \succ c \succ \ldots & : & \text{agents from } \mathcal{D}_1, \\ b \succ c \succ a \succ \ldots & : & \text{agents from } \mathcal{D}_2, \\ c \succ a \succ b \succ \ldots & : & \text{agents from } \mathcal{D}_3. \end{array}$$

Since π is tops-inclusive, the alternatives a, b, c are all π-optimal and, assuming that all three subsets are π-decisive, we will have $a \succ_R c \succ_R b \succ_R a$, which contradicts to the transitivity of $\succ_R$ proved in Lemma 3. $\square$

Lemma 9 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two π-decisive subsets of $\mathcal{N}$ such that $\mathcal{D}_1 \cup \mathcal{D}_2 \neq \mathcal{N}$. Then the union $\mathcal{D}_1 \cup \mathcal{D}_2$ is π-decisive.*

Proof: Suppose first that $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint. As $\mathcal{D}_1 \cup \mathcal{D}_2 \neq \mathcal{N}$, then $\mathcal{M} = \mathcal{N} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2) \neq \emptyset$. By Lemma 8 $\mathcal{M}$ is not π-decisive. But then $\mathcal{D}_1 \cup \mathcal{D}_2 = N \setminus \mathcal{M}$ is π-decisive by Corollary 3.

Now let us assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ have a nonzero intersection. We may also assume that this intersection is different from both of the sets because otherwise the result is trivial. Let us consider any profile such that for some alternatives $a, b, c \in I_m$

$$\begin{array}{lcl} a \succ b \succ c \succ \dots & : & \text{agents from } \mathcal{D}_1 \cap \mathcal{D}_2, \\ a \succ c \succ b \succ \dots & : & \text{agents from } \mathcal{D}_1 \setminus \mathcal{D}_2, \\ b \succ a \succ c \succ \dots & : & \text{agents from } \mathcal{D}_2 \setminus \mathcal{D}_1, \\ c \succ b \succ a \succ \dots & : & \text{agents from } \mathcal{M}, \end{array}$$

where $\mathcal{M} = \mathcal{N} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$. We note that $a, b, c \in \pi(R)$ as π is tops-inclusive. Then $a \succ_R b$ since $\mathcal{D}_1$ is π-decisive and $b \succ_R c$ since $\mathcal{D}_2$ is π-decisive. By transitivity of $\succ_R$ on $\pi(R)$ we get $a \succ_R c$ and hence $\mathcal{D}_1 \cup \mathcal{D}_2$ is π-decisive. $\square$

Corollary 4 *There exists a π-decisive subset $\mathcal{D}$ of $\mathcal{N}$ of cardinality $n-1$.*

Proof: This is the same to say that one of the singletons is indecisive. Suppose to the contrary that all of them are π-decisive. Then by Lemma 9 all proper subsets of $\mathcal{N}$ are π-decisive. This is impossible since by Corollary 3 a subset and its complement cannot be simultaneously π-decisive. $\square$

Lemma 10 *Let $\emptyset \neq \mathcal{D}_1 \subseteq \mathcal{D} \subseteq \mathcal{D}_2 \neq \mathcal{N}$ with $\mathcal{D}_1$ and $\mathcal{D}_2$ being π-decisive. Then $\mathcal{D}$ is π-decisive.*

Proof: Let us consider any profile such that for some alternatives $a, b, c \in I_m$

$$\begin{array}{lcl} a \succ b \succ c \succ \dots & : & \text{agents from } \mathcal{D}_1, \\ b \succ a \succ c \succ \dots & : & \text{agents from } \mathcal{D} \setminus \mathcal{D}_1, \\ b \succ c \succ a \succ \dots & : & \text{agents from } \mathcal{D}_2 \setminus \mathcal{D}, \\ c \succ b \succ a \succ \dots & : & \text{agents from } \mathcal{N} \setminus \mathcal{D}_2. \end{array}$$

Since $a, b, c \in \pi(R)$, we get $a \succ_R b$ as $\mathcal{D}_1$ is decisive and $b \succ_R c$ as $\mathcal{D}_2$ is decisive. By the transitivity of $\succ_R$ on $\pi(R)$ we get $a \succ_R c$ which means that $\mathcal{D}$ is decisive. $\square$

Definition 12 *Let F be an SCF. An agent $k \in \mathcal{N}$ will be called an π-dictator, if for every profile R and for every pair of two distinct alternatives $a, b \in \pi(R)$ it is true that aR_kb implies $a \succ_R b$; an agent $k \in \mathcal{N}$ will be called an π-antidictator, if for every profile R and for every pair of two distinct alternatives a, b with $a, b \in \pi(R)$ it is true that aR_kb implies $b \succ_R a$.*

The following two propositions are obvious.

Proposition 7 *An agent $k \in \mathcal{N}$ is an π-dictator, if all coalitions in $\mathcal{N}$ containing k are π-decisive. An agent $k \in \mathcal{N}$ is an π-antidictator, if all coalitions in $\mathcal{N}$ not containing k are π-decisive.*

Proposition 8 *Let F be an π-ian SCF. Then an π-dictator is an ordinary dictator.*

Proof: It follows from the fact that π is tops-inclusive.

Now we are ready to formulate the main results of this paper.

Theorem 1 *Suppose $m \geq 3$. Let π be any neutral, hereditary and tops-inclusive SCC and let F be an SCF which is $\mathcal{F}$-consistent for some π-complete set $\mathcal{F}$ of SCFs. Then there exists either an π-dictator or an π-antidictator.*

Proof: Without loss of generality, we assume that $\mathcal{D} = \{1, \dots, n-1\}$ is π-decisive. The existence of it is guaranteed by Corollary 4. By Lemma 7 there is an π-decisive sigleton in $\mathcal{D}$; and we may assume that it is $\{1\}$. By Lemma 10 all subsets of $\mathcal{D}$, which contain $\{1\}$, are π-decisive.

Now the key question is whether or not $\{1, n\}$ is π-decisive. If, yes, then by Lemma 10 all proper subsets containing $\{1\}$ are π-decisive. It remains to prove that $\mathcal{N}$ itself is π-decisive, it would mean that agent 1 is an π-dictator.

We note first that if $\pi(R) \subseteq P_1(R)$ for all profiles R, then $\mathcal{N}$ is trivially π-decisive. If not, then $\pi(R)$ contains all second preferences. Let us consider any profile of the following type

$$\begin{array}{lcl} a \succ b \succ c \succ \dots & : & \text{agents from } \mathcal{N} \setminus \{2,3\} \\ b \succ a \succ c \succ \dots & : & \text{agent } 2, \\ a \succ c \succ b \succ \dots & : & \text{agent } 3. \end{array}$$

Then $a, b, c \in \pi(R)$ as $\pi(R)$ contains all first and second preferences. We get $a \succ_R b$ as $\mathcal{N} \setminus \{2\}$ is π-decisive and $b \succ_R c$ as $\mathcal{N} \setminus \{3\}$ is π-decisive.

By transitivity we get $a \succ_R c$ which by Lemma 4 means that $\mathcal{N}$ is decisive. Thus agent 1 is an π-dictator.

Suppose now that $\{1, n\}$ is not π-decisive. Then it immediately follows that all agents $2, 3, \ldots, n{-}1$ are π-decisive. Indeed, if, for example, agent 2 is not π-decisive, then $(n{-}1)$-element subset $\{1, 3, \ldots, n\}$ is π-decisive and by Lemma 10 the pair $\{1, n\}$ is π-decisive. Now by Lemma 9 it follows that every nonempty subset of $\mathcal{D}$ is π-decisive. Then n would be an π-antidictator if and only if an empty set is π-decisive.

We note first that if $\pi(R) \subseteq P_1(R)$ for all profiles R, then $\emptyset$ is trivially π-decisive. If not, then $\pi(R)$ contains all second preferences. Let us consider any profile of the following type

$$
\begin{array}{lcl}
a \succ b \succ c \succ \ldots & : & \text{agents from } \mathcal{N} \setminus \{n{-}1, n{-}2\} \\
b \succ a \succ c \succ \ldots & : & \text{agent n-1,} \\
a \succ c \succ b \succ \ldots & : & \text{agent n-2.}
\end{array}
$$

Then $a, b, c \in \pi(R)$ as $\pi(R)$ contains all first and second preferences. We get $b \succ_R a$ as $\{n{-}1\}$ is π-decisive and $c \succ_R b$ as $\{n{-}2\}$ is π-decisive. By transitivity we get $c \succ_R a$ which by Lemma 4 means that $\emptyset$ is π-decisive. Thus agent n is an π-antidictator. $\square$

Theorem 2 *Suppose $m \geq 3$. Let π be any neutral, hereditary and tops-inclusive SCC and F be an π-ian SCF which is $\mathcal{F}$-consistent for some complete set $\mathcal{F}$ of π-ian SCFs. Then either F is dictatorial and there exists an agent $i \in \mathcal{N}$ such that $F(R) = \max R_i$ or F is π-antidictatorial and there exists an agent $i \in \mathcal{N}$ such that $F(R) = \min R_i$ on $\pi(R)$.*

Proof: It follows from Theorem 1 and Propositions 2 and 8.

Now we are going to single out some interesting cases which fall under this general result.

According to Koray [2] an SCF is said to be universally consistent if it is $\mathcal{F}$-consistent for any set of SCFs $\mathcal{F}$. The following corollary generalizes the main result of [2].

Corollary 5 *Let F be an universally consistent (not necessarily neutral) SCF. Then it is dictatorial or antidictatorial.*

Definition 13 *Let F be an SCF. We say that an agent k is a topsonly-antidictator for F, if, for every profile R, $F(R)$ is the lowest ranking alternative relative to R_k among the first preferences of R.*

Corollary 6 *Let F be an SCF which always chooses an alternative among the first preferences of the profile. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be the set of all dictatorial SCFs. Then F is $\mathcal{D}$-consistent if and only if it is dictatorial or topsonly-antidictatorial.*

Proof: To obtain this result one has to set $\pi(R)$ to be the set of all first preferences of R. Then it is clear that $\mathcal{D}$ is a complete set of π-ian SCFs for this SCC π.$\square$

Definition 14 *Let F be an SCF. We say that an agent k is a q-Paretian antidictator for F, if, for every profile R, $F(R)$ is the lowest ranking q-Pareto optimal alternative relative to R_k.*

Corollary 7 *Let $\mathcal{P}_q$ be the class of all q-Paretian SCFs and F be an q-Paretian SCF. Then F is $\mathcal{P}_q$-consistent if it is dictatorial or else q-Pareto antidictatorial.*

Proof: Here the set $\pi(R)$ should be the set of all q-Pareto optimal elements of R. $\square$

References

[1] Arrow, K.J., Social choice and individual values, Wiley, N.Y.,1951, 1963.

[2] Koray, S., Self-selective social choice functions verify Arrow and Gibbard-Satterthwaite theorems, Econometrica **68** (2000), 981–995.

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