Scattering on the annulus

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Abstract

We generalise the asymptotic formula for the scattering matrix in [2] to the case of non-simple spectrum. This asymptotic formula is used to identify a simple family of switches and investigate the properties of a member of the family using numerical techniques.

1 Introduction

Here we generalise the idea in [2] where the scattering matrix of a non-compact graph is written in terms of the spectral properties of its compact part. In this paper the authors consider an annulus to which n semi-infinite rays are attached using a certain self-adjoint boundary condition. Under the assumption that the spectrum of the compact graph is simple, the authors of [2] provide an asymptotic formula for the scattering matrix at an eigenvalue of the compact graph in the limit of weak interaction between the rays and the compact part of the graph. To be precise it is shown that the scattering matrix can be written in terms of a projection onto the space spanned by the vector of values of the eigenfunction at the nodes of the rays.
Here we lift the restriction that the spectrum of the compact graph is simple. We find that their formula generalises in the obvious way where now we have a projection onto a space which may be more than one-dimensional. Furthermore we show that all but the first two terms of the asymptotic expansion of the scattering matrix depend on the orthogonal complement of the above projection. This allows us to prove the corollary that if the projection is the identity (ie. we have perfect reflection) then the scattering matrix is the identity independent of the interaction between the rays and the compact part of the graph.

In the final section we use this asymptotic formula to identify a family of switches. Using a simple numerical simulation we investigate the properties of a member of this family and compare it to a similar switch where the switching property is achieved by simply raising a potential barrier to the flow of current.
We note that our construction was anticipated by Exner et al [3]. In this paper the authors consider a 'quantum interference transistor' which uses the same basic construction as our switch. The difference is that, in our approach, we have used the asymptotic formula described above to guide our choice of *resonance* parameters [2, 8, 7].
An interesting property of our switch is that as the size of the switch decreases the efficiency of the switch increases without bound—unlike a switch based on simply raising a potential barrier where the efficiency has an upper bound due to tunneling. This is due to our assumption that the switch operates in the resonance case [2, 8, 7].

2 Asymptotics of the scattering matrix of a non-compact graph in terms of the spectral properties of its compact part

In this section we generalise the result in the report [2] which describes the asymptotics of the scattering matrix of a non-compact graph in terms of the spectrum of its compact part, in the case where the spectrum is simple. Here we extend this result to the case of spectrum with arbitrary finite multiplicity.
Consider the case of an arbitrary compact graph Γ_c with n semi-infinite rays attached to points on the *edges* of Γ_c. The interaction between each ray and the compact graph is specified by a particular self-adjoint boundary condition (see equation (2) below) at the point of attachment. Here we will be investigating the limit as the interaction between the compact graph and the rays goes to zero.
The component of the i-th scattering wave solution on the j-th ray is denoted by ψ_{ij}, and has the form

$$\psi_{ij} = \delta_{ij} e^{-ikx_j} + S_{ij} e^{ikx_j}. \tag{1}$$

It is assumed that the potential on the rays is zero.
We denote the i-th scattering wave solution on the compact part of the graph by ψ_{i0}. The scattering wave solutions satisfy the following self-adjoint (theorem 1 of [2]) boundary conditions at each connection point a_j

$$\begin{aligned} [\psi'_{i0}]|_{a_j} &= -\beta \psi'_{ij}|_0 \\ \psi_{ij}|_0 &= \overline{\beta} \psi_{i0}|_{a_j} \end{aligned} \tag{2}$$

where $j = 1, \ldots, n$. The notation $[\cdot]$ denotes the jump in the value of the function. It is clear that these boundary conditions are invariant with respect

to the orientation chosen on the edge of the compact graph.
It is clear (theorem 5 of [2]) that we can use the ansatz

$$\psi_{i0} = \sum_{j=1}^{n} \alpha_{ij}(k) g_0(x, a_j, k)$$

for the scattered wave on the compact subgraph where $g_0(x, y, k)$ is the Green's function of the (unperturbed) compact graph and the $a_j \in \Gamma_c$ is the connection point of the j-th ray.
The scattering matrix can be found from the above equations (theorem 5 of [2]) by eliminating α_{ij} and solving

$$S(k) = -\left[\mathbb{I} + ik|\beta|^2 G\right]\left[\mathbb{I} - ik|\beta|^2 G\right]^{-1}. \tag{3}$$

Here the matrix G is given by the unperturbed Green's function on the compact graph evaluated at the nodes of each of the rays

$$G_{ij} = g_0(a_i, a_j).$$

Now since the unperturbed compact graph has only discrete spectrum, we may expand the Green's function (theorem 2 of [2]) in terms of an absolutely and uniformly convergent series

$$g_0(x, y) = \sum_i \frac{\xi_i(x)\xi_i(y)}{\lambda_i - \lambda}$$

of orthonormalised eigenfunctions, $\{\xi_i\}$. Consequently we can write G as

$$G_{ij} = \sum_i \frac{\xi_i(a_i)\xi_i(a_j)}{\lambda_i - \lambda}.$$

Let us choose a particular eigenvalue, λ_l. In general λ_l will not be a simple eigenvalue; let us assume that it has a p-dimensional eigenspace

$$\mathcal{R}_l = \bigvee \{\xi_{l,i}\}_{i=1}^{p} \subset \mathcal{H}.$$

Here $\mathcal{H}$ is just the space of complex functions on Γ_c. We define the mapping $\mathcal{P} : \mathcal{H} \to \mathbb{C}^n$, where n is the number of rays, by taking the values of an element of $\mathcal{H}$ at the nodes of each of the n rays. Let us use the notation

$$\mathcal{P}(\psi) = |\psi\rangle \in \mathbb{C}^n.$$

In particular, we will be interested in the map of the eigenspace $R_l = \mathcal{P}(\mathcal{R}_l)$ which is in general an m-dimensional subspace of $\mathbb{C}^n$, $m \leq p$ —take for instance the case where one of the eigenfunctions $\xi_{l,i}$ is zero on the nodes of each of the rays. We claim

Proposition 2.1 *It is possible to choose an orthonormal basis $\{\phi_i\}_{i=1}^p$ for $\mathcal{R}_l$ which forms an* orthogonal, *but not necessarily normalised, basis for R_l under the map $\mathcal{P}$.*

Proof: Given an orthonormal basis $\{\xi_{l,i}\}_{i=1}^p$ for $\mathcal{R}_l$ we note that any unitary matrix $U \in \mathsf{U}(p)$ gives us another orthonormal basis via

$$\phi_i = \sum_{j=1}^{p} U_{ij} \xi_{l,j}$$

Mapping by $\mathcal{P}$ and then forming the inner product in $\mathbb{C}^n$

$$\langle \phi_i | \phi_j \rangle = \sum_{r,s=1}^{p} \bar{U}_{ir} \langle \xi_r | \xi_s \rangle U_{js}$$

so that finding an orthogonal basis for R_l amounts to finding the unitary matrix U which diagonalises the hermitian matrix $A_{rs} = \langle \xi_r | \xi_s \rangle$. □

This observation is useful in that it allows us to write the matrix G in 'diagonal' form

$$\begin{aligned} G &= \frac{1}{\lambda_l - \lambda} \left[|\phi_1\rangle\langle\phi_1| + \cdots + |\phi_m\rangle\langle\phi_m| \right] + \sum_{i \neq l} \frac{|\xi_i\rangle\langle\xi_i|}{\lambda_i - \lambda} \\ &= \frac{D_l}{\lambda_l - \lambda} + \mathcal{K}_l \end{aligned} \tag{4}$$

Using this notation, we can easily prove the following:

Theorem 2.1 *If λ_l is an eigenvalue of the compact graph Γ_c then, for vanishing coupling between Γ_c and the rays ($|\beta| \sim 0$), we have*

$$\begin{aligned} S(\lambda_l) &= -\mathbb{I} + 2P_l - 2 \sum_{s=1} (ik_l |\beta|^2 P_l^{\perp} \mathcal{K}_l P_l^{\perp})^s \\ &= -\mathbb{I} + 2P_l + O(|\beta|^2) \end{aligned} \tag{5}$$

where P_l is the orthogonal projection onto R_l.

Proof: Using the notation of equation (4), the scattering matrix has the form

$$S(\lambda) = -\left[\mathbb{I} + \frac{ik|\beta|^2 D_l}{\lambda_l - \lambda} + ik|\beta|^2 \mathcal{K}_l \right] \left[\mathbb{I} - \frac{ik|\beta|^2 D_l}{\lambda_l - \lambda} - ik|\beta|^2 \mathcal{K}_l \right]^{-1}.$$

Since $D_l = D_l^{\star}$, the matrix $E_l = \mathbb{I} - \frac{ik|\beta|^2 D_l}{\lambda_l - \lambda}$ has an inverse. Using the basis of $\mathbb{C}^n$ where the first k vectors are the normalised vectors $|\phi_i\rangle/\pi_i$, where we have used $\pi_i \equiv \sqrt{\langle\phi_i|\phi_i\rangle}$, D_l is diagonalised with the first k diagonal entries equal to π_i^2 and the remaining $n-m$ entries zero. Consequently in this basis

$$\begin{aligned} E_l^{-1} &= \begin{pmatrix} 1-\frac{ik|\beta|^2\pi_1^2}{\lambda_l-\lambda} & & & & & \\ & \ddots & & & 0 & \\ & & 1-\frac{ik|\beta|^2\pi_k^2}{\lambda_l-\lambda} & & & \\ & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\lambda_l-\lambda}{\lambda_l-\lambda-ik|\beta|^2\pi_1^2} & & & & & \\ & \ddots & & & 0 & \\ & & \frac{\lambda_l-\lambda}{\lambda_l-\lambda-ik|\beta|^2\pi_k^2} & & & \\ & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}. \end{aligned}$$

From this representation we can easily see that the following limit applies

$$\lim_{\lambda\to\lambda_l} E_l^{-1} = P_l^{\perp} = \mathbb{I} - P_l. \tag{6}$$

Using the same basis we see that

$$E_l^{\star}E_l^{-1} = \begin{pmatrix} \frac{\lambda_l-\lambda+ik|\beta|^2\pi_1^2}{\lambda_l-\lambda-ik|\beta|^2\pi_1^2} & & & & & \\ & \ddots & & & 0 & \\ & & \frac{\lambda_l-\lambda+ik|\beta|^2\pi_k^2}{\lambda_l-\lambda-ik|\beta|^2\pi_k^2} & & & \\ & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

This gives us the limit

$$\lim_{\lambda\to\lambda_l} E_l^{\star}E_l^{-1} = P_l^{\perp} - P_l = \mathbb{I} - 2P_l. \tag{7}$$

Now the denominator of the scattering matrix can be written

$$\begin{aligned} \left[\mathbb{I} - \frac{ik|\beta|^2 D_l}{\lambda_l-\lambda} - ik|\beta|^2\mathcal{K}_l\right]^{-1} &= \left[[\mathbb{I} - ik|\beta|^2\mathcal{K}_l E_l^{-1}]E_l\right]^{-1} \\ &= E_l^{-1}\left[\mathbb{I} - ik|\beta|^2\mathcal{K}_l E_l^{-1}\right]^{-1}. \end{aligned}$$

Again, the matrix $\mathbb{I} - ik|\beta|^2\mathcal{K}_l E_l^{-1}$ has an inverse for $\lambda \sim \lambda_l$ since $\mathcal{K}_l = \mathcal{K}_l^{\star}$. This gives the following expression for the scattering matrix

$$S(\lambda) = -\left[E_l^{\star}E_l^{-1} + ik|\beta|^2\mathcal{K}_l E_l^{-1}\right]\left[\mathbb{I} - ik|\beta|^2\mathcal{K}_l E_l^{-1}\right]^{-1}$$

$$
\begin{aligned}
&= -\left[E_l^\star E_l^{-1} + ik|\beta|^2\mathcal{K}_l E_l^{-1}\right]\sum_{s=0}(ik|\beta|^2\mathcal{K}_l E_l^{-1})^s \\
&= \left[-\mathbb{I} - \sum_{j=1}^{m}\frac{2ik|\beta|^2|\phi_j\rangle\langle\phi_j|}{(\lambda_l-\lambda)-ik|\beta|^2\pi_j^2} - ik|\beta|^2\mathcal{K}_l E_l^{-1}\right]\sum_{s=0}(ik|\beta|^2\mathcal{K}_l E_l^{-1})^s,
\end{aligned}
$$

where in the last two lines we assume that $|\beta|$ is small enough so that the expansion of the inverse is valid. Then, using the limits of equations (6,7), we immediately get the desired relation

$$
\begin{aligned}
S(\lambda_l) &= -\left[\mathbb{I} - 2P_l + ik_l|\beta|^2\mathcal{K}_l P_l^\perp\right]\sum_{s=0}(ik_l|\beta|^2\mathcal{K}_l P_l^\perp)^s \\
&= -\mathbb{I} + 2P_l - 2\sum_{s=1}(ik_l|\beta|^2 P_l^\perp\mathcal{K}_l P_l^\perp)^s \\
&= -\mathbb{I} + 2P_l + O(|\beta|^2). \qquad \square
\end{aligned}
$$

This formula may look surprising for $|\beta| = 0$ as it implies that there may be non-zero transmission in the case of zero connection between the rays. Actually the transmission coefficients are not continuous with respect to λ uniformly in β [2]. The physically significant parameters of the system are obtained by averaging of functions of the transmission coefficients with respect to the Fermi distribution—we discuss this in connection with the example considered in the next section.

Corollary 2.1 *If λ_l is an eigenvalue of Γ_c such that $P_l = \mathbb{I}$ then the above formula is independent of $|\beta|$, ie.*

$$S(\lambda_l) = \mathbb{I}$$

Proof: If $P_l = \mathbb{I}$ then $P_l^\perp = 0$. Putting this into the formula for $S(\lambda_l)$ in the theorem gives the result. $\square$

3 Example of a simple quantum switch

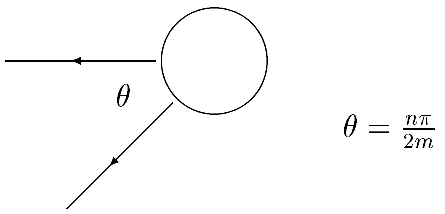

Figure 1: The family of switches

Let us consider a device with two leads/rays incident on an annulus of circumference 2π where the angle between the rays is $n\pi/2m$, n and m are

integers (see figure 1). The potential is a constant on the annulus. It is clear that the eigenfunctions on the annulus are just the trigonometric functions and the multiplicity of the spectrum is two.
Let us consider two states of the device corresponding to differing constant values of the potential, $q \equiv q_c$ and $q \equiv q_o$. For $q \equiv q_c$ the scattering matrix has zero reflection coefficient (a closed switch) and for $q \equiv q_o$ it has zero transmission coefficient (an open switch). We need to identify some $\lambda = \lambda_l$ which is an eigenvalue on the annulus for both states $q \equiv q_c$ and $q \equiv q_o$. Furthermore, from equation (5), we require that for $q \equiv q_c$, R_l is spanned by either one of the vectors $(1,1)$ or $(1,-1)$, while for $q \equiv q_o$, R_l is the whole of $\mathbb{C}^2$.
For λ_l to be an eigenvalue in both states it must satisfy

$$\begin{aligned} \sqrt{\lambda_l - q_c} &= n_c \\ \sqrt{\lambda_l - q_o} &= n_o \end{aligned} \tag{8}$$

where n_c and n_o are integers. The eigenspaces on the unperturbed annulus will be spanned by the functions

$$\begin{array}{ll} \text{For } q \equiv q_c & \{\cos(n_c x), \sin(n_c x)\} \\ \text{For } q \equiv q_o & \{\cos(n_o x), \sin(n_o x)\} \end{array}$$

Projecting these eigenspaces onto $\mathbb{C}^2$ to get R_l gives

$$\begin{array}{ll} \text{For } q \equiv q_c & R_l = \bigvee \left\{ \begin{pmatrix} 1 \\ \cos(n_c n\pi/2m) \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(n_c n\pi/2m) \end{pmatrix} \right\} \\ \text{For } q \equiv q_o & R_l = \bigvee \left\{ \begin{pmatrix} 1 \\ \cos(n_o n\pi/2m) \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(n_o n\pi/2m) \end{pmatrix} \right\}. \end{array}$$

Then it is easy to see that to get the required subspaces we need

$$\begin{array}{ll} \dfrac{n_c n\pi}{2m} = \dfrac{l\pi}{2} & l \text{ is even} \\ \dfrac{n_o n\pi}{2m} = \dfrac{l\pi}{2} & l \text{ is odd.} \end{array}$$

We must choose n, m, λ_l and q_c, q_o so that n_c and n_o from equation (8) satisfies

$$\frac{n_c n}{m} \text{ is even,} \qquad \frac{n_o n}{m} \text{ is odd.} \tag{9}$$

In fact any n_o such that $n_o n/m$ is not even will do although we consider the stronger condition here.
Clearly, we have described a family of quantum switches with the property that at some prescibed energy λ_l the switch has full transmission in the closed state ($|T| = 1$) and full reflection in the open state ($|R| = 1$). However, for practical purposes this only gives the behaviour of the switch as the temperature approaches zero. For finite temperature we need to consider the

properties of the switch averaging over the Fermi distribution.

Following Landauer [5, 6, 4] the effective conductance of an elementary scatterer may be written

$$\sigma_s = \frac{e^2}{h}\frac{|T|^2}{|R|^2} = \frac{e^2}{h}\frac{|T|^2}{1-|T|^2}.$$

Combining this with the basic conductance of the quantum wire $\sigma_0 = e^2/h$ [7] in accordance with the Matiessen rule

$$\sigma^{-1} = \sigma_0^{-1} + \sigma_s^{-1}$$

we can write the quantum conductance as

$$\sigma = \frac{e^2}{h}|T|^2.$$

The conductance may be estimated for finite temperature by taking the integral with respect to the Fermi distribution over the continuous spectrum

$$\begin{aligned} \hat{\sigma} &= \frac{1}{\mu(\sigma_c)}\int_{\sigma_c(\mathcal{L})}\sigma(\lambda)d\mu(\lambda) \\ &= \frac{e^2}{h}\frac{1}{\mu(\sigma_c)}\int_{\sigma_c(\mathcal{L})}|T|^2(\lambda)d\mu(\lambda). \end{aligned} \tag{10}$$

Here the distribution $d\mu$—the Fermi distribution—describes the occupancy of energy levels

$$d\mu = \frac{1}{2\tau}\mathrm{sech}^2((\lambda-\lambda_F)/2\tau)d\lambda.$$

Also $\sigma_c(\mathcal{L})$ is the continuous spectrum of the operator, λ_F is the Fermi energy and τ represents the temperature. The distribution $d\mu$ is peaked around the Fermi energy, the width of the peak depending on the temperature τ, and the integral is normalised by the measure of the spectrum, $\mu(\sigma_c)$.

The properties of our switch are calculated in the resonance case [2, 8, 7] when the Fermi energy in the wires λ_F coincides with the eigenvalue of the unperturbed operator on the annulus λ_l. A consequence of this is that, as we consider an annulus of smaller area, so the de Broglie wavelength represented by λ_l becomes smaller and consequently we have to design the wires so that the Fermi energy increases. To see precisely how the Fermi energy scales with respect to the radius of the annulus let us consider the Schrödinger equation

$$-\frac{\hbar^2}{2m_e}\frac{d^2\psi}{dx^2} + V\psi = E_F\psi$$

where E_F is the Fermi energy in units of energy. Let us suppose that the annulus has radius r and we rescale the above equation so that it becomes dimensionless and the radius is normalised to unity. Normalising the radius is achieved by making the change of variables

$$\xi = \frac{x}{r}$$

and then multiplying through by the factor $2m_e r^2/\hbar^2$ gives us the dimensionless equation

$$-\frac{d^2\psi}{d\xi^2} + \frac{2m_e r^2}{\hbar^2} V\psi = \frac{2m_e r^2}{\hbar^2} E_F \psi.$$

Then we see that λ_l, which is just the coefficient on the right hand side, is

$$\lambda_l = \frac{2m_e r^2}{\hbar^2} E_F$$

and so the Fermi energy scales as

$$E_F = \frac{\lambda_l}{2m_e} \left(\frac{\hbar}{r}\right)^2, \tag{11}$$

ie. the wires have to be chosen so that the Fermi energy is inversely proportional to the area of the annulus.

In calculating properties of the device, in particular when we integrate over the Fermi distribution, we see that the important parameter is the ratio between the deviation of the energy from the Fermi energy and the temperature in units energy. That is the Fermi distribution depends on the energy and temperature as

$$\frac{\lambda - \lambda_F}{2\tau} = \frac{E - E_F}{2k_B T_K}$$

where E_F is the Fermi energy in units energy, T_K is the temperature in Kelvin and k_B is the Boltzmann constant. Multiplying top and bottom by the factor $\frac{1}{2m_e}\left(\frac{\hbar}{r}\right)^2$ we get the following relationship between T_K and τ

$$k_B T_K = \frac{\tau}{2m_e} \left(\frac{\hbar}{r}\right)^2. \tag{12}$$

We interpret equations (11,12) in the following way. The Fermi energy in the wires is assumed to match the eigenvalue of the unperturbed operator on the annulus—the resonance case. Consequently, as a smaller annulus is considered the wires must be chosen so that the Fermi energy is proportionaly higher.
Now suppose we fix the operating temperature T_K and consider annuli of differing radii. As the area enclosed by the annulus gets smaller, so E_F increases and, as we have fixed T_K, τ decreases. Clearly in the limit $\tau \to 0$

the distribution becomes the Dirac distribution at the Fermi energy, so the integral simply becomes

$$|T|^2(\lambda_l).$$

But we have constructed our switch so that at $\lambda = \lambda_l$ this quantity is zero in the open state and unity in the closed state. In other words, for fixed operating temperature T_K the efficiency (the ratio between the conductance in the closed state and the conductance in the open state) of the switch increases without bound as the area enclosed by the annulus goes to zero.
This switch has quite different properties from a switch based on simply raising a potential barrier to create the open state. Here the efficiency of the switch has an upper bound due to quantum tunneling effects.

Let us now consider the simplest member of the family of switches described above. We choose $m = n = \lambda_l = 1$, $q_c = -3$ and $q_o = 0$. It is easy to see that this choice satisfies the condition (9) and, for $q = q_o = 0$, the scattering matrix has identically zero transmission coefficient for *all* values of β—this last result holds generaly as can be seen from corollary 2.1.
To describe the behavior of this device we consider the averaged quantum conductance given in equation (10). We reproduce some of the steps in the calculation of $|T|^2$ here. Let us define the scattering solution by

$$\begin{aligned}
\psi_1 &= e^{-ikx} + Re^{ikx} \\
\psi_2 &= a_2 \cos(k_q x) + b_2 \sin(k_q x) \\
\psi_3 &= a_3 \cos(k_q x) + b_3 \sin(k_q x) \\
\psi_4 &= Te^{ikx}
\end{aligned}$$

where $k_q = \sqrt{k^2 - q}$. The scattering solutions on the rays are ψ_1, ψ_4; ψ_3 is the scattering solution on the interior edge of length $\pi/2$ and ψ_2 is the scattering solution on the interior edge of length $3\pi/2$. We assume that $x = 0$ in ψ_2, ψ_3 corresponds to the node of the ray on which ψ_1 is defined. The boundary conditions at the nodes of the graph (equation(2)) give us

$$\begin{aligned}
1 + R &= \bar{\beta} a_2 = \bar{\beta} a_3 \\
T &= \bar{\beta}[a_2 \cos(3k_q\pi/2) + b_2 \sin(3k_q\pi/2)] \\
&= \bar{\beta}[a_3 \cos(k_q\pi/2) + b_3 \sin(k_q\pi/2)]
\end{aligned}$$

so we may put $a = a_2 = a_3$. The first order derivative terms give us

$$\begin{aligned}
ik\beta[1 - R] &= b_2 k_q + b_3 k_q \\
ik\beta T &= [b_2 k_q \cos(3k_q\pi/2) - a k_q \sin(3k_q\pi/2)] \\
&\quad + [b_3 k_q \cos(k_q\pi/2) - a k_q \sin(k_q\pi/2)].
\end{aligned}$$

We use the first of these boundary conditions to solve for R and then using the unitarity of the scattering matrix we have

$$|T|^2 = 1 - |\bar{\beta} a - 1|^2$$

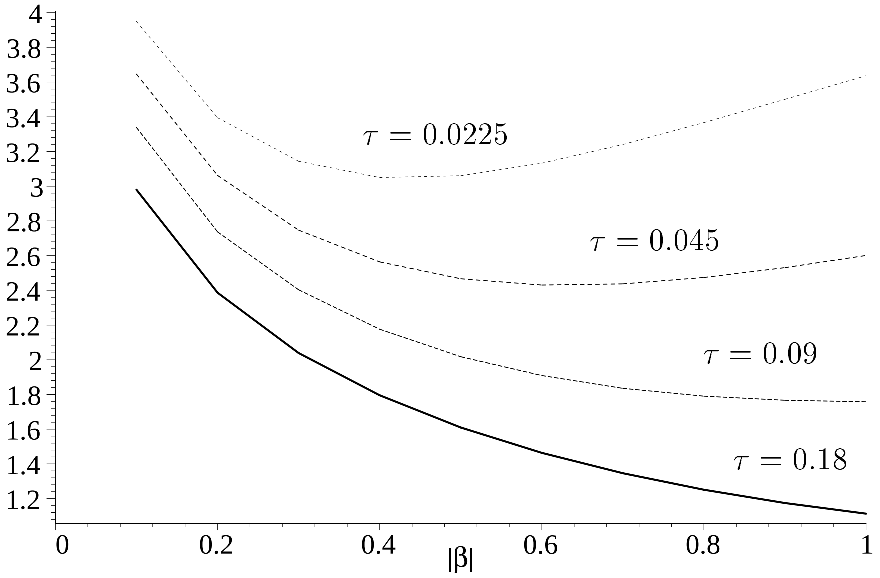

Figure 2: $\log_{10}(\hat{\sigma}_c/\hat{\sigma}_o)$ versus $|\beta|$

It is easy to eliminate T and R in the remaining boundary conditions so that we can solve for a

$$a = \frac{2ik\beta\,(ik|\beta|^2\sin(3k_q\pi/2)\sin(k_q\pi/2) - k_q\sin(2k_q\pi))}{-2ik|\beta|^2k_q\sin(2k_q\pi) - 2k_q^2\,(1-\cos(2k_q\pi)) + (ik|\beta|^2)^2\sin(3k_q\pi/2)\sin(k_q\pi/2)}$$

to finally obtain

$$\begin{aligned}
|T|^2 &= \frac{p_0 + p_1\cos(k_q\pi) + p_2\cos(2k_q\pi) + p_3\cos(3k_q\pi)}{q_0 + q_1\cos(k_q\pi) + q_2\cos(2k_q\pi) + q_3\cos(3k_q\pi) + q_4\cos(4k_q\pi)} \\
p_0 &= 32|\beta|^4k^2k_q^2 \\
p_1 &= 16|\beta|^4k^2k_q^2 \\
p_2 &= -32|\beta|^4k^2k_q^2 \\
p_3 &= -16|\beta|^4k^2k_q^2 \\
q_0 &= 2|\beta|^8k^4 + 24|\beta|^4k^2k_q^2 + 48k_q^4 \\
q_1 &= -2|\beta|^8k^4 + 8|\beta|^4k^2k_q^2 \\
q_2 &= |\beta|^8k^4 - 16|\beta|^4k^2k_q^2 - 64k_q^4 \\
q_3 &= -2|\beta|^8k^4 - 8|\beta|^4k^2k_q^2 \\
q_4 &= |\beta|^8k^4 - 8|\beta|^4k^2k_q^2 + 16k_q^4.
\end{aligned}$$

This is used in equation (10) for the averaged quantum conductance to produce graphs of this quantity with respect to $|\beta|$ and at different values of the temperature, τ with Fermi energy $\lambda_F = \lambda_l = 1$—see figures 2-4. These plots were produced using Maple—the source has been included in the Appendix.

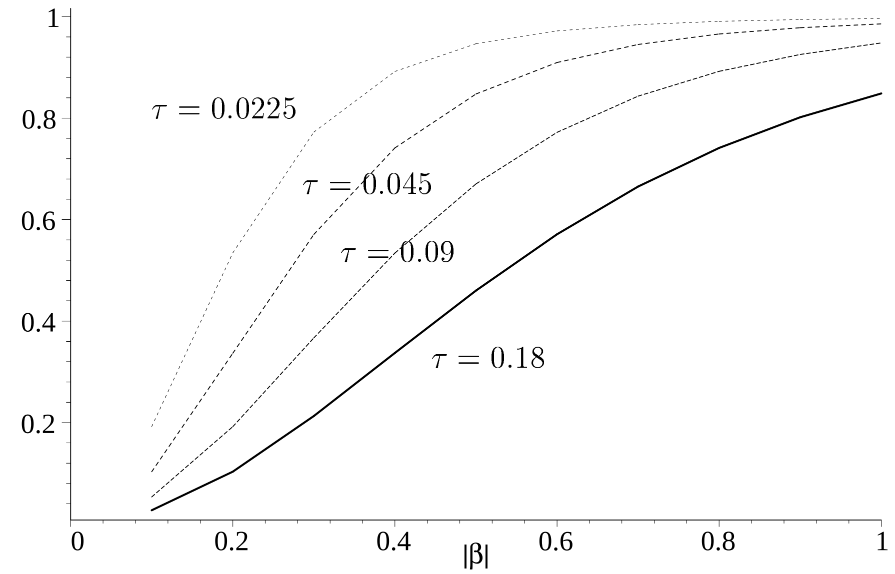

Figure 3: $h\hat{\sigma}_c/e^2$ versus $|\beta|$

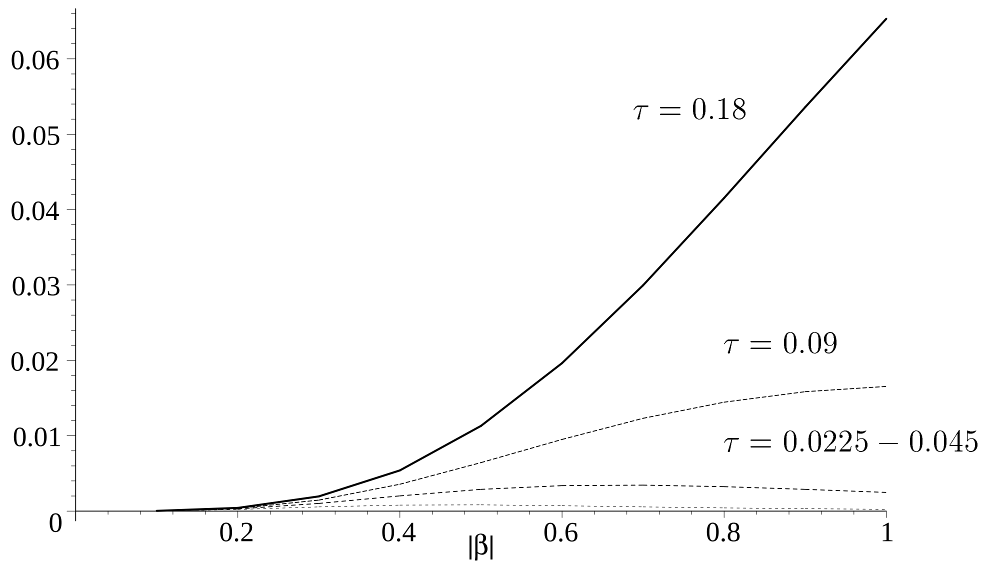

Figure 4: $h\hat{\sigma}_o/e^2$ versus $|\beta|$

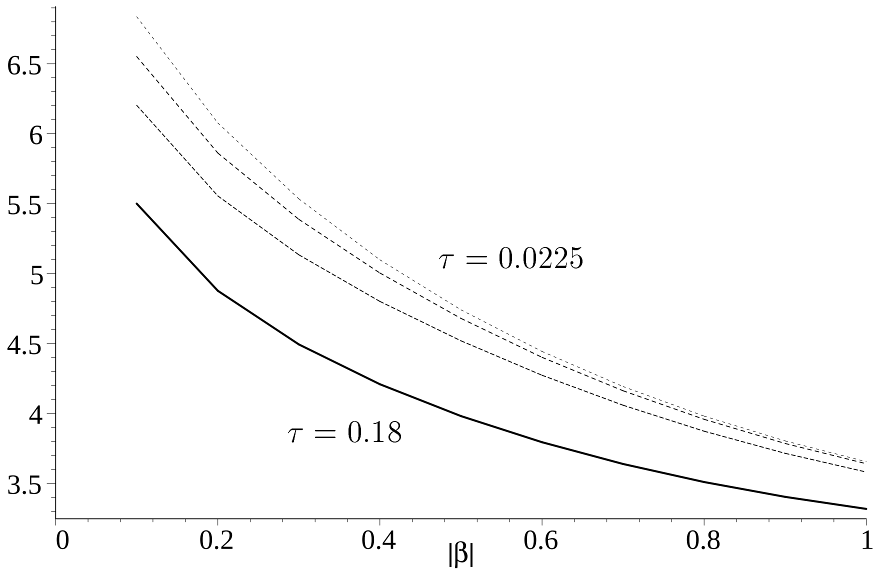

Figure 5: $\log_{10}(\hat{\sigma}_c/\hat{\sigma}_o)$ versus $|\beta|$

In these figures we show the averaged current in the closed state $\hat{\sigma}_c$, the averaged current in the open state $\hat{\sigma}_o$ (in units of e^2/h), and the ratio $\hat{\sigma}_c/\hat{\sigma}_o$.

In order to interpret these results, we should choose a realistic range of values for the radius r (so that the resonance case can be achieved using suitable materials) and then, using equations (1112), deduce the operating temperature T_K associated with the values of τ chosen in the simulation. For the relevant details of properties of semiconducting materials see [1, 7, 9].

For the purposes of comparison we consider a switch similar in structure to the above example, but where the open state is achieved by *raising* a potential barrier to the flow of electrons.
Constructing the switch using an annulus with two incident rays we choose, $m = 1$ and $n = 2$, (the angle between the rays is π radians) $\lambda_l = k_l = 1$, $q_c = 0$ and $q_o = 3$—these parameters are chosen so that they are similar to the parameters in the first example. The modulus of the transmission coefficient is given by

$$|T|^2 = \frac{16(k^2-q)k^2|\beta|^4}{(4(k^2-q)+k^2|\beta|^4)^2 - (4(k^2-q)-k^2|\beta|^4)^2\cos^2(k_q\pi)}.$$

We use this in equation (10) to produce graphs of the conductance with respect to $|\beta|$ and at different values of the temperature, τ with Fermi energy $\lambda_F = \lambda_l = 1$—see figures 5-7. The additional Maple source is also included in the Appendix.

We note that the switch based on raising a potential barrier (figures 5-7) preforms much better than the switch based on interference effects (figures

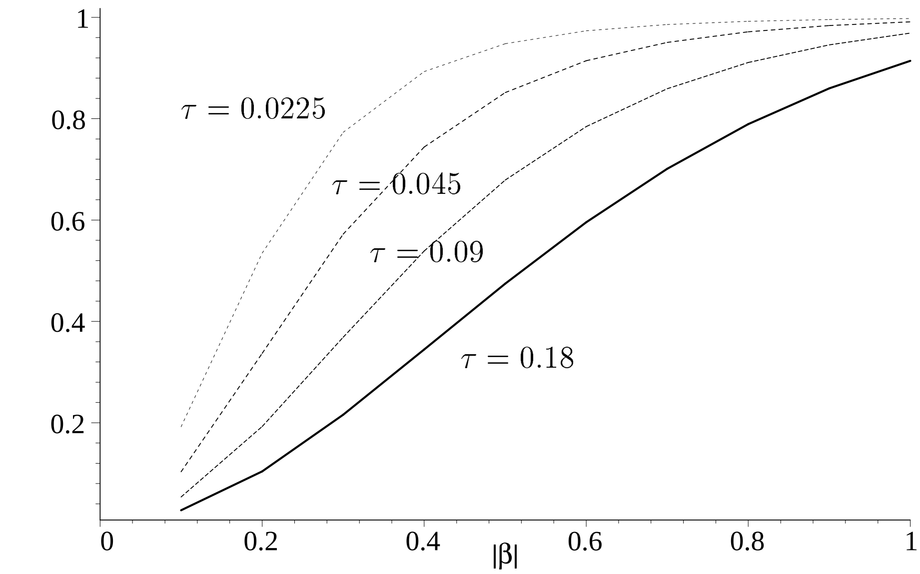

Figure 6: $h\hat{\sigma}_c/e^2$ versus $|\beta|$

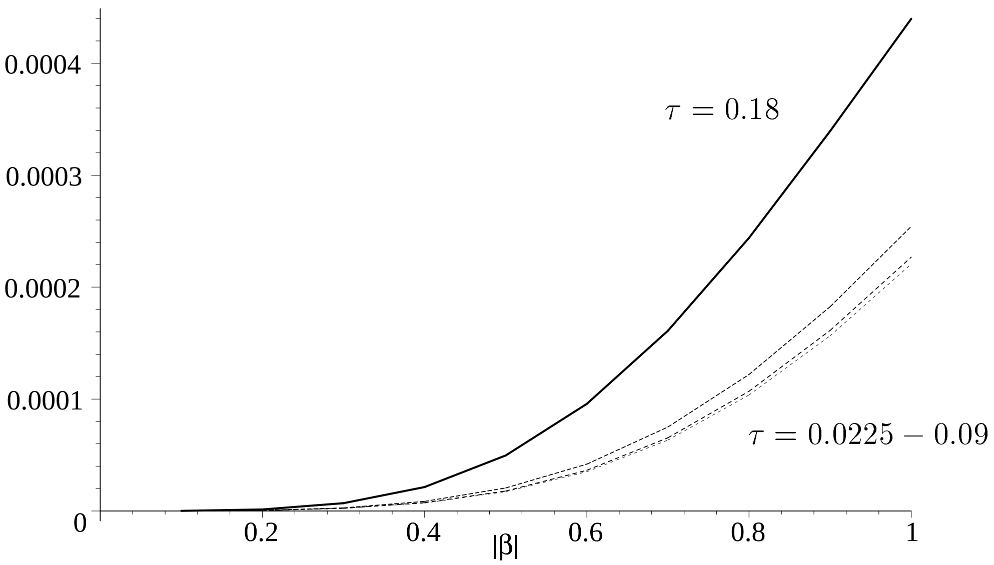

Figure 7: $h\hat{\sigma}_o/e^2$ versus $|\beta|$

2-4)—at least in the temperature ranges that we consider—especialy in the limit of weak coupling between ring and rays, small $|\beta|$. Both switches have good properties in the closed state (figures 3,6), however the open state—possibly due to tunneling effects—appears to be more difficult to achieve. This also appears to explain why, in the limit of small $|\beta|$, the properties of the switches improve: weak coupling between the ring and rays improves the open state of the switches. On the other hand, in the limit $|\beta| \to 1$ the ratio $\hat{\sigma}_c/\hat{\sigma}_o$ for the second example rapidly decreases to a bound due to tunneling which may be calculated from the transmission coefficient

$$lim_{\tau\to 0} \left.\frac{\hat{\sigma}_c}{\hat{\sigma}_o}\right|_{|\beta|=1} \approx 4.57 \times 10^3,$$

see figure 5. The first switch, as we have shown, does not have this bound and consequently for sufficiently low temperature or small radius we conjecture that it will have better properties.

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Appendix: Maple Code

The first segment of code produces the numerical simulation for the first model of a switch.

```
lc1:=b4*x;
lc2:=x+3;
kc2:=lc2^(1/2);
pc0:= 32*lc1*lc2;
pc1:= 16*lc1*lc2;
pc2:=-32*lc1*lc2;
pc3:=-16*lc1*lc2;

qc0:= 2*lc1^2 + 24*lc1*lc2 + 48*lc2^2;
qc1:=-2*lc1^2 + 8*lc1*lc2;
qc2:= lc1^2 - 16*lc1*lc2 - 64*lc2^2;
qc3:=-2*lc1^2 - 8*lc1*lc2;
qc4:= lc1^2 - 8*lc1*lc2 + 16*lc2^2;
mc:=pc0 + pc1*cos(Pi*kc2) + pc2*cos(2*Pi*kc2) + pc3*cos(3*Pi*kc2);
```

```
nc:=qc0 + qc1*cos(Pi*kc2) + qc2*cos(2*Pi*kc2) + qc3*cos(3*Pi*kc2) +
qc4*cos(4*Pi*kc2);
Tc:=simplify(expand(mc/nc));

lo1:=b4*x;
lo2:=x;
ko2:=lo2^(1/2);
po0:= 32*lo1*lo2;
po1:= 16*lo1*lo2;
po2:=-32*lo1*lo2;
po3:=-16*lo1*lo2;

qo0:= 2*lo1^2 + 24*lo1*lo2 + 48*lo2^2;
qo1:=-2*lo1^2 + 8*lo1*lo2;
qo2:= lo1^2 - 16*lo1*lo2 - 64*lo2^2;
qo3:=-2*lo1^2 - 8*lo1*lo2;
qo4:= lo1^2 - 8*lo1*lo2 + 16*lo2^2;
mo:=po0 + po1*cos(Pi*ko2) + po2*cos(2*Pi*ko2) + po3*cos(3*Pi*ko2);
no:=qo0 + qo1*cos(Pi*ko2) + qo2*cos(2*Pi*ko2) + qo3*cos(3*Pi*ko2) +
qo4*cos(4*Pi*ko2);
To:=simplify(expand(mo/no));

Ef:=1;
d:=(1/(2*T))*( sech((x-Ef)/(2*T)) )^(2);
N:=1 + tanh(Ef/(2*T));

b_arr:=[0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1];
T_arr:=[0.0225,0.045,0.09,0.18];
m:=10;
n:=4;
a:=max(0,Ef-10*T);
b:=Ef+10*T;

data_arr:=array(1..4,1..m,1..n);

for i to m do:
b4:=b_arr[i]^4:
for j to n do:
T:=T_arr[j]:
aCc:=evalf( Int(Tc*d, x=a..b, 5, _NCrule) ):
aCo:=evalf( Int(To*d, x=a..b, 5, _NCrule) ):
data_arr[1,i,j]:=b_arr[i]:
data_arr[2,i,j]:=log[10](aCc/aCo):
data_arr[3,i,j]:=aCc/N:
```

```
data_arr[4,i,j]:=aCo/N:
od:
od;

save data_arr, ‘ring4.m‘;
```

The second segment of code produces the numerical simulation for the switch realised by a simple potential barrier on an annulus.

```
lc1:=b4*x;
lc2:=x;
kc2:=lc2^(1/2);
mc:= 16*lc1*lc2;
nc:= (4*lc2 + lc1)^2 - ((4*lc2 - lc1)*cos(Pi*kc2))^2;
Tc:=simplify(expand(mc/nc));

lo1:=b4*x;
lo2:=x-3;
ko2:=lo2^(1/2);
mo:= 16*lo1*lo2;
no:= (4*lo2 + lo1)^2 - ((4*lo2 - lo1)*cos(Pi*ko2))^2;
To:=simplify(expand(mo/no));

Ef:=1;
d:=(1/(2*T))*( sech((x-Ef)/(2*T)) )^(2);
N:=1 + tanh(Ef/(2*T));

b_arr:=[0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1];
T_arr:=[0.0225,0.045,0.09,0.18];
m:=10;
n:=4;
a:=max(0,Ef-10*T);
b:=Ef+10*T;

data_arr:=array(1..4,1..m,1..n);

for i to m do:
b4:=b_arr[i]^4:
for j to n do:
T:=T_arr[j]:
aCc:=evalf( Int(Tc*d, x=a..b, 5, _NCrule) ):
aCo:=evalf( Int(To*d, x=a..b, 5, _NCrule) ):
data_arr[1,i,j]:=b_arr[i]:
data_arr[2,i,j]:=log[10](aCc/aCo):
data_arr[3,i,j]:=aCc/N:
```

```
data_arr[4,i,j]:=aCo/N:
od:
od;

save data_arr, ‘ringe4.m‘;
```

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