A Lattice-valued Banach-Stone Theorem

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Abstract

Let X and Y be two compact Hausdorff spaces, and E be a Banach lattice. We show that if there is a non-vanishing preserving Riesz isomorphism $\Phi : C(X, E) \to C(Y)$, then X is homeomorphic to Y and E is Riesz isomorphic to $\mathbb{R}$.

1 Introduction

Let X and Y be compact Hausdorff spaces. The Banach-Stone theorem says that if the spaces of the real-valued continuous functions $C(X)$ and $C(Y)$ are linearly isometric then X and Y are homeomorphic. The conclusion is still valid if $C(X)$ and $C(Y)$ are algebra isomorphic or Riesz isomorphic [3]. If we weaken the compactness of X and Y to realcompactness, then $C(X)$ and $C(Y)$ being algebra isomorphic or Riesz isomorphic implies that X and Y are homeomorphic [4] [6] [9]. Many versions of the Banach-Stone theorem have been obtained in different ways. We need to mention the papers [1], [2] and [5]. By using the biseparating maps, the authors proved in [1] that if T is a linear biseparating map from $C(X)$ onto $C(Y)$ and X and Y are realcompact then there exist $a \in C(Y)$ with $a(y) \neq 0$ for all $y \in Y$ and a homeomorphism $h : Y \to X$ such that $(Tf)(y) = a(y)f(h(y))$ for all $f \in C(X)$, $y \in Y$. Such a map T is sometimes called a *strong Banach-Stone map.* It was proved in [5] that for compact Hausdorff spaces X and Y, Banach spaces E and F, a

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surjective linear isometry $T : C(X, E) \to C(Y, F)$ is a strong Banach-Stone map if and only if T is biseparating.

Throughout this paper, X and Y are compact Hausdorff spaces, E and F are Banach lattices. For a subset $A \subseteq X$, cl A denotes the closure of A. Recall that a linear map $T : E \to F$ is a *Riesz homomorphism* if T satisfies $T(x \vee y) = Tx \vee Ty$ for all $x, y \in E$. A bijective Riesz homomorphism is called a *Riesz isomorphism*. Let $C(X, E)$ denote the space of all continuous functions from X into E which is endowed with the order

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \text{for all } x \in X.$$

Each element f of $C(X, E)$ is endowed with the sup norm, i.e.,

$$||f|| = \sup_{x \in X} ||f(x)||.$$

Obviously, $C(X, E)$ is a Banach lattice. A function f is called *non-vanishing* if $f(x) \neq 0$ for each $x \in X$. In addition, $\mathbb{R}$ denotes the Banach lattice of reals with the usual norm and order.

The motivation of this paper is to study function spaces from the lattice point of view. Especially, we shall consider the following question: Suppose that E and F are Banach lattices and $C(X, E)$ and $C(Y, F)$ are Riesz isomorphic, are X and Y homeomorphic? Unfortunately, the general answer for this question is negative, as shown by the following simple example.

Example 1.1. *Let $X = \{0\}$, $E = \mathbb{R}^2$ with sup norm, $Y = \{0, 1\}$ with the discrete topology and $F = \mathbb{R}$. Then $C(X, E)$ is both Riesz and isometrically isomorphic to $\mathbb{R}^2$. Define a surjection $\Phi : C(X, E) \to C(Y)$ by*

$$\Phi(a, b)(y) = \begin{cases} a & \text{if } y = 0, \\ 2b & \text{if } y = 1, \end{cases}$$

for all $(a, b) \in \mathbb{R}^2$. Then Φ is a Riesz isomorphism. Clearly, X and Y are not homeomorphic, and E and $\mathbb{R}$ are not Riesz isomorphic. Furthermore, it is easy to verify that Φ is not an isometry.

The above example indicates that more conditions on Riesz isomorphisms between $C(X, E)$ and $C(Y, F)$ are needed to make X and Y homeomorphic in the case of $F = \mathbb{R}$. To seek such conditions, we first study the existence and uniqueness of supports for a Riesz homomorphism from $C(X,E)$ to $\mathbb{R}$. Then the following main theorem is proved: If there is a non-vanishing preserving Riesz isomorphism Φ from $C(X,E)$ onto $C(Y)$, then X is homeomorphic to Y and E is Riesz isomorphic to $\mathbb{R}$. We also show that the condition: "non-vanishing preserving" is not strong in the sense that any Riesz isomorphism from $C(X)$ onto $C(Y)$ automatically satisfies it.

2 Supports for Riesz homomorphisms

Let X and Y be compact Hausdorff spaces, and E and F be Banach lattices. For $h \in C(X)$ and $u \in E$, let $h \otimes u \in C(X,E)$ be defined as $(h \otimes u)(x) = h(x)u$ for each $x \in X$. We call $a \in X$ a *support* for a Riesz homomorphism $\Phi : C(X,E) \to F$ if

$$\Phi(f) = \Phi(\mathbf{1}_X \otimes f(a)) \text{ for all } f \in C(X,E),$$

where $\mathbf{1}_X \in C(X)$ is defined by $\mathbf{1}_X(x) = 1$ for all $x \in X$. In this section, we shall show that any non-trivial homomorphism from $C(X,E)$ into $\mathbb{R}$ has a unique support. Since a Riesz homomorphism between two Banach lattices is a positive operator, the following lemma is a corollary from Proposition 1.3.5 of [8].

Lemma 2.1. *Let X be a compact Hausdorff space, and E and F be Banach lattices. Then any Riesz homomorphism $\Phi : C(X,E) \to F$ is norm-continuous.*

Theorem 2.2. *Let X be a compact Hausdorff space, and E be a Banach lattice. Then any non-trivial Riesz homomorphism $\Phi : C(X,E) \to \mathbb{R}$ has a unique support.*

Proof. Let $\Phi : C(X,E) \to \mathbb{R}$ be a non-trivial Riesz homomorphism. Let the set $A \subseteq X$ be defined as

$$A = \{a \colon \Phi(f) = 0 \text{ for all } f \in C(X,E) \text{ which vanishes on some } U \in \Sigma(a)\},$$

where $\Sigma(a)$ denotes the family of all open neighbourhoods of the point a in X. We shall show that A is a singleton whose unique element is the support for Φ. Suppose that $A = \emptyset$. Then for any $x \in X$, there exist $f_x \in C(X,E)$ and $U_x \in \Sigma(x)$ such that $f_x(U_x) = \{0\}$, but $\Phi(f_x) \neq 0$. Thus, we have

$$\Phi(|f_x|) = |\Phi(f_x)| > 0.$$

The family $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X, so it has a finite subcover $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$. Let

$$f = \bigwedge_{i=1}^{n} \ |f_{x_i}|.$$

Then $f \equiv 0$, which contradicts the fact that $\Phi(f) \neq 0$. Therefore, $A \neq \emptyset$.

Next we show that any $a \in A$ is a support for Φ. Without loss of generality, it is sufficient to prove that for any $a \in A$ and any $f \in C(X, E)$, $f(a) = 0$ implies $\Phi(f) = 0$. Suppose the contrary, then there are $a \in A$ and $f \in C(X, E)$ such that $f(a) = 0$, but $\Phi(f) \neq 0$. By Lemma 2.1, Φ is norm-continuous, so there exists $\varepsilon > 0$ such that the open ball $B(f, \varepsilon) \subseteq C(X, E) \setminus \Phi^{-1}(0)$. Since the function $x \mapsto ||f(x)||$ is also continuous, we can choose open neighbourhoods U and V of a such that $a \in \text{cl } V \subseteq U$ and $||f(x)|| < \varepsilon/2$ for all $x \in U$. By Urysohn's lemma, there exists a continuous function $h : X \to [0, 1]$ such that $h(\text{cl } V) = \{0\}$ and $h(X \setminus U) = \{1\}$. Define $hf \in C(X, E)$ by $(hf)(x) = h(x)f(x)$ for each $x \in X$. Now, following from

$$\begin{aligned} ||f - hf|| &= \sup\nolimits_{x \in X} ||f(x) - (hf)(x)|| \\ &= \sup\nolimits_{x \in U} ||f(x) - (hf)(x)|| \\ &\leq \varepsilon/2, \end{aligned}$$

we have $hf \in B(f, \varepsilon)$. It follows that $\Phi(hf) \neq 0$. On the other hand, we have $(hf)(V) = \{0\}$ which implies $\Phi(hf) = 0$. A contradiction occurs.

It remains to show that A contains exactly one element. Suppose that A contains at least two points. Let $a_1, a_2 \in A$ with $a_1 \neq a_2$. First, choose closed neighbourhoods V_i of a_i $(i = 1, 2)$ such that $V_1 \cap V_2 = \emptyset$, then select open neighbourhoods U_i of a_i $(i = 1, 2)$ such that $V_i \subseteq U_i$ and $U_1 \cap U_2 = \emptyset$. By Urysohn's lemma, there is a continuous function $f_i : X \to [0, 1]$ such that

$$f_i(V_i) = \{0\} \text{ and } f_i(X \setminus U_i) = \{1\} \ (i = 1, 2).$$

Since Φ is non-trivial, by the definition of a support, there must exist some $u \in E$ such that $u \neq 0$ and $\Phi(\mathbf{1}_X \otimes u) \neq 0$. Observe that $(f_1 \otimes u)(V_1) = \{0\}$ and $(f_2 \otimes u)(V_2) = \{0\}$. So $\Phi(f_1 \otimes u) = 0$ and $\Phi(f_2 \otimes u) = 0$. It follows from $f_1 \vee f_2 = \mathbf{1}_X$ that

$$\Phi(\mathbf{1}_X \otimes u) = \Phi(f_1 \otimes u) \vee \Phi(f_2 \otimes u) = 0,$$

which is a contradiction. The proof is completed. $\square$

Corollary 2.3. [3] *Let X be a compact Hausdorff space. For any Riesz homomorphism $\Phi : C(X) \to \mathbb{R}$ with $\Phi(\mathbf{1}_X) = 1$ there exists a unique $a \in X$ such that $\Phi(f) = f(a)$ for each $f \in C(X)$.*

3 Banach-Stone theorems

Let X and Y be compact Hausdorff spaces, and E and F be Banach lattices. For each $y \in Y$, let $\hat{y} : C(Y, F) \to F$ be the evaluation map defined by

$$\hat{y}(g) = g(y) \quad \text{for all } g \in C(Y, F).$$

It is easy to see that if $\Phi : C(X, E) \to C(Y, F)$ is a Riesz homomorphism, then for each $y \in Y$, $\hat{y} \circ \Phi : C(X, E) \to F$ is also a Riesz homomorphism. For each $u \in E$, let

$$\hat{\Phi}(y)(u) = \hat{y} \circ \Phi(\mathbf{1}_X \otimes u).$$

Then $\hat{\Phi}(y)$ is a linear operator from E to F.

A map $T : C(X, E) \to C(Y, F)$ is said to be *non-vanishing preserving* if $T(f)$ is non-vanishing for each non-vanishing function $f \in C(X, E)$. It is easy to verify that if a Riesz homomorphism $\Phi : C(X, E) \to C(Y)$ is non-vanishing preserving then all $\hat{y} \circ \Phi$ are non-trivial Riesz homomorphisms from $C(X, E)$ to $\mathbb{R}$. Thus, by applying Theorem 2.2, there is a unique support x_y for $\hat{y} \circ \Phi$, i.e., for each $f \in C(X, E)$

$$\hat{y} \circ \Phi(f) = \hat{y} \circ \Phi(\mathbf{1}_X \otimes f(x_y)).$$

Let the map $\phi : Y \to X$ be defined by $\phi(y) = x_y$ for each $y \in Y$. We shall call ϕ the *associate map* of Φ.

Lemma 3.1. *Let X and Y be compact Hausdorff spaces, and E be a Banach lattice. If a surjective Riesz homomorphism $\Phi : C(X, E) \to C(Y)$ is non-vanishing preserving, then $\hat{\Phi}(y)$ is a Riesz isomorphism of E onto $\mathbb{R}$ for each $y \in Y$.*

Proof. For each $y \in Y$ we only need to show $\hat{\Phi}(y)$ is bijective. To do this, select $w \in E$ with $w \neq 0$. Since

$$\hat{\Phi}(y)(w) = \Phi(\mathbf{1}_X \otimes w)(y) \neq 0,$$

$\hat{\Phi}(y)$ is injective. Now, let $v \in \mathbb{R}$. Since Φ is surjective, there is $f \in C(X, E)$ such that $\Phi(f) = v\mathbf{1}_Y$. Set $u = f \circ \phi(y)$. Then

$$\hat{\Phi}(y)(u) = \hat{y} \circ \Phi(\mathbf{1}_X \otimes u) = \hat{y} \circ \Phi(f) = v,$$

which implies that $\hat{\Phi}(y)$ is surjective. □

Lemma 3.2. *Let X and Y be compact Hausdorff spaces, and E be a Banach lattice. If a Riesz homomorphism $\Phi : C(X, E) \to C(Y)$ is non-vanishing preserving, then the associate map $\phi : Y \to X$ of Φ is continuous.*

Proof. Let $U \subseteq X$ be an open set, and $y_0 \in \phi^{-1}(U)$. By Urysohn's lemma, there exists a continuous function $h : X \to [0, 1]$ satisfying $h(\phi(y_0)) = 1$ and $h(X \setminus U) = \{0\}$. Choose $u \in E$ such that $u \neq 0$. By hypothesis of the lemma, we get $\Phi(\mathbf{1}_X \otimes u)(y_0) \neq 0$. Let $g = \Phi(h \otimes u)$, and $V = \{y \in Y : |g(y)| > 0\}$. Then

$$g(y_0) = \Phi(h \otimes u)(y_0) = \hat{y}_0 \circ \Phi(\mathbf{1}_X \otimes u) \neq 0.$$

It follows that $y_0 \in V$. Next, we show $V \subseteq \phi^{-1}(U)$. For any $y \notin \phi^{-1}(U)$, we have $\phi(y) \notin U$. Thus $h(\phi(y)) = 0$. Then

$$g(y) = \Phi(h \otimes u)(y) = \hat{y} \circ \Phi(\mathbf{1}_X \otimes ((h \otimes u) \circ \phi(y))) = 0,$$

which implies that $y \notin V$. Therefore, ϕ is continuous. □

Theorem 3.3. *Let X and Y be compact Hausdorff spaces, and E be a Banach lattice. If there exists a non-vanishing preserving Riesz isomorphism $\Phi : C(X, E) \to C(Y)$, then X and Y are homeomorphic and E and $\mathbb{R}$ are Riesz isomorphic. More precisely, for any $y \in Y$, $\hat{\Phi}(y)$ is a Riesz isomorphism from E onto $\mathbb{R}$ and the associate map ϕ of Φ is a homeomorphism from Y onto X.*

Proof. By Lemma 3.1, for any $y \in Y$, $\hat{\Phi}(y)$ is a Riesz isomorphism from E onto $\mathbb{R}$. Let ϕ be the associate map of Φ. We first show that ϕ must be surjective. If there is some $x_0 \in X$ such that $\phi(y) \neq x_0$ for all $y \in Y$, then for each $y \in Y$, we can choose an open neighbourhood U_y of $\phi(y)$ such that $x_0 \notin \mathrm{cl}\, U_y$. By Lemma 3.2, ϕ is continuous, thus $\{\phi^{-1}(U_y) : y \in Y\}$ is an open cover of Y. Let $\{y_1, y_2, ..., y_n\}$ be the finite set of Y such that

$$Y = \bigcup_{i=1}^{n} \phi^{-1}(U_{y_i}).$$

Since $x_0 \notin \text{cl } U_{y_i}$, there is a continuous function $f_i : X \to [0,1]$ satisfying $f_i(x_0) = 1$ and $f_i(\text{cl } U_{y_i}) = \{0\}$ $(1 \leq i \leq n)$. Let $f = \bigwedge_{i=1}^{n} f_i$. Then $f(\text{cl } U_{y_i}) = \{0\}$ for each $1 \leq i \leq n$ and $f(x_0) = 1$. Choose any $u \in E$ with $u \neq 0$ and let $h = f \otimes u$. Now consider $\Phi(h)$. For any $y \in Y$, $\phi(y) \in U_{y_i}$ for some $1 \leq i \leq n$. Then

$$\Phi(h)(y) = \hat{\Phi}(y)(h \circ \phi(y)) = \hat{\Phi}(y)(0) = 0.$$

Thus, $\Phi(h) = 0$. Since Φ is an isomorphism, $h \equiv 0$. But $h(x_0) = u \neq 0$, which is a contradiction.

Next we show that Φ^{-1}, the inverse of Φ, is also non-vanishing preserving. Assume that $g \in C(Y)$ is non-vanishing, but $f = \Phi^{-1}(g)$ is vanishing at some point $x_0 \in X$, i.e., $f(x_0) = 0$. Since ϕ is surjective, we can select $y_0 \in Y$ with $\phi(y_0) = x_0$. It follows that

$$g(y_0) = \Phi(f)(y_0) = \hat{\Phi}(y_0)(f \circ \phi(y_0)) = 0.$$

However, $g(y_0) \neq 0$, since g is non-vanishing. This is a contradiction.

Let ψ be the associate map of Φ^{-1}. To complete the proof, it suffices to show that $\mathbf{1}_X = \phi \circ \psi$. Suppose that there exists some point $z \in X$ with $z \neq \phi \circ \psi(z)$. We may choose a function $f \in C(X, E)$ such that $f(z) = 0$ and $f(\phi \circ \psi(z)) \neq 0$. Then

$$\hat{\Phi^{-1}}(z) \circ \hat{\Phi}(\psi(z))(f(\phi \circ \psi(z))) = \Phi^{-1}(\Phi(f))(z) = f(z) = 0.$$

By Lemma 3.1, $\hat{\Phi^{-1}}(z)$ and $\hat{\Phi}(\psi(z))$ are Riesz isomorphisms between E and $\mathbb{R}$, and $f(\phi \circ \psi(z)) \neq 0$. Then we have

$$\hat{\Phi^{-1}}(z) \circ \hat{\Phi}(\psi(z))(f(\phi \circ \psi(z))) \neq 0,$$

which is impossible. By a similar argument, we can show $\mathbf{1}_Y = \psi \circ \phi$. Thus the associate map ϕ of Φ is a homeomorphism from Y onto X. This completes the proof. $\square$

Remark 3.4. (1) The Riesz isomorphism Φ defined in Example 1.1 is not non-vanishing preserving, since $(0, 1)$ is non-vanishing and $\Phi(0, 1)$ is vanishing at $y = 0$.

(2) Any surjective Riesz homomorphism $\Phi : C(X) \to C(Y)$ is non-vanishing preserving. To achieve this, let $u = \Phi(\mathbf{1}_X) \in C(Y)$ and $v \in C(X)$ such that $\Phi(v) = \mathbf{1}_Y$. Since X is compact, each function in $C(X)$ is bounded. There exists a positive number c for which $v \leq c\mathbf{1}_X$. Then

$$\mathbf{1}_Y = \Phi(v) \leq c\Phi(\mathbf{1}_X) = cu.$$

It follows that $u(y) > 0$ for each $y \in Y$. Let $f \in C(X)$ be non-vanishing. Again, by the compactness of X, there is a positive number c_0 such that $|f| \geq c_0\mathbf{1}_X$. Then

$$|\Phi(f)| = \Phi(|f|) \geq c_0\Phi(\mathbf{1}_X) = c_0 u > 0.$$

Therefore, $\Phi(f)$ is non-vanishing. Hence, as a corollary of Theorem 3.3, we have the following classical result.

Corollary 3.5. [3] *Let X and Y be compact Hausdorff spaces. If $C(X)$ and $C(Y)$ are Riesz isomorphic, then X and Y are homeomorphic.*

References

[1] J. Araujo, E. Beckenstein and L. Narici, Biseparating maps and homeomorphic real-compactifications, *J. Math. Anal. Appl.* **192** (1995), 258–265.

[2] E. Beckenstein, L. Narici and A. Todd, Automatic continuity, *Manusripta Math.* **62** (1988), 257-275.

[3] E. de Jonge and A. van Rooij, Introduction to Riesz Spaces, Mathematical Centre Tracts **78**, Amsterdam, 1978.

[4] L. Gillman and M. Jerison, Rings of continuous functions, Springer-Verlag, New York-Heidelberg, 1976.

[5] S. Hernandez, E. Beckenstein and L. Narici, Banach-Stone theorem and separating maps, *Manuscripta Math.* **86** (1995), 409-416.

[6] C. B. Huijsmans and B. de Pagter, Subalgebras and Riesz subspaces of an f-algebra, *Proc. London Math. Soc.* (3) **48** (1984), 161–174.

[7] W. Luxemberg and A. Zaanen, Riesz Spaces **I**, North-Holland Publishing Co., Amsterdam-London, 1971.

[8] P. Meyer-Nieberg, Banach Lattices, Springer-Verlag Berlin, Heidelberg, New York, 1991.

[9] H. Xiong, A characterization of Riesz spaces which are Riesz isomorphic to $C(X)$ for some completely regular space X, *Nederl. Akad. Wetensch. Indag. Math.* **51** (1989), 87–95.

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