On Spaces with Quasi–Regular–G_δ–Diagonals*

A.M. Mohamad

June 29, 2000

Abstract

This paper studies spaces with quasi–regular–G_δ–diagonal. It is shown that if X is a normal space, then the following are equivalent:

1. X admits a development satisfying the 3–link property.
2. X is a $w\Delta$ with quasi–regular–G_δ–diagonal.
3. X is a $w\Delta$ with regular–G_δ–diagonal.
4. X is K–semimetrizable via a semimetric satisfying (AN).
5. There is a semimetric d on X such that:
   a. if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences both converging to the same point, then $\lim d(x_n, y_n) = 0$, and
   b. if x and y are distinct points of X and $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences converging to x and y, respectively, then there are integers L and M such that if $n > L$, then $d(x_n, y_n) > \frac{1}{M}$.

1 Introduction

Diagonal properties play a very important role as a factor of metrizability and developability. For example, a countably compact space with a quasi–G_δ–diagonal is compact metrizable (Shiraki [11]), a linearly ordered topological space (= $LOTS$) with G_δ–diagonal is metrizable (Lutzer [5]), a paracompact

*The author acknowledges the support of the Marsden Fund Award UOA 611, from the Royal Society of New Zealand.
AMS (1991) Subject Classification: 54E30, 54E35.
Keywords and phrases: quasi–regular–G_δ–diagonal; $w\Delta$–space; developable; 3–link property.

Cech–complete space with a G_δ–diagonal is metrizable (Nagata [9]), a locally compact locally connected space with G^*_δ–diagonal is developable (Gruenhage [3]), a locally compact locally connected space with quasi–G^*_δ–diagonal is quasi–developable (Gartside and Mohamad [2]), a locally compact locally connected space with a quasi–regular–G_δ–diagonal is metrizable (Gartside and Mohamad [2]), a space X is $w\Delta$ and has a quasi–G^*_δ–diagonal if and only if it is developable (Mohamad [8]).

The purpose of this paper is to investigate the effects of a quasi–regular–G_δ–diagonal.

A space X has a **quasi–regular–$\boldsymbol{G_\delta}$–diagonal** [2] if and only if there is a countable sequence $\langle U_n : n \in \ N\rangle$ of open subsets in X^2, such that for all $(x, y) \notin \Delta$, there is $n \in \ N$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$.

Recall that a subset H of the space X is a **regular $\boldsymbol{G_\delta}$–set** if there is a sequence $\langle U_n : n \in \ N\rangle$ of open sets in X such that $H = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$. We say that X has a **regular–$\boldsymbol{G_\delta}$–diagonal** [3] if $\Delta = \{(x, x) : x \in X\}$ is a regular G_δ–set in X^2.

A space X has a **quasi–$\boldsymbol{G^*_\delta}$–diagonal** [7] if there exists a countable family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets of X such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$.

A space X is called **quasi–developable** (**developable**) if X has a countable family $\langle \mathcal{G}_n : n \in \ N\rangle$ of collections of open subsets (open covers) of X such that the sets of the form $st(x, \mathcal{G}_n)$ [i.e. the union of all sets in $\mathcal{G}_n$ which contain x] form a local base at x for all $x \in X$. The sequence $\langle \mathcal{G}_n : n \in \ N\rangle$ is called a quasi–development (development) for X. A space X is called Moore if X is a regular developable. A development $\langle \mathcal{G}_n : n \in \ N\rangle$ for the space X is said to satisfy the **3–link property** if whenever p and q are distinct points of X, there is an integer n such that no member of $\mathcal{G}_n$ intersects both $st(p, \mathcal{G}_n)$ and $st(q, \mathcal{G}_n)$.

A space X has a property **$(\boldsymbol{AN})$** [4] if each point of X has a neighborhood of arbitrarily small diameter.

For terminologies which are not defined in this paper, the readers should consult books [3] and [1].

2 The main results

Theorem 2.1 *A space X has quasi–regular–G_δ–diagonal if and only if there is a sequence $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of open subsets of X such that*

1. *for all $x \in X$ we have $\{x\} = \bigcap_{n \in c(x)} \overline{st(x, \mathcal{G}_n)}$ where $c(x) = \{n \in \mathbb{N} : x \in \mathcal{G}_n^*\}$, where $\mathcal{G}_n^* = \bigcup\{G : G \in \mathcal{G}_n\}$;*

2. *for all $x, y \in X$ with $x \neq y$, there are an integer n and open sets U and V containing x and y respectively such that $x \in \mathcal{G}_n^*$ and no member of $\mathcal{G}_n$ meets both U and V.*

Proof. Suppose that X has a quasi–regular–G_δ–diagonal. Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable sequence of open subsets in X^2 such that for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{G : G$ is open in X and $G \times G \subset U_n\}$. Then,

claim 1 for all $x \in X$ we have $\{x\} = \bigcap_{n \in c(x)} \overline{st(x, \mathcal{G}_n)}$.

Let $x, y \in X$ with $x \neq y$. Then, there is some $n \in \mathbb{N}$ such that $(x, x) \in U_n$, but $(x, y) \notin \overline{U_n}$. This implies that $y \notin \overline{st(x, \mathcal{G}_n)}$ and $st(x, \mathcal{G}_n) \neq \emptyset$.

claim 2 for all $x, y \in X$ with $x \neq y$, there are an integer n and open sets U and V containing x and y respectively such that no member of $\mathcal{G}_n$ meets both U and V. Let $x, y \in X$ with $x \neq y$. Then there is an integer n such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. Let U and V be open sets in X that contain x and y respectively such that $U \times V$ does not intersect U_n. To see that no member of $\mathcal{G}_n$ intersects both U and V, suppose instead that G is a member of $\mathcal{G}_n$ such that $r \in G \cap U$ and $s \in G \cap V$. It follows that $(r, s) \in U_n \cap (U \times V)$ which is a contradiction.

Conversely, suppose that $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ is a sequence of collections of open subsets of X as described in the theorem. Define $U_n = \bigcup\{G \times G : G \in \mathcal{G}_n\}$ for each $n \in \mathbb{N}$. Then the sets U_n are open in X^2. Further, if $(x, y) \in X^2$ such that $x \neq y$, then there are an integer n and open sets U and V containing x and y respectively such that no member of $\mathcal{G}_n$ meets both U and V. It must be the case that U_n does not intersect $U \times V$. Therefore $(x, y) \notin \overline{U_n}$. ■

Corollary 2.2 *[2] Every space with a quasi–regular–G_δ–diagonal has a quasi–G_δ^*–diagonal.*

Theorem 2.3 *[8] A space X is $w\Delta$ and has a quasi–G_δ^*–diagonal if and only if it is developable.*

Theorem 2.4 *A normal perfect space X has a regular–G_δ–diagonal if and only if it has a quasi–regular–G_δ–diagonal.*

Proof. Every regular–G_δ–diagonal is a quasi–regular–G_δ–diagonal. Conversely suppose that X has a quasi–regular–G_δ–diagonal. Then there is a countable family $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of collections of open subsets of X such that, for any distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$$

and for all $x, y \in X$ with $x \neq y$, there are an integer n and open sets U and V containing x and y respectively such that no member of $\mathcal{G}_n$ meets both U and V and $x \in \mathcal{G}_n^*$.

For each $n \in \mathbb{N}$, the set $\mathcal{G}_n^*$ is open and hence, since X is perfect, $\mathcal{G}_n^*$ is an F_σ-set. Thus $\mathcal{G}_n^* = \bigcup_j F_{n,k}$, where $F_{n,k}$ is a closed subset of X, for each $k \in N$.

By normality, find $U_{n,k}$ open with $F_{n,k} \subseteq U_{n,k} \subseteq \bar{U}_{n,k} \subseteq \mathcal{G}_n^*$.

For each ordered pair (n, k) of natural numbers, let

$$\mathcal{H}_{n,k} = \mathcal{G}_n \cup \{X - \bar{U}_{n,k}\}.$$

Then $\langle \mathcal{H}_{n,k} : n, k \in \mathbb{N} \rangle$ is a countable family of open covers of X.

Now, $x \in \mathcal{G}_n^*$, so $x \in F_{n,k}$ for some k. It follows, $x \in U_{n,k}$. Put $U' = U_{n,k} \cap U$. We know that, there is no element of $\mathcal{G}_n$ meets both U and V. So, there is no element of $\mathcal{G}_n$ meets both U' and V. Since $X - \bar{U}_{n,k}$ does not meet U', there is no element of $\mathcal{H}_{n,k}$ meets both U' and V. Hence, from Theorem 1 [12], X has a regular–G_δ–diagonal. ∎

Question 2.5 *Does every perfect space with a quasi–regular–G_δ–diagonal have a regular–G_δ–diagonal?*

In the proof of Theorem 2.4, we did not use perfect normality, but the following condition: every open set is a union of countably many regular closed sets (recall that a set C is a regular closed if $C = \overline{V}$ for some open set V). Let us call such a space regularly perfect.

Question 2.6 *Is there an example of regularly perfect space which is not perfectly normal?*

Question 2.7 *Is there an example of regular and perfect space which is not regularly perfect?*

Theorem 2.8 *A normal space X is $w\Delta$ with quasi–regular–G_δ–diagonal if and only if it admits a development satisfying the 3–link property.*

Proof. The sufficiency of the condition is obvious.

Necessity. Suppose that X is a $w\Delta$ with quasi–regular–G_δ–diagonal. Then, by Corollary 2.2 and Theorem 2.3, X is developable. Then, X is a perfect. By Theorem 2.4, X has a regular–G_δ–diagonal. Applying [12, Theorem 2] will complete the proof. ■

From Theorem 2.8 and [4, Theorem 5.4.] we get the following result:

Corollary 2.9 *Let X be a normal space. Then the following conditions are equivalent:*

1. *X admits a development satisfying the 3–link property.*
2. *X is a $w\Delta$ with quasi–regular–G_δ–diagonal.*
3. *X is a $w\Delta$ with regular–G_δ–diagonal.*
4. *X is K–semimetrizable via a semimetric satisfying (AN).*
5. *There is a semimetric d on X such that:*

   a. *if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences both converging to the same point, then lim $d(x_n, y_n) = 0$, and*

   b. *if x and y are distinct points of X and $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences converging to x and y, respectively, then there are integers L and M such that if $n > L$, then $d(x_n, y_n) > \frac{1}{M}$.*

3 Pseudocompatness and quasi–regular–G_δ–diagonals

Recall that a space X is pseudocompact if it is Tychonoff and every continuous real–valued function defined on X is bounded.

In [10] Reed observes that a pseudocompact Moore spaces with regular–G_δ–Diagonal is metrizable. Note that pseudocompact spaces with regular–G_δ–Diagonal are Moore spaces by the following result of McArthur in [6]:

Lemma 3.1 *Let X be a pseudocompact space. Suppose $\langle U_n : n \in \mathbb{N} \rangle$ is a decreasing sequence of open sets such that $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n} = \{x\}$ for a point $x \in X$. Then the sets U_n form a local neighborhood base at x.*

Lemma 3.2 *Let X be a pseudocompact space with quasi–G^*_δ–diagonal. Then X is quasi–developable.*

Proof. Let $\langle \mathcal{V}_n : n \in \mathbb{N} \rangle$ be a quasi–G^*_δ–diagonal sequence for X. Without loss of generality we may assume that $\mathcal{V}_1 = \{X\}$. Set $c_{\mathcal{V}}(x) = \{n : st(x, \mathcal{V}_n) \neq \emptyset\}$. Then $\bigcap_{n \in c_{\mathcal{V}}(x)} \overline{st(x, \mathcal{V}_n)} = \{x\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ set

$$\mathcal{G}_F = \{\bigcap_{i \in F} V_i : V_i \in \mathcal{V}_i\}.$$

We show that $\{\mathcal{G}_{_F} : F \in \mathcal{F}\}$ is a quasi–development of X. For each $n \in \mathbb{N}$ put $F_n = c_{\mathcal{V}}(x) \cap \{1, 2, ..., n\},$. Then $F_n \neq \emptyset$. Note that $st(x, \mathcal{G}_{F_m}) \subseteq st(x, \mathcal{V}_n)$ for each $m \geq n$ and $n \in c(x)$. Note also that

$$\bigcap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_{F_n})} = \bigcap_{n \in \mathbb{N}} st(x, \mathcal{G}_{F_n}) = \{x\}.$$

By Lemma 3.1, $\{st(x, \mathcal{G}_{F_n}) : n \in \mathbb{N}\}$ forms a local neighborhood base at x. Hence, $\{st(x, \mathcal{G}_F) : F \in \mathcal{F}\} - \emptyset$ forms a local neighborhood base at x. ■

Question 3.3 *Is every pseudocompact space with a quasi–regular–G_δ–diagonal metrizable?*

Acknowledgement: The author is grateful to Prof. David Gauld for his kind help and valuable comments and suggestions on this paper. The author wishes to thank Prof. J. Nagata for his useful remarks on this paper.

References

[1] R. Engelking, *General Topology*, Berlin, 1989.

[2] P.M. Gartside and A.M. Mohamad, *Cleavability of manifolds*, Topology Proc. (to appear).

[3] G.Gruenhage, *Generalized metric spaces*, Handbook of Set-theoretic Topology (1984), 423–501.

[4] K. Lee, *Spaces in which compacta are uniformly regular G_δ*, Pacific J. Math., 81 (1979), 435–445.

[5] D. Lutzer, *A metrization theorem for linearly orderable spaces*, Proc. Amer. Math. Soc., 22 (1969), 557–558.

[6] W. McArthur, G_δ–diagonals and metrization theorems, Pacific J. Math., 44 (1973), 613–617.

[7] A.M. Mohamad, *Generalization of G^*_δ-diagonals and $w\Delta$–spaces*, Acta Math. Hungar 80 (1998), 285–291.

[8] A. M. Mohamad, *Developable Spaces and problems of Fletcher and Lindgren*, submitted.

[9] J. Nagata, *On a necessary and sufficient condition of metrizability*, J. Inst. Polytech. Osaka City Univ. Ser. A, Math, 1 (1950), 93–100.

[10] G.M. Reed, Set–theoretic problems in Moore spaces, In: J. Van Mill, G.M. Reed (eds.), Open problems in topology, North–Holland, 1990, 163–181

[11] H. Shiraki, *A note on spaces with a uniform base*, Proc. Japan. Acad., 47 (1971), 1036–1041.

[12] P. Zenor, *Spaces with regular–G_δ–diagonals*, Pacific J. Math. 40 (1972), 759–763.