Characterizations of Moore and Semi–stratifiable Spaces *

Abdul M. Mohamad

September 29, 1999

Abstract

In this paper, we study Moore and semi–stratifiable spaces. We give characterizations of developable and semi–stratifiable spaces. We prove that: a regular space X is semi–stratifiable if and only if it is a β, quasi–semi–stratifiable and the following are equivalent for a regular $w\Delta$–space X:

(a) X is a Moore space;

(b) X is a hereditarily weakly θ–refinable space with a quasi–G_δ–diagonal;

(c) X is a quasi–G_δ^*–diagonal;

(d) X is a quasi–semi–stratifiable space;

(e) X is a quasi–α–space.

1 Definitions

Throughout this paper "space" will always mean "T_1 topological space".

Let (X,τ) be a space and let $g\colon \mathbb{N}\times X \to \tau$ be a map such that $c(x) = \{n \in \mathbb{N} : g(n,x) \neq \emptyset\}$ (it is used elsewhere with a similar meaning) is infinite and $x \in \bigcap_{n\in c(x)} g(n,x)$. g is called **quasi–*COC*–map** (= quasi–countable open covering map) for X if the following condition is satisfied: for each $i,j \in c(x), g(i,x) \subset g(j,x)$ if $i > j$. g is called ***COC*–map** (= countable open covering map) for X if the following conditions are satisfied:

*The author acknowledges the support of the Marsden Fund Award UOA 611, from the Royal Society of New Zealand. AMS (1991) Subject Classification: 54E30, 54E35. Keywords and phrases: semi–stratifiable space; weakly γ–space; weakly β–space; metrizable; Moore space; quasi–α–space; quasi–G_δ^*–diagonal.

(a) $x \in \bigcap_{n\in\mathbb{N}} g(n,x)$ for all $x \in X$;

(b) $g(n+1,x) \subset g(n,x)$ for all $n \in \mathbb{N}$ and $x \in X$.

Consider the following conditions on g.

(1) $n \in c(x)$ whenever $x \in g(n,y)$.

(2) The collection $\{g(n,x) : n \in c(x)\}$ is a local basis at the point x.

(3) If $x \in g(n,x_n)$ for every $n \in c(x)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(4) If $x \in g(n,x_n)$ for every $n \in \mathbb{N}$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(5) If $x \in g(n,y)$, then $y \in g(n,x)$.

(6) If $x \in g(n,x_n)$ for every $n \in c(x)$, then the sequence $\langle x_n \rangle$ has a cluster point.

(7) If $x \in g(n,x_n)$ for every $n \in \mathbb{N}$, then the sequence $\langle x_n \rangle$ has a cluster point.

(8) $\bigcap_{n\in c(x)} g(n,x) = \{x\}$

(9) $\bigcap_{n\in\mathbb{N}} g(n,x) = \{x\}$

(10) If $y \in g(n,x)$ then $g(n,y) \subseteq g(n,x)$.

(11) If $y_n \in g(n,x)$ and $x_n \in g(n,y_n)$ for each $n \in c(x)$ then x is a cluster point of the sequence $\langle x_n \rangle$.

(12) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n,y_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(13) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n,y_n)$, then the sequence $\langle x_n \rangle$ has a cluster point.

(14) If $x_n \in g(n,x)$ for every $n \in c(x)$, then the sequence $\langle x_n \rangle$ has a cluster point.

A space X is called **developable**; **semi–stratifiable**; $\boldsymbol{w\Delta}$; $\boldsymbol{\beta}$; $\boldsymbol{\alpha}$ if X has a COC–map g satisfies (12); (4); (13); (7); (9) and (10).

Bennett proved that quasi–developable spaces can be characterized by a quasi–COC–map g satisfying conditions (2), (3) and (5) [1, Theorem 1] and by [8, Lemma 2.1], quasi–$w\Delta$–spaces can be characterized by a quasi–COC–map g satisfying conditions (5), (6) and (14). A quasi–COC–map satisfying conditions (2), (3) and (5) will be called a quasi–developable–map and one satisfying conditions (5), (6) and (14) will be called a quasi–$w\Delta$–map.

Lee defines **quasi–semi–stratifiable spaces** to be those possessing a quasi–COC–map g that satisfies conditions (1) and (3)[6, Definition 2.3]. A quasi–COC–map which satisfies the conditions (1) and (3) will be called a quasi–semi–stratifiable map.

Definition 1.1 *A space* (X, τ) *is said to be a* **quasi–$\boldsymbol{\alpha}$–space***;* **weak–$\boldsymbol{\beta}$–space***;* **weak–$\boldsymbol{\gamma}$–space** *if there is a quasi–COC–map g that satisfies (1), (8) and (10); (1) and (6); (11) respectively. A quasi–COC–map g which satisfies these respective conditions will be called a quasi–α–map; weak–β–map; weak–γ–map respectively.*

Let $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a sequence of families of open subsets of X, for example $g : \mathbb{N} \times X \to \tau$ may be a function as above and $\mathcal{G}_n = \{g(n, x) \ / \ x \in X\}$. Define $c(x) = c_{\mathcal{G}}(x) = \{n : x \in \mathcal{G}_n^*\}$ where $\mathcal{G}_n^* = \bigcup\{G : G \in \mathcal{G}_n\}$. A space X has a quasi–G_δ^*–diagonal [7] if there is such a sequence $\mathcal{G}$ such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$.

A space is Moore (resp. quasi–Moore) if and only if it is regular developable (quasi–developable).

For terminologies which are not defined in this paper, the readers should consult books [5] and [2].

2 Main Results

Theorem 2.1 *The following are equivalent for a regular space X.*

(a) X is a semi–stratifiable space;

(b) X is a β–space with a quasi–G_δ^*–diagonal;

(c) X is a quasi–α, β–space.

Proof. Every regular semi–stratifiable space is an α and has a G_δ^*–diagonal, so, it is clear that (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c). To prove (b) $\Rightarrow$ (a), let $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ be a quasi–G_δ^*–diagonal sequence of X and let $g : \mathbb{N} \times X \to \tau$ be a β–map of X. Then $\bigcap_{n\in c_{\mathcal{V}}(x)} \overline{st(x, \mathcal{V}_n)} = \{x\}$.

Define a map $h : \mathbb{N} \times X \to \tau$ by

$$h(n,x) = \begin{cases} g(n,x) \cap st(x,\mathcal{V}_n) & \text{if } x \in \mathcal{V}_n^*. \\ g(n,x) & \text{if } x \notin \mathcal{V}_n^* \ . \end{cases}$$

Let $r(n,x) = \bigcap_{i=1}^{n} h(i,x)$. We prove that $r(n,x)$ is a semi–stratifiable–map. Let $x \in r(n,x_n)$. It is clear that r is a β–map, so, $\langle x_n \rangle$ has a cluster point, say p. Suppose that $x \neq p$. Choose k large enough that $x \in \overline{st(x,\mathcal{V}_k)}$ but $p \notin \overline{st(x,\mathcal{V}_k)}$.

For each $n \geq k$,

$$x_n \in st(x,\mathcal{V}_k).$$

Thus the open neighborhood $X - \overline{st(x,\mathcal{V}_k)}$ of p contains at most $k-1$ members of the sequence $\langle x_n : n \in \mathbb{N} \rangle$, which contradicts the fact that p is a cluster point of $\langle x_n \rangle$.

To prove (c) $\Rightarrow$ (a), let g be a β–map for X and f be a quasi–α–map for X. Define

$$h(n,x) = \begin{cases} g(n,x) \cap f(n,x) & \text{if } n \in c(x). \\ g(n,x) & \text{if } n \notin c(x) \ . \end{cases}$$

Let $k(n,x) = \bigcap_{i=1}^{n} h(i,x)$. We shall show that the map k satisfies the conditions for a semi–stratifiable–map. Clearly the first and second conditions are satisfied. To check the third condition, let $x \in k(n,x_n)$, for $n \in \mathbb{N}$. Then for $n \in \mathbb{N}$ $x \in g(n,x_n)$ and so $\langle x_n \rangle$ has a cluster point y. Suppose $x \neq y$. Now $\bigcap_{n \in c(y)} f(n,y) = \{y\}$ and so there is $n_o \in \mathbb{N}$ such that $x \notin f(n_o,y)$. Since y is a cluster point of the sequence $\langle x_n \rangle$, there is a $m \geq n_o$ such that $x_m \in f(n_o,y)$ and so, $n_o \in c(x_m)$. Since f is a quasi–α map for X, $x_m \in f(n_o,y)$ implies $f(n_o,x_m) \subseteq f(n_o,y)$. But $x \in k(m,x_m) \subseteq f(n_o,x_m)$ and so, $x \in f(n_o,y)$ which is a contradiction. Thus $x = y$ and x is a cluster point of $\langle x_n \rangle$. ∎

Theorem 2.2 *The following are equivalent for a regular $w\Delta$–space X.*

(a) X is a Moore space;

(b) X is a hereditarily weakly θ–refinable space with a quasi–G_δ–diagonal;

(c) X is a quasi–G_δ^*–diagonal;

(d) X is a semi–stratifiable space;

(e) X is a quasi–semi–stratifiable space;

(f) X is a G_δ^*–diagonal;

(g) X is a α–space;

(h) X is a quasi–α–space.

Proof. Every Moore space is hereditarily θ–refinable space with a quasi–G_δ–diagonal, so (a)$\Rightarrow$ (b). The implication (b)$\Rightarrow$ (c) follows from [7, Theorem 2.6]. The implication (c)$\Rightarrow$ (d) follows from the theorem above and the fact that every $w\Delta$–space is an β. The implication (d)$\Rightarrow$ (e) is trivial. The implication (e)$\Rightarrow$ (c) follows from the fact that every regular quasi–semi–stratifiable space has a quasi–G^*_δ–diagonal [7]. The implication (d)$\Rightarrow$ (f) follows from the fact that every regular semi–stratifiable space has a G^*_δ–diagonal. The implication (f)$\Rightarrow$ (a) is Hodel's theorem [5, Theorem 3.3 (a space is a Moore space if and only if it is a regular $w\Delta$ with a G^*_δ–diagonal)]. The implication (a)$\Rightarrow$ (g) follows from the fact that every developable space is α. The implication (g)$\Rightarrow$ (h) is obvious. The implication (h) $\Rightarrow$ (e) is the [8, Theorem 3.4: a regular space X is a quasi–semi–stratifiable if and only if it is a quasi–α, quasi–β–space]. ■

Counterexamples involving weakening of the hypotheses in Theorem 2.2 are given in [3] and [4] as follows.

Example 2.3 *There is a p–adic analytic manifold which is separable, submetrizable, quasi–developable, but not perfect (see [3, Example 3.7]). This example also can serve as a quasi–semi–stratifiable space (which is weak–β–space) which has a G^*_δ–diagonal but which is not semi–stratifiable.* ■

Example 2.4 *There is a quasi–developable manifold which has a G_δ–diagonal but not a G^*_δ–diagonal (see [4, Example 2.2]) This example also can serve as a quasi–$w\Delta$ manifold which is not $w\Delta$. (It is not even a β–manifold).* ■

Proposition 2.5 *A regular weakly γ–space is a Moore space if and only if it is a β–space.*

Proof. Every Moore space is β. Conversely let f be a β map and g a weakly γ–map for X. Define

$$h(n,x) = \begin{cases} g(n,x) \cap f(n,x) & \text{if } n \in c(x). \\ f(n,x) & \text{if } n \notin c(x)\ . \end{cases}$$

Let $r(n,x) = \bigcap_{i=1}^{n} h(i,x)$. We show that r is a developable map. Let $\{x, x_n\} \subseteq r(n, y_n)$, for all $n \in \mathbb{N}$. Now $x \in f(n, y_n)$, for all $n \in c(x)$ so $\langle y_n \rangle$ has a cluster point, say y. Let $\langle y_{n_k} \rangle$ be a subsequence of $\langle y_n \rangle$ such that

$y_{n_k} \in g(k,y)$ for all $k \in c(x)$. Now $x_{n_k} \in r(n_k, y_{n_k}) \subseteq g(n_k, y_{n_k}) \subseteq g(k, y_{n_k})$, so we have $y_{n_k} \in g(k,y)$ and $x_{n_k} \in g(k, y_{n_k})$ for all $k \in c(y)$. Thus y is a cluster point of $\langle x_{n_k} \rangle$. On the other hand $y_{n_k} \in g(k,y)$ and $x \in g(k, y_{n_k})$ for all $k \in c(y)$, so y is a cluster point of $\langle x \rangle$. Therefore $x = y$ from which it follows that x is a cluster point of $\langle x_n \rangle$. ■

Acknowledgement: The author is grateful to Prof. David Gauld and Dr. David McIntyre for their kind help and valuable comments and suggestions on this paper.

References

[1] H.Bennett,*A short proof that compact quasidevelopable spaces are metrizable*, Proc. Amer. Math. Soc. 86, No. 4, (1982), 667–669.

[2] R. Engelking, *General Topology*, Berlin, 1989.

[3] P.M. Gartside and A.M. Mohamad, *Diversity of p–adic analytic manifolds*, Topology Appl. (to appear).

[4] P.M. Gartside, C. Good, R. Knight and A.M. Mohamad, *Quasi–developable manifolds*, Topology Appl. (to appear).

[5] G.Gruenhage, *Generalized metric spaces*, Handbook of Set–theoretic Topology (1984), 423–501.

[6] I.H.Lee, *On quasi–semidevelopable spaces*, J. Korean Math. Soc. 12, No. 1, (1975), 71–77.

[7] A. M. Mohamad, *Generalization of G^*_δ–diagonals and $w\Delta$–spaces*, Acta Math. Hungar, 80 (1998), 285–291.

[8] A. M. Mohamad, *Some conditions which imply quasi–developability*, Bull. Cal. Math. Soc., 90 (1998), 325–330

[9] A. M. Mohamad, *Developable Spaces and problems of Fletcher and Lindgren*, submitted.

The Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland
New Zealand.
mohamad@math.auckland.ac.nz