A relation between the spectrum of the Laplacean and the geometry of a compact graph

Mark Stuart Harmer
email: harmer@math.auckland.ac.nz
Department of Mathematics
University of Auckland
New Zealand

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Abstract

For the Laplacean on a compact graph with edges of commensurate length and flux-conserved boundary conditions we provide a description of the spectrum in terms of the geometry of the graph.

1 Introduction

We consider the Laplacean on a compact graph with edges of commensurate length and flux-conserved boundary conditions. For this simple operator we are able to find the relationship between the spectrum and the geometry of the graph on which the operator is defined.
Specifically, suppose we are given a compact graph Γ_c which we decompose into the set of subgraphs $\Gamma_{d(i)}$ each with one vertex. This set is specified by a projection P, while the way in which they are connected to form Γ_c is specified by an 'adjacency' matrix φ. Then we are able to show that the spectrum of the operator corresponds to the spectrum of the matrix $P^{\perp}\varphi P^{\perp}$ up to the dimension of a couple of the eigenspaces.

2 The spectrum on a compact graph with edges of commensurate length

Here we consider the case of minus the Laplacean with flux-conserved boundary conditions [2, 1, 3] on a compact graph Γ_c with edges of commensurate length. By flux-conserved boundary conditions we mean that at each vertex, ν, of the graph the functions on the graph are continuous

$$\phi_1|_\nu = \phi_2|_\nu = \cdots = \phi_d|_\nu$$

and conserve flux around the vertex

$$\sum_{i=1}^{d} \phi_i'|_\nu = 0.$$

In the above formulae we assume that the vertex ν of Γ_c has degree d, ϕ_i is the value of the function on the i-th edge incident on ν, ϕ_i' is the x-derivative of ϕ_i and in evaluating the derivative we assume that the orientation of each edge incident on ν is the same—ie. all edges are either leaving or entering the vertex ν.
This operator is also considered in the paper by Carlson [1], where the author also establishes some basic results on the spectrum—see below. In particular it is shown [1]that this operator (minus the Laplacean with flux-conserved boundary conditions on a compact graph) is self-adjoint and has non-negative eigenvalues of finite multiplicity.
We assume that Γ_c is connected (for a disconnected graph each connected component may be treated separately). These are the only restrictions we place on Γ_c, particularly Γ_c may have multiple edges, edges with end-points connected to the same vertex and vertices with degree one. In the case of a vertex with degree one we need to be specific about the boundary conditions as there is obviously some ambiguity in assigning flux-conserved boundary conditions to such a vertex. For our purposes, we always assume that a vertex with degree one has Neumann boundary conditions (ie. zero slope).

As Γ_c has edges of commensurate length we can rescale and split edges so that Γ_c has only edges of length one. Let us suppose that this rescaled graph, which for convenience we also denote as Γ_c, has m vertices and p edges. Consider splitting Γ_c into m subgraphs $\Gamma_{d(1)}, \ldots, \Gamma_{d(m)}$ each consisting of a single vertex with $d(i)$ edges attached where $d(i)$ is the degree of the i-th vertex of Γ_c. Precisely we split each edge of Γ_c in half getting m subgraphs $\Gamma_{d(1)}, \ldots, \Gamma_{d(m)}$ each with one vertex and $d(i)$ edges which each have length $1/2$.
In this way the vertices are indexed by the integers $\{1, \ldots, m\}$. We would also like to index the $2p$ edges of the subgraphs $\Gamma_{d(1)}, \ldots, \Gamma_{d(m)}$. We do this in the obvious way; the first $d(1)$ edges are from the subgraph $\Gamma_{d(1)}$, the next $d(2)$ edges are from $\Gamma_{d(2)}$, etc. In general, the edges in the range

$$\mathcal{E}_i \equiv \{D(i) + 1, \ldots, D(i) + d(i)\}$$

where

$$D(i) = d(1) + \cdots + d(i-1),$$

are from subgraph $\Gamma_{d(i)}$.
Using this scheme of indexing the edges the following $2p \times 2p$ 'adjacency' matrix naturally appears

$$\varphi_{ij} = \begin{cases} 1 & : \text{if edge } i \text{ and } j \text{ are linked in } \Gamma_c \\ 0 & : \text{otherwise} \end{cases} .$$

This is not really the adjacency matrix. Indeed the adjacency matrix is not strictly defined for a graph with multiple edges—which we have allowed—and, furthermore, from the adjacency matrix we can recover the structure of the graph to which it belongs while φ will not allow us to recover Γ_c unless we also have the $d(i)$.
Some properties of φ are immediate. It is idempotent and therefore has only eigenvalues ± 1. It has p eigenvalues $+1$ and p eigenvalues -1 as can be seen by considering the matrices $\varphi \pm \mathbb{I}$ both of which have rank p. The eigenvectors are easily constructed from the form of φ (if $\varphi_{ij} = 1$ then $e_i \pm e_j$, where $\{e_l\}$ is the usual basis of $\mathbb{C}^{2p}$, is an eigenvector with eigenvalue ± 1).

Choosing an arbitrary edge $j \in \mathcal{E}_i$ we can write the general solution on this edge as

$$\psi_j = \alpha_i \cos(kx) + \beta_j \sin(kx).$$

The subscript i is an index of the vertex and the subscript j an index of the edge. This solution is continuous and in order for it to satisfy conservation of flux at the vertex we need

$$\sum_{j \in \mathcal{E}_i} \beta_j = 0. \tag{1}$$

All that remains is to ensure that these solutions match at the points at which the subgraphs $\Gamma_{d(1)}, \ldots, \Gamma_{d(m)}$ link. Recalling that each edge of Γ_c has length 1, these matching conditions are expressed using φ as

$$\begin{aligned} \alpha_i \cos(k/2) + \beta_j \sin(k/2) &= \alpha_r \cos(k/2) + \varphi_{js}\beta_s \sin(k/2) \\ \alpha_i \sin(k/2) - \beta_j \cos(k/2) &= -\alpha_r \sin(k/2) + \varphi_{js}\beta_s \cos(k/2), \end{aligned} \tag{2}$$

where we assume summation over repeated indices. The index r is uniquely specified by j.
We consider three cases:

I. $\sin(k/2) \neq 0$, $\cos(k/2) \neq 0$. Then writing $z = \tan(k/2)$ the matching conditions become

$$\begin{aligned} \alpha_i + \beta_j z &= \alpha_r + \varphi_{js}\beta_s z \\ \alpha_i z - \beta_j &= -\alpha_r z + \varphi_{js}\beta_s. \end{aligned}$$

Eliminating α_r and solving for α_i gives us

$$2\alpha_i = \beta_j \left[\frac{1}{z} - z\right] + \varphi_{js}\beta_s \left[\frac{1}{z} + z\right]$$

or, using

$$\zeta = \frac{\frac{1}{z} - z}{\frac{1}{z} + z} = \cos(k),$$

we write this as

$$\frac{2\alpha_i}{\frac{1}{z} + z} = \beta_j \zeta + \varphi_{js}\beta_s.$$

This gives us $d(i)$ equations for each vertex i of Γ_c. Actually, since the left hand side $(2\alpha_i/((1/z)+z))$ of each equation is the same, we can eliminate the α_i to get $d(i)-1$ equations in the β_j. Specifically,

$$\beta_{D(i)+1}\zeta + \varphi_{D(i)+1,s}\beta_s = \cdots = \beta_{D(i)+d(i)}\zeta + \varphi_{D(i)+d(i),s}\beta_s.$$

II. $\sin(k/2) = 0$. Here equations (2) become

$$\begin{aligned} \alpha_i &= \alpha_r \\ -\beta_j &= \varphi_{js}\beta_s. \end{aligned}$$

This gives us

$$\beta_{D(i)+1} + \varphi_{D(i)+1,s}\beta_s = \cdots = \beta_{D(i)+d(i)} + \varphi_{D(i)+d(i),s}\beta_s = 0$$

plus the fact that all of the α_i are equal (this condition merely adds one to the dimension of the eigenspace).

III. $\cos(k/2) = 0$. Here equations (2) become

$$\begin{aligned} \beta_j &= \varphi_{js}\beta_s \\ \alpha_i &= -\alpha_r. \end{aligned}$$

This gives us

$$\beta_{D(i)+1} - \varphi_{D(i)+1,s}\beta_s = \cdots = \beta_{D(i)+d(i)} - \varphi_{D(i)+d(i),s}\beta_s = 0$$

plus the fact that the α_i 'alternate'. This condition adds one to the dimension of the eigenspace iff there are no cycles of odd length in Γ_c.

These equations along with equation (1) characterise the discrete spectrum.

Let us introduce the projections P_i in $\mathbb{C}^{d(i)}$ onto the subspace spanned by the vector $(1,\ldots,1)^T$ (the same projection which we used above in our discussion of flux-conserved boundary conditions). Then we can define the projection

$$P \equiv P_1 \otimes P_2 \otimes \cdots \otimes P_m \in \mathbb{C}^{2p}$$

where the P_i appear down the diagonal. Defining this projection allows us to write equation (1) in the simple form

$$P\beta = 0$$

where $\beta \in \mathbb{C}^{2p}$ is the vector of coefficients of the general solution. Not only that but we can use $P^{\perp}$ to express the matching conditions, equations (2), derived for the above three cases in equally simple forms. Let us consider case I above, then it is not hard to see that for this case the matching conditions become

$$P^{\perp}(\varphi + \zeta)\beta = 0,$$

or, using the fact that $P\beta = 0$, this becomes

$$(P^{\perp}\varphi P^{\perp} + \zeta)\beta = 0.$$

We summarise these results for the spectrum of minus the Laplacean in the following theorem:

Theorem 2.1 *The spectrum of minus the Laplacean with flux-conserved boundary conditions on an arbitrary compact graph with edges of length one corresponds to the spectrum of the matrix*

$$P^{\perp}\varphi P^{\perp}$$

inside the open interval $(-1, 1)$ *where the eigenvalues* ζ_l *of this matrix are related to the eigenvalues of the original operator* $\lambda_l = k_l^2$ *via*

$$-\zeta_l = \cos k_l.$$

Moreover, the dimension of the eigenspaces of the matrix (in $P^{\perp}\mathbb{C}^{2p}$*) and the operator coincide.*
At the endpoints of the interval we get: for $\zeta = -1$*, ie.* $k = 2\pi n$*, the dimension of the eigenspace of the operator coincides with the dimension of the kernel of*

$$\varphi + \mathbb{I}$$

(in $P^{\perp}\mathbb{C}^{2p}$*) plus one—consequently this eigenvalue always appears.*
For $\zeta = 1$*, ie.* $k = (2n+1)\pi$*, the dimension of the eigenspace of the operator coincides with the dimension of the kernel of*

$$\varphi - \mathbb{I}$$

(in $P^{\perp}\mathbb{C}^{2p}$*) and, as long as there are no cycles of odd length, plus one .*

Really, each eigenvalue ζ_l of the matrix corresponds to a family, $\{(k_l + 2\pi n)^2\}$ where $n \in \mathbb{Z}$, of eigenvalues of the original operator. This fact is established in the paper by Carlson [1] where, furthermore, it is shown that there are at most $2p$ families of eigenvalues, counting multiplicity. We are able to refine that result here.

Corollary 2.1 *The number of families of eigenvalues of the operater, counting multiplicity, does not exceed* $2p - m + 2$ *when there are no cycles of odd length in* Γ_c *and* $2p - m + 1$ *if there are.*

Proof: The matrix $P^{\perp}\varphi P^{\perp}$ has rank at most $2p-m$. We only need to note that at the endpoints, ± 1, of the interval this matrix overestimates the number of eigenvalues. $\square$

Corollary 2.2 *The dimension of the eigenspace of the operator corresponding to $\zeta = -1$, ie. $k = 2\pi n$, is in the range $[p-m+1, p+1]$.*
The dimension of the eigenspace of the operator corresponding to $\zeta = 1$, ie. $k = (2n+1)\pi$, is in the range $[p-m+1, p+1]$ if there are no cycles of odd length or the range $[p-m, p]$ is there are.

Proof: These statements follow simply from the fact that the nullity of the matrices $\varphi \pm \mathbb{I}$ is p and the rank of $P^{\perp}$ is $2p-m$. Hence, the spaces

$$\ker(\varphi \pm \mathbb{I}) \cap P^{\perp}\mathbb{C}^{2p}$$

have dimension at least $p-m$. $\square$

This description allows us to describe the spectrum of the operator on a compact graph in terms of purely geometrical properties, φ and P, of the graph. This may be useful in investigating the spectral problem, ie. the problem of recovering the structure of the graph from the spectrum.

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References

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