On Asymptotic Strategy-Proofness of Classical Social Choice Rules

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Abstract: We show that, when the number of participating agents n tends to infinity, all classical social choice rules are asymptotically strategy-proof with the proportion of manipulable profiles being of order $O(1/\sqrt{n})$.

1. Introduction

The well-known impossibility theorem of Gibbard [6] and Satterthwaite [10] states that every non-dictatorial social choice function is manipulable. Nitzan [8] numerically showed the severity of this problem for relatively small societies (up to 90 agents). Nevertheless there always has been hope that for large societies manipulability might not be an issue if the social choice function is chosen correctly. In other words, it is important to know which social choice functions are nonmanipulable asymptotically, i.e. for which of them the proportion of profiles, at which a successful attempt to manipulate might take place, tends to zero as the number of agents grows.

This question was studied by Peleg [9] who proved that every representable positionalist voting system (in the sense of Gärdenfors [5]) is asymptotically nonmanipulable. This class of voting systems includes, for example, the plurality rule and the Borda count. Fristrup and Keiding [3, 4], sharpened this result for the plurality rule by proving that the proportion of manipulable profiles for the plurality is bounded from above by $K/\sqrt{n}$, where n is the number of participating agents and K is a number which depends only on the number of alternatives but not on n.

The author proved that the same asymptotic behavior of the proportion of manipulable profiles can be also observed for the run-off procedure [11] and the majoritarian compromise [12]. Since STV or Hare's rule can be treated in the same way as the run-off procedure, there are two main types of rules left for which asymptotic strategy-proofness has not been established yet or was established without estimates on the speed of convergence of the proportion of manipulable profiles to zero. These are rules based on the majority relation and the scoring rules, with Borda being the main unresolved case.

The main result of this paper is that, for all rules based on the majority relation and for all faithful scoring rules, – the most important of which being the Borda rule, – the proportion of manipulable profiles is of order $O\left(1/\sqrt{n}\right)$.

As the result of this paper shows, we can state now that all classical rules are asymptotically strategy-proof. But this is not true for all rules. We refer the reader to [9] to Example 4.1 which shows that not all rules are asymptotically strategy-proof. In fact, the rule given there is even anonymous, monotonic and Paretian.

We note also that in [11] it was proved that for the plurality the asymptotic lower bound for the proportion of manipulable profiles is also $k/\sqrt{n}$, for some $k > 0$ not depending on n. No other lower bounds have been discovered so far.

2. Definitions and Basic Concepts

Let A and $\mathcal{N}$ be two finite sets of cardinality m and n respectively. The elements of A will be called alternatives, the elements of $\mathcal{N}$ agents. We assume that the agents have preferences over the set of alternatives. By $\mathcal{L} = \mathcal{L}(A)$ we denote the set of all linear orders on A; they represent the preferences of agents over A. The elements of the cartesian product

$$\mathcal{L}(A)^n = \mathcal{L}(A) \times \ldots \times \mathcal{L}(A) \quad (n \text{ times})$$

are called n-profiles or simply profiles. They represent the collection of preferences of an n-element society of agents $\mathcal{N}$. If a linear order $R_i \in \mathcal{L}(A)$ represents the preferences of the i-th agent, then by aR_ib, where $a, b \in A$, we denote that this agent prefers a to b.

A family of mappings $F = \{F_n\}$, $n \in \mathbb{N}$,

$$F_n \colon \mathcal{L}(A)^n \to A,$$

is called a social choice function (SCF). Historically SCFs were often called rules.

Definition 1 *Let F be a SCF and let $R = (R_1, \ldots, R_n)$ be a profile. We say that the profile R is manipulable for F if there exists a linear order $R_i^{'}$ such that for a profile $R^{'} = (R_1, \ldots, R_i^{'}, \ldots, R_n)$, where $R_i^{'}$ replaces R_i, we have $F(R')R_iF(R)$. We also say that the profile R is unstable if there exists a linear order $R_i^{'}$ such that $F_n(R^{'}) \neq F_n(R)$.*

Every manipulable profile is unstable, but the reverse is not always true.

Let us define the following two indices as suggested in [8]. Given the rule F, we define the index of manipulability

$$K_F(n,m) = \frac{d_F(n,m)}{(m!)^n}, \tag{1}$$

where $d_F(n,m)$ is the total number of all manipulable profiles, and the index of instability

$$L_F(n,m) = \frac{e_F(n,m)}{(m!)^n}, \tag{2}$$

where $e_F(n,m)$ is the total number of all unstable profiles. We note that under the assumption that $\mathcal{L}_F(A)^n$ is a discrete probability space with the uniform distribution (the so-called "impartial-culture assumption"[8]), the indices $K_F(m,n)$ and $L_F(m,n)$ become the probabilities of drawing at random a manipulable or an unstable profile, respectively.

Clearly $K_F(m,n) \leq L_F(m,n)$. Therefore any upper bound for $L_F(n,m)$ is an upper bound for $K_F(m,n)$.

3. The Rules Based on the Majority Relation

Definition 2 *Let $R = (R_1, \ldots, R_n)$ be a profile. The majority relation $M(R)$ on A (which clearly depends on the given profile) is the binary relation on A such that for any a_k and a_ℓ in A we have $a_kM(R)a_\ell$ if and only if $card(\{i \mid a_kR_ia_\ell\} > card(\{i \mid a_\ell R_ia_k\}$ or $card(\{i \mid a_kR_ia_\ell\} = card(\{i \mid a_\ell R_ia_k\}$ and a_k is earlier in A than a_ℓ, i.e., $k < \ell$. Of course, the second case can occur only for even n.*

Mathematically speaking, the majority relation is a tournament on A, i.e., complete and asymmetric binary relation.

A great number of voting rules are based on computing the majority relation. Using these rules, given a profile R, one computes $M(R)$ first and then implements one or another algorithm to determine the winner using the information contained in $M(R)$. Such rules are also called tournament solutions (see the book [7] devoted to them).

Theorem 1 *Let F be a rule based on the majority relation. Then there is a constant C_m which depends only on m but not on n such that*

$$L_F(n,m) \leq \frac{C_m}{\sqrt{n}}.$$

Proof: Let us consider two alternatives $a, b \in A$. Suppose $aM(R)b$. A participating agent will be able to change this to $bM(R)a$ only in the case when $card(\{i \mid bR_ia\} = \frac{n}{2}$ for an even n or $card(\{i \mid bR_ia\} = \lfloor \frac{n}{2} \rfloor$ for an odd n. Since, as is well-known,

$$\binom{n}{n/2} < \frac{2^n}{\sqrt{n}},$$

the probability of this happening will be not greater than

$$\frac{\binom{n}{n/2}}{2^n} < \frac{1}{\sqrt{n}}.$$

Since we have $\frac{1}{2}m(m-1)$ pairs of alternatives, then

$$L_F(n,m) \leq \frac{\frac{1}{2}m(m-1)}{\sqrt{n}}.$$

The theorem is proved.

4. The Scoring Rules

The following important class of SCFs was introduced by Gärdenfors [5]. We follow [9] in defining them.

Definition 3 *A representation function is a function* $f\colon \mathcal{L}(A)\times A\to \mathrm{I\!R}$ *such that*

$$aR_ib \Longrightarrow f(R_i,a)\geq f(R_i,b).$$

It is called faithful if

$$(aR_ib \text{ and } a\neq b) \Longrightarrow f(R_i,a)>f(R_i,b).$$

Definition 4 *A representation function* $f\colon \mathcal{L}(A)\times A\to \mathrm{I\!R}$ *is called positionalist, if* $f(R,a)$ *depends only on the cardinality of the lower contour set* $L(R,a)=\{b\in A\mid aRb \text{ and } a\neq b\}$ *of* a *relative to* R.

The simpliest positionalist representation function which defines the Borda rule can be defined by $f(R_i,a)=card\,(L(R_i,a))$. It is easy to see that it is faithful.

Let f be a representation function and $R\in\mathcal{L}(A)^n$ be a profile. We define the *score function* $Sc_f\colon \mathcal{L}(A)^n\times A\to \mathrm{I\!R}$ by

$$Sc_f(R,a)=\sum_{i=1}^{n} f(R_i,a),\quad a\in A. \tag{3}$$

Definition 5 *A SCF* F *is called (faithfully) representable if there exists a (faithful) representation function* f *such that for every* $R\in\mathcal{L}(A)^n$ *we have* $F(R)=a_i$ *if and only if*

$$j<i \implies Sc_f(R,a_i)>Sc_f(R,a_j), \tag{4}$$
$$j>i \implies Sc_f(R,a_i)\geq Sc_f(R,a_j). \tag{5}$$

When f *is positionalist, the SCF* F *is also called a (faithful) scoring rule (or point-voting scheme).*

Every scoring rule F with a representation function f is characterised by the vector $W_f=(w_1,\ldots,w_m)$ of weights such that $w_i=f(R,a)$ if and only if $card(L(R,a))=m-i$. The weights must satisfy the condition

$$w_1\geq w_2\geq\ldots\geq w_m=0.$$

and it is clear that the scoring rule is *faithful* if $w_i\neq w_{i+1}$ for all $i=1,2,\ldots,m{-}1$.

For a profile $R \in \mathcal{L}(A)^n$ ans an alternative $a \in A$ the score of a can be now computed as follows. Let $I_a = (i_1, \ldots, i_m)$ be the vector such that the number i_k shows how many times the alternative a was ranked kth. Then

$$Sc_f(R, a) = W_f \cdot I_a = \sum_{\ell=1}^{m} w_\ell i_\ell.$$

The most commonly used scores are the plurality score $Sc_P(R, a)$, when P is the plurality representation function, which is defined by the vector of weights $W_P = (1, 0, \ldots, 0)$ and the Borda score $Sc_B(a)$, where B is the Borda representation function defined by the vector of weights $W_B = (m-1, m-2, \ldots, 1, 0)$.

Let us consider a multiset

$$I_{n,q} = \{\underbrace{1, \ldots, 1}_{q}, \underbrace{2, \ldots, 2}_{q}, \ldots, \underbrace{n, \ldots, n}_{q}\} \tag{6}$$

with its subsets partially ordered by inclusion. It is well-known that the collection of all subsets of $I_{n,q}$ of middle size, $\lfloor qn/2 \rfloor$, is a maximal antichain in $I_{n,q}$ (see, for example, [1]); if n and q are both odd, then the subsets of size $\lceil qn/2 \rceil$ also form a maximal antichain.

In [2] Anderson proved that the length $s(n)$ of a maximal antichain in $I_{n,q-1}$ satisfies the inequality

$$c_q \frac{q^{n-1}}{\sqrt{n}} \le s(n) \le C_q \frac{q^{n-1}}{\sqrt{n}},$$

for some constants c_q, C_q, which depend on q but not on n. This result will be used later.

Let $R = (R_1, \ldots, R_n)$ be a profile on A and $A' = A \setminus \{a_m\}$. Then we define the restrictions $Q_i = R_i|_{A'}$ of the linear orders R_i on A' and the restricted profile $Q = (Q_1, \ldots, Q_n)$ on A'. We will also say that R is an extention of Q.

Let us now consider a profile Q on A' and let $E(Q)$ be the set of all possible extentions of Q.

Definition 6 *Let R and S belong to $E(Q)$. We say that $R \le S$ if for all $i = 1, 2, \ldots, n$ the following inclusion for the lower contour sets hold:*

$$L(R_i, a_m) \subseteq L(S_i, a_m).$$

Clearly this defines a partial order on $E(Q)$.

Lemma 1 *The poset $(E(Q), \leq)$ is isomorphic to the poset $(\mathcal{P}(I_{n,m-1}), \subseteq)$ of all subsets of $I_{n,m-1}$.*

Proof: Suppose that R is an extention of Q and $card(L(R_i, a_m)) = t_i$. Then our isomorphism should assign to this particular extention the following subset of $I_{n,m-1}$:

$$\{\underbrace{1, \ldots, 1}_{t_1}, \underbrace{2, \ldots, 2}_{t_2}, \ldots, \underbrace{n, \ldots, n}_{t_n}\}.$$

The proof of this isomorphism is obvious.

Lemma 2 *Let F be a faithful scoring rule with a representation function f. Let Q be a profile on A'. Let us denote $a_m = a$ and let $b \in A'$. Then any two distinct extentions $R, S \in E(Q)$ such that*

$$Sc_f(R, a) - Sc_f(R, b) = Sc_f(S, a) - Sc_f(S, b) \tag{7}$$

are not comparable relative to $\leq$.

Proof: Suppose $R \leq S$ and $R \neq S$. Then $Sc_f(R, a) < Sc_f(S, a)$ because the rule is faithful. At the same time for every i there must be $L(R_i, b) \supseteq L(S_i, b)$. Hence $Sc_f(R, b) \geq Sc_f(S, b)$ and the equation (7) cannot hold. This proves the lemma.

Lemma 3 *Let F be a faithful scoring rule with a representation function f. Let $a, b \in A$ and k is an integer. Then, for some $C_m > 0$ depending on m but not on n, there are no more than $C_m(m!)^n/\sqrt{n}$ profiles $R \in \mathcal{L}(A)^n$ such that*

$$Sc_f(R, a) - Sc_f(R, b) = k. \tag{8}$$

Proof: We will show that the same constant C_m as in Anderson's theorem works. Let $A' = A \setminus \{a\}$. There are $(m-1)!$ profiles on A' and we have m^n extentions for each of them to a profile on A. Let us take an arbitrary profile Q on A'. Then by Lemma 2 any two extentions with the property (8) will not be comparable. Thus by Lemma 1 and by the aforementioned result of Anderson we have no more than $C_m m^n/\sqrt{n}$ such extentions. In total we cannot have more than

$$\frac{C_m m^n}{\sqrt{n}} \cdot (m-1)! = \frac{C_m (m!)^n}{\sqrt{n}}$$

profiles for which (8) is satisfied. The lemma is proved.

Theorem 2 *For any faithful scoring rule F with a representation function f there exists a constant D_m depending on m but not on n such that*

$$L_F(n,m) \leq \frac{D_m}{\sqrt{n}}. \tag{9}$$

Proof: Let $a, b \in A$. Then by Lemma 3 the total number of profiles satisfying

$$Sc_f(R,a) - Sc_f(R,b) = k. \tag{10}$$

for $k = -2m+2, -2m+3, \ldots, 0, \ldots, 2m-3, 2m-2$ will be not greater than

$$(4m-3)\frac{C_m (m!)^n}{\sqrt{n}}.$$

Thus the total number of profiles where the difference between the scores of some two alternatives is smaller than $2m-1$ will be not greater than

$$(4m-3)\binom{m}{2}\frac{C_m (m!)^n}{\sqrt{n}}.$$

Since every manipulable and even every unstable profile is among those counted, we see that (9) holds with $D_m = (4m-3)\binom{m}{2}C_m$.

This proves the theorem.

Corollary 1 *All faithful scoring rules, including Borda, are asymptotically strategy proof with the proportion of manipulable profiles being of order $O(1/\sqrt{n})$, when the number of agents n tends to infinity.*

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