Infinite Games Associated with Asymmetric Topology and Applications

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Abstract

In this paper, we study two types of topological games, $\mathcal{G}(x)$-games and $\mathcal{G}(\mathcal{F})$-games, and topological spaces defined by them, namely $\mathcal{G}$-spaces and game-compact spaces. It is shown these games are associated with κ-semi-stratifiabilty, which is the duality of quasi-metrizability. Finally, we apply these games and relevant properties to study multi-valued maps. Consequently, the Choquet-Dolecki theorem on multi-valued maps is deduced. Main results of Hansell et al in [**?**] are generalized.

1 Introduction

Traditionally, symmetry is a very important principle in much of Mathematics. However, in the second half of the 20th century, it has been shown that certain asymmetric structures play important roles in some parts of Mathematics and in other disciplines, such as topography and theoretical computer science. As a matter of fact, asymmetric distance functions had already been

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considered by Hausdorff when in his classical book "Set-Theory". It was Wilson [?] who introduced the term quasi-metric and noted that convergences in quasi-metric spaces arise in three natural ways.

Basically, if one omits in the definition of a (pseudo-)metric the symmetry condition, one reaches the notion of a *quasi-(pseudo-) metric*. Many classical counterexamples in topology like the Sorgenfrey line, the Niemytzki plane, the Michael line or the Pixley-Roy space over the reals are quasi-metrizable and not metrizable. Similarly, if one omits in the usual definition of an entourage uniformity the symmetry condition, one obtains the concept of a (filter) *quasi-uniformity*. There are two monographs [?] and [?] published in the area of asymmetric topology. In these monographs, basic properties of quasi-metric spaces and quasi-uniform spaces, and associated topological properties were discussed. The research in this area has been developed rapidly, and a theory of quasi-uniform spaces which is similiar to the theory of uniformities has been established. Recently, this theory has been applied by Cao, Künzi, Reilly and Romaguera to study hyperpaces and function spaces in [?], [?], [?], [?] and [?] etc.

Another way to obtain asymmetric structures is to consider the "duals" of existing asymmetyric ones. For example, one may consider when a quasi-metric space is metrizable. A property $\mathbf{P}$ is called a *dual of quasi-metrizability* if $\mathbf{P}$ is necessary and sufficient to make a quasi-metric space metrizable. In this paper, we mainly consider two such dual properties, namely quasi-Nagataness and κ-semi-stratifiability. There are two types of topological games associated with κ-semi-stratifiability, namely the $\mathcal{G}(x)$-game and the $\mathcal{G}(\mathcal{F})$-game. Two classes of spaces, $\mathcal{G}$-spaces and game-compact spaces are defined by using these games. We shall show that every κ-semi-stratifiable $\mathcal{G}$-space is stratifiable, which extends a well-known result of Lutzer [?]. As an application, we obtain factorizations of Nagata spaces. Furthermore, we shall show that the class of game-compact spaces includes all κ-semi-stratifiable spaces, all function spaces $C_p(X)$ for $\mathcal{G}$-spaces X, and all Banach spaces with their weak topology. Among other things, we also prove that $C_p(\omega_1+1)$ is game-compact, but not Dieudonné-complete; and that $C_p(\beta\omega_2)$ is isocompact which does not have a G_δ-diagonal.

Finally, we apply these games to study kernels and active boundaries of multi-valued maps. We shall show that for a usc multi-valued map Φ from a topological space X into a topological space Y, if x is a $\mathcal{G}$-point then the active boundary of Φ at x is compact. This result generalizes Theorem 2 of [?]. Furthermore, $Frac\Phi(x)$ is a compact kernel for Φ at x if Φ is closed in the G_δ-topology on Y. Consequently the Choquet-Dolecki theorem from [?] and [?] is extended in terms of topological games.

2 Notations

In general, all topological spaces are assumed to be regular T_1 and for undefined notations, see [?] and [?]. Only some essential preliminaries are introduced here. Let X be a topological space. A subset of X is said to be a *G_δ-set* if it is the intersection of a countable family of open sets in X. The *G_δ-topology* on X is the topology generated by taking all the G_δ-sets in X as a base. The *diagonal* $\Delta(X)$ of X is defined by $\Delta(X) = \{(x,x) : x \in X\}$. If $\Delta(X)$ is a G_δ-set of $X \times X$, then we say that X has a *G_δ-diagonal*, equivalently, X has a G_δ-diagonal if and only if there is a sequence $(\mathcal{G}_n : n \in \omega)$ of open covers of X such that $\bigcap_{n\in\omega} \mathrm{st}(x, \mathcal{G}_n) = \{x\}$ for each $x \in X$. Such a sequence $(\mathcal{G}_n : n \in \omega)$ is called a *G_δ-diagonal sequence* for X. Moreover, if there is a sequence $(\mathcal{G}_n : n \in \omega)$ of open covers of X such that $\bigcap_{n\in\omega} \overline{\mathrm{st}(x, \mathcal{G}_n)} = \{x\}$ for all $x \in X$, then X is said to have a *G^*_δ-diagonal*. In this case, we call $(\mathcal{G}_n : n \in \omega)$ a *G^*_δ-diagonal sequence* for X. Furthermore, X is *Dieudonné-complete* if X is Tychonoff and there exists a complete uniformity on X. It is well-known that a Tychonoff space X is Dieudonné-complete if and only if it is homeomorphic to a closed subspace of a Cartesian product of metrizable spaces.

Recall that a space X is *semi-stratifiable* if there is a function G which assigns to each $n \in \omega$ and to each closed set $H \subseteq X$, an open set $G(n,H)$ such that (1) $H = \bigcap_{n\in\omega} G(n,H)$; and (2) $H \subseteq K$ implies $G(n,H) \subseteq G(n,K)$ for each $n \in \omega$. If, in addition, for each compact set C and each closed set H with $C \bigcap H = \emptyset$ there is an $n \in \omega$ such that $C \bigcap G(n,H) = \emptyset$ then X is called *κ-semi-stratifiable.* Moreover, if we have $H = \bigcap_{n\in\omega} \overline{G(n,H)}$, then X is *stratifiable.* In the literature, a first countable and stratifiable space is called a *Nagata space.* It was shown in [?] that every stratifiable space is κ-semi-stratifiable, and that every κ-semi-stratifiable space is semi-stratifiable. However, neither converse implication holds. Recall that a point x of X is a *q-point* if there is a sequence $(U_n : n \in \omega)$ of neighbourhoods of x such that if $x_n \in U_n$ for all $n \in \omega$ then $(x_n : n \in \omega)$ has a cluster point in X (not necessarily equal to x). If each point of X is a q-point then X is called a *q-space*.

A multi-valued map $\Phi : X \to Y$ is *upper semi-continuous*, abbreviated as usc, at $x \in X$ if for each open set $V \subseteq Y$ with $\Phi(x) \subseteq V$ there is an open set U containing x such that $\Phi(U) \subseteq V$. We say Φ is usc if it is usc at each point $x \in X$. For a Tychonoff space X, let $C(X)$ be the family of all continuous real-valued functions defined on X. We shall denote by $C_p(X)$ the space of $C(X)$, endowed with the topology of pointwise convergence on X. Obviously, a basic neighbourhood of a function $f \in C_p(X)$ is

$$W(x_0, x_1, ..., x_n; \epsilon)(f) = \bigcap_{i=0}^{n} \{g \in C(X) : \ | f(x_i) - g(x_i)| < \epsilon\},$$

where $\epsilon > 0$ and $x_0, x_1, ..., x_n \in X$. For a point $x \in X$, $\mathcal{U}(x)$ denotes the set of all neighbourhoods of x. Let $\mathbf{N}$ be the set of all positive integers, and $\omega = \{0\} \bigcup \mathbf{N}$. Finally, ω_1 (ω_2) is the first (second) uncountable ordinal.

3 Duals of quasi-metrizability

Since asymmetric topologies usually come from certain topological spaces by loosening the requirement of 'symmetry', they appear quite frequently in pairs. In studying this, the concept of g-functions is a powerful tool. By definition, a *g-function* on a space X is a map $g : \omega \times X \to \tau(X)$ such that $\bigcap_{n\in\omega} g(n, x) = \{x\}$ for $x \in X$; and $g(n + 1, x) \subseteq g(n, x)$ for $(n, x) \in \omega \times X$. Recall that a space X is a *γ-space* [**?**] if there is a g-function g on X such that if $x_n \to p$ and $y_n \in g(n, x_n)$ for all $n \in \omega$, then p is a cluster point of $(y_n : n \in \omega)$. Every quasi-metrizable space is a γ-space, but the converse is not true. A space X is called *quasi-Nagata* [**?**] if there is a g-function g on X such that if $y_n \to p$ and $y_n \in g(n, x_n)$ for each $n \in \omega$, then $(x_n : n \in \omega)$ has a cluster point in X. It was first proved in [**?**] that a γ-space X is metrizable if and only if X is a quasi-Nagata space. Hence, quasi-Nagataness is a dual of quasi-metrizability.

Proposition 3.1. *(a) The class of quasi-Nagata spaces is both hereditary to closed subspaces and countably productive.*

(b) The perfect image of a quasi-Nagata spaces is quasi-Nagata.

The following characterzation of κ-semi-stratifiable spaces, which was given by Gao [**?**], shows that every κ-semi-stratifiable space is quasi-Nagata. The space ω_1 with the order opology is quasi-Nagata, but it cannot be κ-semi-stratifiable. Otherwise, ω_1 is a countably compact space with a G_δ-diagonal, and thus is metrizable. This is impossible, since ω_1 is not even quasi-metrizable.

Lemma 3.2. *A space X is κ-semi-stratifiable if and only if there is a g-function $g : \omega \times X \to \tau(X)$ on X such that if $x_n \to p$ and $x_n \in g(n, y_n)$ for each $n \in \omega$, then p is a cluster point of $(y_n : n \in \omega)$.*

Let X be a space, and $x \in X$. Consider the following $\mathcal{G}(x)$*-game* [**?**] [**?**] played in X between players (α) and (β). Player (α) goes first and chooses a point $x_0 \in X$. Player (β) then responds by choosing $U_0 \in \mathcal{U}(x)$. Following this, (α) must select another (possibly the same) point $x_1 \in U_0$ and in turn (β) must again respond to this by choosing (possibly the same) $U_1 \in \mathcal{U}(x)$. This proceedure is repeated infinitely many times. We shall say that (β) *wins* the $\mathcal{G}(x)$-game if the sequence $(x_n : n \in \omega)$ has a cluster point in X. Otherwise, (α) is said to have *won* the game. By a *strategy* $s = (s_n : n \in \omega)$ for (β), we mean a sequence of $\mathcal{U}(x)$-valued functions. We shall call a finite sequence $(x_0, x_1, ..., x_n)$ or an infinite sequence $(x_0, x_1, ...)$ an s*-sequence* if $x_{i+1} \in s_i(x_0, x_1, ..., x_i)$ for $0 \leq i \leq n-1$; or $x_{n+1} \in s_n(x_0, x_1, ..., x_n)$ for all $n \in \omega$. A strategy $s = (s_n : n \in \omega)$ for (β) is called *a winning strategy* if each infinite s-sequence has a cluster point in X. Finally, we call x a $\mathcal{G}$*-point* of X if (β) has a winning strategy for the $\mathcal{G}(x)$-game. In addition, if every point of X is a $\mathcal{G}$-point, then X is called a $\mathcal{G}$*-space*. Note that the notion of $\mathcal{G}$-space is a common generalization of the concepts of q-space and W-space of Gruenhage in [**?**].

Theorem 3.3. *Let X be a κ-semi-stratifiable $\mathcal{G}$-space. Then X is stratifiable.*

Proof. Let $g : \omega \times X \to \tau(X)$ be a g-function for X as in Lemma **??**. Suppose that $x \in X$ and H is a closed set in X such that $x \notin H$. We claim that $x \notin \overline{\bigcup_{h \in H} g(n, h)}$ for some $n \in \omega$. Assume the contrary, then $x \in \overline{\bigcup_{h \in H} g(n, h)}$ for each $n \in \omega$. By regularity of X, we first choose $G \in \mathcal{U}(x)$ such that $\overline{G} \bigcap H = \emptyset$. Since X is a $\mathcal{G}$-space, the player (β) has a winning strategy $s = (s_n : n \in \omega)$ for the $\mathcal{G}(x)$-game. Player (α) goes first and chooses a point (arbitrarily) $x_0 \in X$. To respond this, player (β) chooses a point $y_0 \in H$ and $s_0(x_0)$ such that

$$s_0(x_0) \bigcap G \bigcap g(0, y_0) \neq \emptyset.$$

Inductively, we may obtain two sequences $(x_n : n \in \omega)$ and $(y_n : n \in \omega)$ such that for each $n \in \omega$, $y_n \in H$, $x_n \in g(n, y_n)$ and

$$x_{n+1} \in s_n(x_0, x_1, ..., x_n) \bigcap G.$$

Since $(x_n : n \in \omega)$ is an s-sequence in X, it must have a cluster point $p \in \overline{G}$. On the other hand, from $x_n \in g(n, y_n)$ above, we conclude that p is also a cluster point for $(y_n : n \in \omega)$. Since H is closed, we have $p \in H$, which is a contradiction. Therefore, by [**?**, Theorem 5.8], X is stratifiable. □

Corollary 3.4. [**?**] If a κ-semi-stratifiable space is first countable, then it is stratifiable.

Recall that a family of subsets $\mathcal{O}$ of a topological space X is *interior-preserving* [**?**] if $\bigcap_{G \in \mathcal{G}} \operatorname{int}(G) = \operatorname{int}(\bigcap_{G \in \mathcal{G}} G)$ for $\mathcal{G} \subseteq \mathcal{O}$. Moreover, the space X is called *orthocompact* [**?**] if every open cover of X has an interior-preserving open refinement. We conclude this section by proving the following factorizations of Nagata spaces.

Theorem 3.5. *The following statements are equivalent for a space X:*
(a) X is a Nagata space;
(b) X is a quasi-Nagata, orthocompact q-space with a G^*_δ-diagonal;
(c) X is a κ-semi-stratifiable q-space.

Proof. The implications $(a) \to (b)$ and $(c) \to (a)$ are either obvious or straightforward. We shall show $(b) \to (c)$. To do this, it suffices to prove each quasi-Nagata orthocompact space X having a G^*_δ-diagonal is κ-semi-stratifiable. Let $(\mathcal{O}_n : n \in \omega)$ be a G^*_δ-diagonal sequence for X. By the orthocompactness of X, each $\mathcal{O}_n$ has an interior-preserving open refinement $\mathcal{G}_n$. For each $n \in \omega$, let

$$g(n, x) = \bigcap_{k \leq n} \{G \in \mathcal{G}_k : x \in G\}.$$

Since each $\mathcal{G}_n$ is interior-preserving, g is a well-defined g-function on X. Let g' be a quasi-Nagata function for X. Define

$$h(n, x) = g(n, x) \bigcap g'(n, x)$$

for all $(n, x) \in \omega \times X$. We wish to show that h satisfies the condition in Lemma **??**. Let $x_n \to p$ and $x_n \in h(n, y_n)$ for each $n \in \omega$. Since X is quasi-Nagata, $(y_n : n \in \omega)$ has a cluster point, say $q \in X$. It suffices to show that $p = q$. For a fixed $n \in \omega$, there are infinitely many integers $k \geq n$ such that $y_k \in g(n, q)$. It follows that

$$x_k \in g(k, y_k) \subseteq \bigcup\{g(n, x) : x \in g(n, q)\} \subseteq \mathrm{st}(q, \mathcal{O}_n).$$

Hence, $p \in \overline{\mathrm{st}(q, \mathcal{O}_n)}$ for each $n \in \omega$. On the other hand, from

$$\bigcap_{n \in \omega} \overline{\mathrm{st}(q, \mathcal{O}_n)} = \{q\},$$

we have $p = q$. Therefore, X is κ-semi-stratifiable. □

4 On game-compact spaces

The $\mathcal{G}(x)$-game in Section **??** is extended to the $\mathcal{G}(\mathcal{F})$-*game* in [**?**], where $\mathcal{F}$ is a filter in a space X as follows: Player (α) goes first and chooses a point $x_0 \in X$. Player (β) must then respond by choosing a member $F_0 \in \mathcal{F}$. Following this, (α) must select another (possibly the same) point $x_1 \in F_0$ and in turn (β) must again respond to this by choosing a member (possibly the same) $F_1 \in \mathcal{F}$. When the players repeat this procedure infinitely many times, (α) and (β) produce a play

$$P = \{(x_n, F_n) : x_{n+1} \in F_n,\ n \in \omega\}$$

of the $\mathcal{G}(\mathcal{F})$-game. We shall say that (β) *wins* this play if the sequence $(x_n : n \in \omega)$ has a cluster point in X. Otherwise, (α) is said to have *won* this play. A strategy $\sigma = (\sigma_n : n \in \omega)$ for (β) is a sequence of $\mathcal{F}$-valued functions. The domain of each function σ_n is the collection of finite sequences $(x_0, x_1, ..., x_n)$ in X with $x_{i+1} \in \sigma_i(x_0, ..., x_i)$ for all $0 \leq i \leq n-1$. We shall call such a finite sequence $(x_0, x_1, ..., x_n)$ or an infinte sequence $(x_0, x_1, ...)$ a σ-*sequence*. A strategy $\sigma = (\sigma_n : n \in \omega)$ for (β) is called *a winning strategy* if each infinite σ-sequence has a cluster point in X. Finally, we shall call the pair $(\mathcal{F}, \sigma)$ a Σ-*filter* if $\mathcal{F}$ is a filter in X and σ is a winning strategy for (β) in the $\mathcal{G}(\mathcal{F})$-game played in X. The space X is defined to be *game-compact* if every Σ-filter $(\mathcal{F}, \sigma)$ in X has a cluster point. It is proved in [**?**] that the class of game-compact spaces has nice stability properties: it is preserved by perfect maps, productive and hereditary for closed subspaces. Also, it includes all metrizable spaces. Our next theorem refines the last statement and shows that the class of game-compact spaces contains an important class of generalized metric spaces.

Lemma 4.1. *Let* $(\mathcal{F}, \sigma)$ *be a* Σ*-filter in a space* X*. Then* $\mathcal{F}$ *is a* σ*-filter in the usual sense, i.e.* $\bigcap_{n\in\omega} \overline{F_n} \neq \emptyset$ *for each sequence* $(F_n : n \in \omega)$ *in* $\mathcal{F}$*.*

Theorem 4.2. *Every* κ*-semi-stratifiable space* X *is game-compact.*

Proof. Let $g : \omega \times X \to \tau(X)$ be a g-function on X as in Lemma **??**. Define

$$\mathrm{st}(n, x) = \bigcup \{g(n, y) : x \in g(n, y)\}.$$

To show that X is game-compact, let $(\mathcal{F}, \sigma)$ be a Σ-filter in X.

Step 1: We first show that for each pair $(F_0, n_0) \in \mathcal{F} \times \omega$, there exist $F_{(F_0,n_0)} \in \mathcal{F}$ and a finite set $\{p_0, p_1, ..., p_n\} \subseteq F_0$ such that

$$\overline{F_{(n_0,F_0)}} \subseteq \textstyle\bigcup_{i=0}^{n} \mathrm{st}(n_0, p_i).$$

Assume this claim is false, then there exists $(F_0, n_0) \in \mathcal{F}$ such that for each $F \in \mathcal{F}$ and each finite set $\{p_0, p_1, ..., p_n\} \subseteq F_0$, we have $\overline{F} \nsubseteq \bigcup_{i=0}^{n} \mathrm{st}(n_0, p_i)$. Let player (α)'s first move be $x_0 = y_0 \in F_0$. Then player (β) responds by $\sigma_0(x_0)$. Since $\overline{\sigma_0(x_0)} \nsubseteq \mathrm{st}(n_0, y_0)$, (α) may choose his next move as

$$y_1 \in \overline{\sigma_0(x_0)} \setminus \mathrm{st}(n_0, y_0) \text{ and } x_1 \in \sigma_0(x_0) \bigcap g(n_0 + 1, y_1).$$

Inductively, we produce sequences $(x_n : n \in \omega)$ and $(y_n : n \in \omega)$ satisfying

$$y_{n+1} \in \overline{\sigma_n(x_0, ..., x_n)} \setminus \textstyle\bigcup_{i=0}^{n} \mathrm{st}(n_0, y_i)$$

and

$$x_{n+1} \in \sigma_n(x_0, ..., x_n) \bigcap g(n_0 + n + 1, y_{n+1})$$

for all $n \in \omega$. Since σ is a winning strategy for (β), $(x_n : n \in \omega)$ has a cluster point. On the other hand, it follows from $x_n \in g(n, y_n)$ for all $n \in \omega$ that the sequence $(y_n : n \in \omega)$ has a cluster point $y \in X$. In particular, we can choose $n_1, n_2 \in \omega$ such that $n_1 < n_2$ and $y_{n_1}, y_{n_2} \in g(n_0, y)$, which contradicts the fact that $y_{n_2} \notin \mathrm{st}(n_0, y_{n_1})$.

Step 2: Suppose that (α)'s first move is $q \in X$. Then player (β)'s first move is $\sigma_0(q)$. Applying the claim in step 1 to the pair $(\sigma_0(q), 0)$, we obtain a finite subset

$$X_0 = \{x_0^0, x_1^0, ..., x_{m_0}^0\} \subseteq \sigma_0(q)$$

and $F_0 \in \mathcal{F}$ such that $\overline{F_0} \subseteq \bigcup_{i=0}^{m_0} \mathrm{st}(0, x_i^0)$. Applying the claim in step 1 inductively, there exist $(F_n : n \in \omega)$ in $\mathcal{F}$ and $(X_n : n \in \omega)$ satisfying $X_n = \{x_0^n, x_1^n, ..., x_{m_n}^n\}$ for all $n \in \omega$; and

$$X_n \subseteq \bigcap_{x_i \in X_i, 0 \leq i \leq n-1} \sigma_n(q, x_0, x_1, ..., x_{n-1})$$

for all $n \geq 1$; and $\overline{F_n} \subseteq \bigcup_{x \in X_n} \mathrm{st}(n, x)$ for all $n \in \omega$. By Lemma **??**, we have $\bigcap_{n \in \omega} \overline{F_n} \neq \emptyset$. Let

$$K = \bigcap_{n \in \omega} \overline{F_n} \setminus \bigcap_{F \in \mathcal{F}} \overline{F}.$$

Without loss of generality, we may assume that $K \neq \emptyset$. Otherwise, the proof is completed.

Step 3: We show that for each point $p \in K$, there exist $n_0 \in \omega$ and $x \in X_{n_0}$ with $p \in \mathrm{st}(n_0, x)$ and $F \in \mathcal{F}$ such that $\mathrm{st}(n_0, x) \bigcap \overline{F} = \emptyset$. Suppose the contrary, then there exists some point $p \in K$ such that for any $F \in \mathcal{F}$, any $n \in \omega$ and any $x \in X_n$ with $p \in \mathrm{st}(n, x)$, we have $\mathrm{st}(n, x) \bigcap \overline{F} \neq \emptyset$. Choose $F_0 \in \mathcal{F}$ such that $p \notin \overline{F_0}$. Inductively, choose two sequences $(x_n : n \in \omega)$ and $(y_n : n \in \omega)$ satisfying $x_n \in X_n$, $p \in \mathrm{st}(n, x_n)$ for each $n \in \omega$; and

$$y_n \in \mathrm{st}(n, x_n) \bigcap \sigma_{n-1}(y_0, y_1, ..., y_{n-1}) \bigcap \overline{F_0}$$

for all $n \geq 1$. Therefore, there exists a sequence $(z_n : n \in \omega)$ with $p, x_n \in g(n, z_n)$ for each $n \in \omega$. Moreover, $(x_n : n \in \omega)$ is a σ-sequence, thus it has a cluster point which is also a cluster point of $(z_n : n \in \omega)$. On the other hand, $(z_n : n \in \omega)$ converges to p. So, p is the unique cluster point of $(x_n : n \in \omega)$, which implies $x_n \to p$. Furthermore, we can choose a sequence $(w_n : n \in \omega)$ such that $x_n, y_n \in g(n, w_n)$ for each $n \in \omega$. By a similar observation, p is a cluster point of $(y_n : n \in \omega)$. It follows that $p \in \overline{F_0}$, which is a contradiction.

Finally, as the set $\bigcup_{n \in \omega} X_n$ is countable, we can choose a countable family $\{H_n : n \in \omega\}$ in $\mathcal{F}$ such that $\bigcap_{n \in \omega} \overline{H_n} \bigcap K = \emptyset$. Let

$$L_n = \overline{F_n} \bigcap \bigcap_{i=0}^{n} \overline{H_i}$$

for all $n \in \omega$. Then we have

$$\bigcap_{n\in\omega} \overline{L_n} = \bigcap_{F\in\mathcal{F}} \overline{F} \neq \emptyset.$$

Therefore, X is game-compact. $\square$

Corollary 4.3. *Every Dieudonné-complete space is game-compact.*

Next we shall show that the class of game-compact spaces includes all function spaces $C_p(X)$ for a large class of X.

Theorem 4.4. *If X is a Tychonoff $\mathcal{G}$-space, then $C_p(X)$ is game-compact.*

Proof. Let $(\mathcal{F}, \sigma)$ be a Σ-filter in $C_p(X)$. Since $\mathbf{R}^X$ is game-compact, we have

$$K = \bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset,$$

where the closure is take in $\mathbf{R}^X$. It suffices to show that K is contained in $C_p(X)$. To this end, assume that there exists some $f_0 \in K \setminus C_p(X)$. Since f_0 is not continous, there must be some point x_0 and $\epsilon > 0$ such that for each $U \in \mathcal{U}(x_0)$ we can choose at least one point $x_U \in U$ satisfying

$$|f_0(x_U) - f_0(x_0)| \geq \epsilon.$$

We will use this to obtain a contradiction. Let $s = (s_n : n \in \omega)$ be a winning strategy for player (β) in the $\mathcal{G}(x_0)$-game. We inductively define an s-sequence $(x_n : n \in \omega)$ in X and a σ-sequence $(f_n : n \in \omega)$ in $C_p(X)$ such that

$$x_n \in \bigcap_{i=1}^{n}\{x \in X : |f_i(x) - f_i(x_0)| \leq \tfrac{1}{n}\}$$

for $n \in \mathbf{N}$ and

$$|f_0(x_n) - f_0(x_0)| \geq \epsilon, \text{ and } f_n \in W(x_0, ..., x_n; \tfrac{1}{n})(f_0)$$

for $n \in \mathbf{N}$. Let $x_\infty \in X$ be a cluster point of $(x_n : n \in \omega)$ and f_∞ be a cluster point of $(f_n : n \in \omega)$ in $C_p(X)$. Then $f_0(x_n) = f_\infty(x_n)$ for all $n \in \omega$, which implies that

$$|f_\infty(x_n) - f_\infty(x_0)| \geq \epsilon$$

for $n \in \omega$. Since f_∞ is continuous, letting $n \to \infty$, then we obtain

$$|f_\infty(x_\infty) - f_\infty(x_0)| \geq \epsilon.$$

On the other hand, we have $f_\infty(x_\infty) = f_\infty(x_0) = f_0(x_0)$. This is a contradiction, which implies that f_0 is continuous. □

Let κ be a cardinal. A map $f : X \to Y$ is called *strictly κ-continuous* [?] if for each $A \subseteq X$ with $|A| \leq \kappa$ there is a continuous map $g : X \to Y$ for which $g|A = f|A$. The *weak functional tightness* [?] of a space X is the smallest infinite cardinal κ such that every real-valued strictly κ-continuous map on X is continuous.

Example 4.5. *$C_p(\omega_1 + 1)$ is game-compact, but not Dieudonné-complete.* Let $\omega_1 + 1$ be endowed with the order topology. Since $\omega_1 + 1$ is compact, by Theorem **??**, $C_p(\omega_1 + 1)$ is game-compact. Define a map $f : \omega_1 + 1 \to \{0, 1\}$ in the following way: $f(\kappa) = 0$ for each $\kappa < \omega_1$ and $f(\omega_1) = 1$. Clearly, f is not continuous. Now, consider an arbitrary set $A \subseteq \omega_1 + 1$ with $|A| \leq \aleph_0$. If $\omega_1 \notin A$, then the constant map $g : \omega_1 + 1 \to \{0, 1\}$ with $g(\kappa) = 0$ for each $\kappa \leq \omega_1$ is continuous and $g|A = f|A$. If $\omega_1 \in A$, let

$$\alpha = \sup\{\kappa : \kappa \in A \setminus \{\omega_1\}\}.$$

The map $h : \omega_1 + 1 \to \{0, 1\}$ defined by $g(\kappa) = 0$ for each $\kappa \leq \alpha + 1$ and $g(\kappa) = 1$ for each $\kappa > \alpha + 1$ is continuous and satisfies $h|A = f|A$. Therefore, we have proved that f is strictly $\aleph_0$-continuous. It follows that the weak functional tightness of $\omega_1 + 1$ is $\aleph_1$. It is well-known that for a Tychonoff space X, $C_p(X)$ is Dieudonné-complete if and only if $C_p(X)$ is realcompact. In [?], Arkhangel'skii proved that $C_p(X)$ is realcompact if and only if the weak functional tightness of X is countable. Therefore, $C_p(\omega_1 + 1)$ is not Dieudonné-complete.

A space X is *isocompact* if every closed countably compact subspace of X is compact. In [?], Arkhangel'skii asked the question: *For which space X, is $C_p(X)$ isocompact?* Although this question has not been completely solved, by using an approach similar to that in Theorem **??**, it was proved in [?] that $C_p(X)$ is isocompact when X is a Tychonoff $\mathcal{G}$-space. Since spaces with a G_δ-diagonal are isocompact, one may ask:

Question 4.6. *Does $C_p(X)$ have a G_δ-diagonal if X is a Tychonoff $\mathcal{G}$-space?*

Our next example shows that $C_p(X)$ does not have a G_δ-diagonal even when X is compact.

Example 4.7. *$C_p(\beta\omega_2)$ is isocompact, but does not have a G_δ-diagonal.* Let $\beta\omega_2$ be the Stone-Čech compactification of the space ω_2 with the discrete topology. By [**?**], $C_p(\beta\omega_2)$ is isocompact. Let

$$\mu\omega_2 = \beta\omega_2 \setminus \bigcup_{\alpha<\omega_2} \overline{\alpha},$$

where the closure is taken in $\beta\omega_2$. Moreover, let

$$X = \{f \in C_p(\beta\omega_2) : f|_{\mu\omega_2} \equiv 0\}.$$

Note that the property of having a G_δ-diagonal is hereditary. Thus it suffices to show that the subspace X does not have a G_δ-diagonal. Suppose that $(\mathcal{G}_n : n \in \omega)$ is a sequence of open covers of X. Let $f : \beta\omega_2 \to [0,1]$ be a map defined as $f(\alpha) = 1$ for all $\alpha < \omega_1$ and $f(\alpha) = 0$ otherwise. Then $f \in X$. For each $n \in \omega$, there is a finite subset $R_n \subseteq \beta\omega_2 \setminus \mu\omega_2$ such that for any $g \in X$, $f|_{R_n} = g|_{R_n}$ implies $g \in \mathrm{st}(f, \mathcal{G}_n)$. Define the map $\varphi : \beta\omega_2 \setminus \mu\omega_2 \to \omega_2$ by letting $\varphi(p)$ be the least $\alpha \in \omega_2$ such that $p \in \overline{\alpha}$. let

$$D_n = [\min\, \varphi(R_n),\ \max\, \varphi(R_n)], \text{ for } n \in \omega; \text{ and } D = \bigcup_{n\in\omega} D_n.$$

Then $R_n \subseteq \overline{D_n}$ for each $n \in \omega$. For points p of R_n, find disjoint neighbourhoods G_p in the subspace $\overline{D_n}$. Define a map $h : \omega_2 \to [0,1]$ as follows. For each $\eta \in D_n$, $h(\eta) = f(\eta)$ if $\eta \in G_p$ for some $p \in R_n$; $h(\eta) = 0$ if η belongs to no set G_p. Finally, set $h(\eta) = 0$ if $\eta \in \omega_2 \setminus D$. Let g be the extension of h to a continuous map on $\beta\omega_2$. Note that $g \in X$. Also note that $g \neq f$, since $\omega_1 \setminus \bigcup_{n\in\omega} R_n \neq \emptyset$. However,

$$\{f, g\} \subseteq \bigcap_{n\in\omega} \mathrm{st}(f, \mathcal{G}_n).$$

Therefore, X does not have a G_δ-diagonal.

5 Applications to multi-valued maps

In 1947, Choquet [**?**] announced that if a multi-valued map Φ from a metrizable space X into a metrizable Y is usc at x, then there exists a compact set

$K \subseteq \Phi(x)$ such that for each open set V containing K there exists a neighbourhood U of x for which $\Phi(U) \subseteq \Phi(x) \bigcup V$. Such a set K is called a *kernel of* Φ *at* x. Dolecki [?] improved the Choquet theorem by providing a natural candidate for a kernel for any multi-valued map, which is the so-called *active boundary of* Φ *at* x, denoted by $Frac\Phi(x)$ and defined by

$$Frac\Phi(x) = \bigcap_{U \in \mathcal{U}(x)} \overline{\Phi(U) \setminus \Phi(x)},$$

Since then, many attempts to generalize the Choquet-Dolecki theorem have been achieved successfully. For example, Hansell et al considered the classes of angelic spaces and Dieudonné-complete spaces in the first countable case and q-point case in [?]. In this section, we deduce the Choquet-Dolecki Theorem by using topological games, and improvements of two main theorems in [?] are presented.

Recall that a space X is a *g-space* [?] if the closure of each relatively countably compact set A of X is compact in X. It is shown in [?] that hereditarily g-spaces are precisely angelic spaces in the usual sense (i.e., spaces satisfying conditions (a), (b) and (c) in [?]). The following theorem was obtained by the author et al in [?] recently.

Theorem 5.1. *Let Φ be a multi-valued map from a space X into a g-space Y. If X is first countable at x and Φ is usc at x, then $Frac\Phi(x)$ is compact. Furthermore, if Y is angelic then $Frac\Phi(x)$ is the samllest kernel of Φ at x.*

Corollary 5.2. [?] *Let Φ be a multi-valued map from a space X into an angelic space Y. If X is first countable at x and Φ is usc at x, then $Frac\Phi(x) \subseteq \Phi(x)$ and $Frac\Phi(x)$ is compact.*

To obtain further generalizations of the Choquet-Dolecki theorem, we need the following lemma, whose proof is similar to that of Lemma 11 in [?].

Lemma 5.3. *Let Φ be a usc multi-valued map from a space X into a space Y. If x is a $\mathcal{G}$-point and Φ is closed in the G_δ-topology on Y then $Frac\Phi(x) \subseteq \Phi(x)$.*

Proof. Suppose that there exists a point ξ with $\xi \in Frac\Phi(x) \setminus \Phi(x)$. As $\Phi(x)$ is closed in the G_δ-topology on Y, we can choose a decreasing sequence $(G_n : n \in \omega)$ of open sets in Y such that

$$\xi \in G \text{ and } \Phi(x) \bigcap G = \emptyset, \text{ where } G = \bigcap_{n \in \omega} \overline{G_n}.$$

By upper semicontinuity of Φ, there exists a closed $W_0 \in \mathcal{U}(x_0)$ such that $\Phi(W_0) \bigcap G = \emptyset$. Let $s = (s_n : n \in \omega)$ be a winning strategy for the player (β) in the $\mathcal{G}(x)$-game. As $\xi \in Frac\ \Phi(x)$, there exist $x_0 \in W_0$ and $y_0 \in \overline{G_0}$ with $y_0 \in \Phi(x_0) \setminus \Phi(x)$. By upper semicontinuity of Φ at x, we can choose a $W_1 \in \mathcal{U}(x)$ such that

$$\overline{W_1} \subseteq s_0(x_0) \bigcap W_0 \bigcap \{z \in X : \Phi(z) \bigcap \{y_0\} = \emptyset\}.$$

Again, as $\xi \in Frac\Phi(x)$, there exist $x_1 \in W_1$ and $y_1 \in \overline{G_1}$ with $y_1 \in \Phi(x_1) \setminus \Phi(x)$. Inductively, we can construct an s-sequence $(x_n : n \in \omega)$ in X, a sequence $(y_n : n \in \omega)$ in Y and a sequence $(W_n : n \in \omega)$ in $\mathcal{U}(x)$ such that $x_n \in W_n$, $\overline{W_{n+1}} \subseteq W_n$ for all $n \in \omega$;

$$y_n \in \overline{G_n} \bigcap (\Phi(x_n) \setminus \Phi(x_0)),$$

for all $n \in \omega$; with

$$\overline{W_{n+1}} \subseteq s_n(x_1, ..., x_n), \text{ and } \Phi(\overline{W_{n+1}}) \bigcap \{y_1, ..., y_n\} = \emptyset$$

for all $n \in \omega$. Let $H = G \bigcup \{y_n : n \in \omega\}$. Then H is a closed set in Y. Since $(x_n : n \in \omega)$ is an s-sequence in X, it must have a cluster point $x_\infty \in \bigcap_{n\in\omega} \overline{W_n}$. It follows from the above that

$$\Phi(x_\infty) \bigcap \{y_n : n \in \omega\} = \emptyset,$$

which implies that $x_\infty \notin \Phi^{-1}(H)$. On the other hand, $x_n \in \Phi^{-1}(H)$ for each $n \in \omega$ and $\Phi^{-1}(H)$ is a closed set in X, thus $x_\infty \in \Phi^{-1}(H)$. A contradiction occurs. Therefore $Frac\Phi(x) \subseteq \Phi(x)$. □

Lemma 5.4. [?] [?] *Let X be a game-compact space. Then for each Σ-filter $(\mathcal{F}, \sigma)$ in X, $K = \bigcap_{F\in\mathcal{F}} \overline{F}$ is compact. Moreover, for each open set W containing K there exists an $F \in \mathcal{F}$ such that $F \subseteq W$.*

Theorem 5.5. *Let Φ be a usc multi-valued map from a space X into a game-compact space Y. If x is a $\mathcal{G}$-point then $Frac\Phi(x)$ is compact. Furthermore, if $\Phi(x)$ is closed in the G_δ-topology on Y, then $Frac\Phi(x)$ is a compact kernel for Φ at x.*

Proof. Let $\mathcal{F}$ be defined as $\mathcal{F} = \{\Phi(U) \setminus \Phi(x) : U \in \mathcal{U}(x)\}$. It is trivial if $\mathcal{F}$ has empty elements. It is shown in [?] [?] that if $\mathcal{F}$ has non-empty elements and x is a $\mathcal{G}$-point then there exists a strategy σ for the player (β)

such that $(\mathcal{F}, \sigma)$ is a Σ-filter on Y. By Lemma **??**, $Frac\Phi(x)$ is compact. Moreover, if $\Phi(x)$ is closed in the G_δ-topology on Y, by Lemma **??**, we have $Frac\Phi(x) \subseteq \Phi(x)$. It follows from the second part of Lemma **??** that $Frac\Phi(x)$ is a kernel for Φ at x. $\square$

Corollary 5.6. [?] *Let Φ be a usc multi-valued map from a space X into a Dieudonné-complete space Y. If x is a q-point then $Frac\Phi(x)$ is compact. Furthermore, if $\Phi(x)$ is closed in the G_δ-topology on Y, then $Frac\Phi(x) \subseteq \Phi(x)$.*

Corollary 5.7. *Let Φ be a usc multi-valued map from a space X into a κ-semi-stratifiable space Y. If x is a $\mathcal{G}$-point, then $Frac\Phi(x)$ is a compact kernel for ϕ at x.*

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