Weak bases and Metrizability*

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Abstract

In this paper we investigate weak bases. We give a characterization of weakly developable spaces and metrization theorems. The metrization results are: a space X is metrizable if and only if X has a $CWBC$–map g satisfying the following conditions:

1. g is a pseudo–strongly–quasi–N–map;
2. for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$;

a space X is metrizable if and only if X has a $CWBC$–map g satisfying the following conditions:

1. if $x \in g(n, y_n)$, $y_n \in g(n, x_n)$, $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for all $n \in \mathbb{N}$, then x_n converges to x;
2. for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.

1 Definitions

A collection $\mathcal{W}$ of subsets of a space X is said to be a **weak base** for X provided that to each $x \in X$, there exists $\mathcal{W}_x \subset \mathcal{W}$ such that:

(i) Each member of $\mathcal{W}_x$ contains x.

(ii) For any two members W_1 and W_2 of $\mathcal{W}_x$, there is a $W_3 \in \mathcal{W}_x$ such that $W_3 \subset W_1 \cap W_2$.

(iii) A subset U of X is open if and only if for every point $x \in U$ there exists a $W \in \mathcal{W}_x$ such that $W \subset U$.

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If to each $x \in X$ we assign a collection $\mathcal{W}_x$ of supersets of $\{x\}$ such that $\mathcal{W} = \bigcup\{\mathcal{W}_x : x \in X\}$ is a weak base by virtue of the collections $\mathcal{W}_x$, then we say that the collection $\mathcal{W}_x$ is a **local weak base** at x for each $x \in X$.

Graded weak bases:

Let (X, τ) be a space, let $g : \mathbb{N} \times X \to \mathcal{P}(X)$ be a function and $\mathcal{G} = \{g(n,x) : n \in \mathbb{N}, x \in X\}$. We call $\mathcal{G}$ a **graded weak base** for X and g is called a **CWBC–map** (= countable weak base covering map) for X and X with a graded weak base is called **weakly first countable** if the following conditions are satisfied:

(a) $x \in \bigcap_{n \in \mathbb{N}} g(n,x)$ for all $x \in X$.

(b) $g(n+1,x) \subset g(n,x)$ for all $n \in \mathbb{N}$ and $x \in X$.

(c) A subset U of X is open if and only if for every $x \in U$ there is an $n \in \mathbb{N}$ such that $g(n,x)$ is contained in U.

The map g is called a **COC–map** (= countable open covering map) for X if conditions (a), (b) and for each $n \in \mathbb{N}, g(n,x)$ is open are satisfied. A space X is called first countable (resp. q) if and only if X has a COC–map g such that if $x_n \in g(n,x)$ for every $n \in \mathbb{N}$, then x is a cluster point of the sequence $\langle x_n \rangle$ (then the sequence $\langle x_n \rangle$ has a cluster point).

Generalizations of developable spaces:

Martin in [4] introduced weakly developable spaces. Let $\mathcal{G} = \{\, \mathcal{G}_n : n \in \mathbb{N}\}$ be a countable family of collections of subsets of a space X. Consider the following conditions on $\mathcal{G}$:

(a) For each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a collection of open sets in X.

(b) Each $\mathcal{G}_n$ is a covering of X;

(c) For each $x \in X$, $\{\, st(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in \mathcal{G}_n^* \,\}$ is a local base at x;

(d) For each $x \in X$ and $n \in \mathbb{N}$, $st(x, \mathcal{G}_n)$ is an open subset of X.

(e) For each $x \in X$, $\{\, st(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in \mathcal{G}_n^* \,\}$ is a local weak base at x;

(f) For any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\};$$

Definition 1.1 *A space X is* **developable** *if there exists a family $\mathcal{G}$ satisfying (a), (b) and (c);*

A space X is **o–semi–developable** *if there exists a family $\mathcal{G}$ satisfying (b), (c) and (d);*

A space X is **semi–developable** *if there exists a family $\mathcal{G}$ satisfying (b) and (c);*

A space X is **weakly–developable space** *if there exists a family $\mathcal{G}$ satisfying (b) and (e);*

A space X has a **quasi–G_δ^*–diagonal** *if there exists a family $\mathcal{G}$ satisfying (a) and (f);*

A regular developable space is called a Moore space. Throughout this paper, every space is T_1.

2 Generalizations of first countable spaces

Recall, a space X is **sequential** [1] if every sequentially open set is open, where a set U is said to be sequentially open if every sequence converging to a point in U is eventually in U. A space is **Frechet** [1] if every accumulation point of a set is the limit of a sequence in the set. Every first countable space is Frechet and every weakly first countable space is sequential.

Lemma 2.1 *A T_2 space X is first countable if and only if X is Frechet and weakly first countable.*

Proof.

It is clear that every first countable space is Frechet and weakly first countable. Now, let $g : \mathbb{N} \times X \to \mathcal{P}(X)$ be a CWBC–map. We claim that for each $x \in X, g(n,x)$ is a neighborhood of x for each $n \in \mathbb{N}$. Suppose it is not, so there is a point $x \in X$ and $m \in \mathbb{N}$ for which $g(m,x)$ is not a neighborhood of x. Then by the Frechet assumption, there is a sequence $\langle x_k \rangle$ which converges to x such that $x_k \in X - g(m,x)$. Let $U = X - \{x, x_1, x_2, x_3, ...\}$, which is open (we are assuming our space is Hausdorff), so that for each $p \in U$ there exist a $g(n,p)$ which is contained in U. Put $V = U \cup \{x\}$, for each $p \in V$ there is a $g(n,p)$, such that $g(n,p) \subset V$, since if $p = x$ then $g(n,p) = g(m,x) \subset V$. By the assumption of weak first countability of g, V is open in X. But x is in V, so that we have the contradiction that $\langle x_k \rangle$ does not converge to x. Thus x is in the interior of $g(m,x)$. Now, put $h(n,x) = Int\ g(n,x)$ for each $n \in \mathbb{N}$ and $x \in X$, then $h : \mathbb{N} \times X \to \tau$ satisfies the first countability condition. ■

From the Lemma 2.1 and the results in [8], we can summarize the relationships between the classes above in the following diagram:

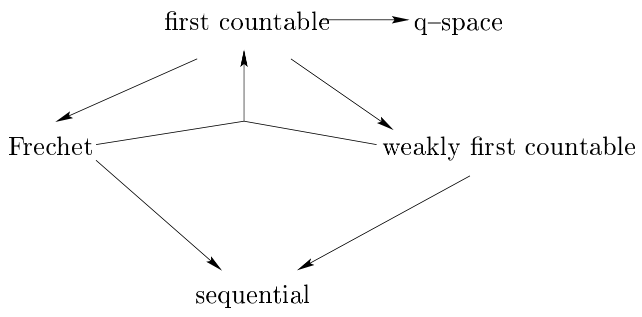

Figure 1: Relationships between some generalized first countable spaces.

Lemma 2.2 *A q space with quasi–G^*_δ–diagonal is first countable.*

Proof. Let f be a q–map and $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ a quasi–G^*_δ–sequence on X. Define g by

$$g(n,x) = \begin{cases} st(x,\mathcal{G}_n) & \text{if } x \in \mathcal{G}^*_n. \\ X & \text{if } x \notin \mathcal{G}^*_n \ . \end{cases}$$

For each $x \in X$ and $n \in \mathbb{N}$, let $h(n,x) = f(n,x) \cap \bigcap_{i=1}^{n} g(i,x)$. Then h is a first countable map. Let $x_n \in h(n,x)$. Then $\langle x_n\rangle$ has a cluster point, say y (because g is q–map). For all $n \in \mathbb{N}, y$ is a cluster point of $\langle x_m : m \geq n\rangle$, so $y \in \overline{h(n,x)}$ as $x_m \in h(n,x)$ for all m. Thus $y \in \bigcap_{n\in\mathbb{N}} \overline{h(n,x)} \subset \bigcap_{n\in\mathbb{N}} \overline{st(x,\mathcal{G}_n)} = \{x\}$, so $y = x$ and x is a cluster point of $\langle x_n\rangle$. ∎

3 Weakly developable spaces

Our next theorem is a characterization of Martin's weak developability concept.

Theorem 3.1 *A space X is weakly developable if and only if there is a CWBC–map $g : \mathbb{N} \times X \longrightarrow X$, such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all n, then then the sequence $\langle x_n\rangle$ has a cluster point.*

Proof. We will prove firstly the sufficiency of the condition. Suppose that there is a CWBC–map $g : \mathbb{N} \times X \longrightarrow X$, such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all n, then the sequence $\langle x_n\rangle$ has a cluster point. For each $i \in \mathbb{N}$, let $\mathcal{G}_i = \{g(j,x) : x \in X, j \geq i\}$. The sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of covers constitutes a weak–development. Suppose, conversely, that $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ is a weak–development for X. Define the map g as follows: for each point x of X let

$g(1,x)$ be some member of $\mathcal{G}_1$ which contains x and, if $n > 1$, let $g(n,x)$ be a member of $\mathcal{G}_n$ such that $x \in g(n,x) \subset g(n-1,x)$. Clearly g is a CWBC–map which satisfies the condition of the theorem. ■

Theorem 3.2 *A regular space is Moore if and only if it is weakly developable and Frechet.*

Proof. Let $g : \mathbb{N} \times X \to \mathcal{P}(X)$ be a weakly developable–map. We can use the same proof as for Lemma 2.1 to prove that for each $x \in X, g(n,x)$ is a neighborhood of x for each $n \in \mathbb{N}$. Thus x is in the interior of $g(n,x)$. Now, put $h(n,x) = Int\ g(n,x)$ for each $n \in \mathbb{N}$ and $x \in X$. Then $h : \mathbb{N} \times X \to \tau$ is a developable map. ■

We can summarize the relationships between some classes of generalizations of developable spaces in the following diagram:

$$\begin{tikzcd}
\text{Developable} \arrow[d] \\
\text{o–semidevelopable} \arrow[d] \\
\text{semidevelopable} \arrow[d] \\
\text{weakly developable} \arrow[d] \\
\text{weakly first countable}
\end{tikzcd}$$

Figure 2: Relationships between some classes of generalizations of developable spaces and weakly developable spaces.

Definition 3.3 *A space X is called a* **pseudo–strongly–quasi–N–space** *if there is a $CWBC$–map $g : \mathbb{N}\times X \to \mathcal{P}(X)$ such that if for each $n \in \mathbb{N}, y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to p in X, then p is a cluster point of the sequence $\langle x_n \rangle$. The $CWBC$–map g is called a* **pseudo–strongly–quasi–N–map** *for X.*

The proof of our next result relies on a metrization theorem of Martin [4].

Theorem 3.4 *(Martin) A necessary and sufficient condition that a topological space X be metrizable is that X has a weak development $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ such that $\{st^2(x,\mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of X.*

Theorem 3.5 *A space X is metrizable if and only if X has a CWBC–map g satisfying the following conditions:*

(1) g is a pseudo–strongly–quasi–N–map;

(2) for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.

Proof. The only if part is obvious. We now prove the if part. Assume that X has a $CWBC$–map g satisfying the conditions (1) and (2). Let $h(n,x) = X - \overline{\{y \in X : x \notin g(n,y)\}}$ and $k(n,x) = g(n,x) \cap h(n,x)$ for each $(n,x) \in \mathbb{N} \times X$. Let $\mathcal{G}_n = \{k(n,x) : (n,x) \in \mathbb{N} \times X\}$. Then $st(x,\mathcal{G}_n) = \bigcup\{k(n,y) : x \in k(n,x)\}$ and $st^2(x,\mathcal{G}_n) = \bigcup\{k(n,y) : k(n,y) \cap st(x,\mathcal{G}_n) \neq \emptyset, (n,x) \in \mathbb{N} \times X\}$.

By condition (2) on g, $x \in h(n,x)$. To see this, let $A = \{y : x \notin g(n,y)\}$, and suppose by contradiction that $x \in \overline{A}$. By (2), there exists $y \in A$ such that $x \in g(n,y)$. But $y \in A$ means that $x \notin g(n,y)$, a contradiction. Therefore, $h(n,x)$ is a neighborhood of x and so is $k(n,x)$. Therefore, in virtue of the Martin metrization theorem 3.4, we only need prove that $\{st^2(x,\mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of X. If $\{st^2(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is not a local weak base for some $x \in X$, then there exists an open neighbourhood U of x such that $st^2(x,\mathcal{G}_n) - U \neq \emptyset$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ take $y_n \in st^2(x,\mathcal{G}_n) - U$. That means we can find $z_n, w_n \in X$ such that $y_n \in k(n,z_n), k(n,z_n) \cap k(n,w_n) \neq \emptyset$ and $x \in k(n,w_n)$. Take $v_n \in k(n,z_n) \cap k(n,w_n)$. By $x \in k(n,w_n) \subseteq g(n,w_n)$ and condition (1), we conclude that $\langle w_n \rangle$ converges to x, and by $v_n \in k(n,w_n) \subseteq h(n,w_n)$ and the definition of h, we get $w_n \in g(n,v_n)$. Using condition (1) again, we have that $\langle v_n \rangle$ converges to x. Similarly, from $v_n \in k(n,z_n) \subseteq g(n,z_n)$, we have that $\langle z_n \rangle$ converges to x, and by $y_n \in k(n,z_n) \subseteq h(n,z_n)$, we get that $\langle y_n \rangle$ converges to x. But $y_n \notin U$ for each $n \in \mathbb{N}$, which is a contradiction. ■

In the next theorem, we use a technique similar to that used in [5, Theorem 2.1].

Theorem 3.6 *A space X is metrizable if and only if X has a CWBC–map g satisfying the following conditions:*

(1) if $x \in g(n,y_n)$, $y_n \in g(n,x_n)$, $x_n \in g(n,y_n)$ and $y_n \in g(n,x)$ for all $n \in \mathbb{N}$, then x_n converges to x;

(2) for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.

Proof. The only if part is obvious. We now prove the if part. Assume that X has a $CWBC$–map g satisfying the conditions (1) and (2). By Theorem 3.5, we need only to prove that X has a pseudo–N–map (because every pseudo–N–map is a pseudo–strongly–quasi–N–map) which satisfies condition (2). For each $p \in X$ and each $n \in \mathbb{N}$ let $h(n,p) = X - \overline{\{y : p \notin g(n,y)\}}$ and $k(n,x) = g(n,x) \cap h(n,x)$. By condition (2) on g, $x \in h(n,x)$. To see this, let $A = \{y : p \notin g(n,y)\}$, and suppose by contradiction that $p \in \overline{A}$. By (2), there exists $y \in A$ such that $p \in g(n,y)$. But $y \in A$ means that $p \notin g(n,y)$, a contradiction. Therefore, $h(n,x)$ is a neighborhood of x and so is $k(n,x)$. So, k is a $CWBC$–map. Now, let $y_n \in k(n,x) \cap k(n,x_n)$ for all $n \in \mathbb{N}$. We have, $y_n \in k(n,x), y_n \in g(n,x)$ and $y_n \in h(n,x)$. From the definition of h, $x \in g(n,y_n)$. It follows that $y_n \in g(n,x)$ and $x \in g(n,y_n)$ (3).

We have, $y_n \in k(n,x_n), y_n \in g(n,x_n)$ and $y_n \in h(n,x_n)$. From the definition of h, $x_n \in g(n,y_n)$. It follows that $y_n \in g(n,x_n)$ and $x_n \in g(n,y_n)$ (4).

If we now combine (3) and (4), we see that $x \in g(n,y_n)$, $y_n \in g(n,x_n)$, $x_n \in g(n,y_n)$ and $y_n \in g(n,x)$ for all $n \in \mathbb{N}$. Hence by condition (1), $\langle x_n \rangle$ converges to x. ■

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References

[1] S.P. Franklin, *Spaces in which sequences suffice. II*, Fund. Math. 57 (1965), 107-115.

[2] G.Gruenhage, *Generalized metric spaces*, Handbook of Set-theoretic Topology (1984), 423-501.

[3] P. M. Gartside and A. M. Mohamad,*Cleavability of manifolds*, to appear.

[4] H. W. Martin, *Weak bases and metrization*, Trans. Amer. Math. Soc. 222 (1976), 338-344.

[5] R. Hodel, *Metrizability of spaces satisfying Nagata's condition*, Math. Japonica 47 (1998), 287-293.

[6] A.M. Mohamad, *Conditions which imply metrizability in some generalized metric spaces*, submitted.

[7] J. Nagata, *Metrizability, generalized metric spaces and g–functions*, Comment. Math. Univ. Carolin. 29 (1988), 715–722.

[8] F. Siwiec, *On defining a space by a weak base*, Pacific J. Math. 52 (1974), 233-245.

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