proposition remark equationsection

Valuation of bonds and options under floating interest rate.

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Abstract

The evolution operators, generators of which contain a numerical parameter forming a Markov process, are considered in connection with problems of financial mathematics. Under certain conditions the exact and explicit expressions for the values of the evolution operators averaged over trajectories of the process and for the corresponding variances are derived. Obtained results are applied for valuation of some financial products with account of floating interest rates.

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1 Introduction: Averaged evolution versus stochastic calculus.

The standard Black-Scholes analysis, [1, 2, 3] usually is based on hypothesis, that asset price follows lognormal random walk and the corresponding risk-free interest rate $r \equiv r_0$ and asset volatility are assumed to be known functions of time. The Black-Scholes algorithm for option pricing is derived under assumption that there are no arbitrage possibilities, which means that all risk- free portfolio earn the same return and trading of the underlying asset goes continuously so that one can sell or buy any share of the asset immediately, if it is necessary. By the fixed interest rate $r = r_0$ these assumption give the following standardized form of Black-Scholes equation for European call option value C normalized with respect to exercise price E in logarithmic scale of prices $S = Ee^x$ and reverted time $\tau : t = T - \frac{2\tau}{\sigma^2}$:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v$$

with the initial condition

$$v(x, 0) = \max(e^x - 1, 0)$$

and exponential asymptotics at $x \to \infty$. Here σ is the volatility of the underlying asset's price and r is the interest rate, see for instance [3]. The analysis of Bond pricing requires random interest rate, see for instance [4, 5]. In this case the interest rate is usually assumed to be governed by some stochastic differential equation defining the local increment of interest rate in dependence of time and absolute value of the interest rate, e.g.

$$dr = w(r,t)dX + u(r,t)dt.$$

For special choice of parameters u, w the equation for Bond price similar to Black- Scholes equation was solved by different authors, see [6] via stochastic calculus approach.

In this note we study the problem of **option pricing** under floating interest rate using remarkable properties of the averaged evolution noticed first by R. Griego and Hersh, see [8]. For the mathematically similar problem of description of quantum evolution on the Markov background these properties were rediscovered and intensely used by physicists for investigation of the Mössbauer Scattering in disperse media, see [9]. The rigorous analysis of the models of quantum systems on Markov background was done in [10] and [11]. In both papers [10, 11] the investigation of the averaged dynamics was successfully reduced to study of the spectral properties of the well-defined nonselfadjoint (dissipative) operator - the generator of semigroup formed by the resolving operator for the Schroedinger equation with random Hamiltonian averaged over all trajectories of the stochastic process.

The actual paper is an attempt to demonstrate that the same approach may be applied to the problems concerning pricing of some financial products under *random* parameters in stochastic differential equations for underlying assets. In Section 2 we consider the Cauchy problem for the equation

$$\begin{cases} \frac{\partial}{\partial t}\psi = -H(t)\psi, \ t > 0, \\ \psi(0) = \psi_0 \end{cases} \tag{1}$$

in the Hilbert space $\mathcal{K}$ with densely defined closed operator function $H(t)$, consequent values of which form some stochastic process. Following the main idea of [8] for the problem (1) we obtain under certain assumptions explicit formulae for the values of quadratic form and for matrix elements of resolving operator averaged over all trajectories of the process and for the corresponding variance at any moment of time t. In Section 3 the obtained general formulae are used in the simplest case $\dim \mathcal{K} = 1$ for calculation (in frameworks of some model) present values of zero-coupon bonds under assumption that spot values of interest rates form a Markov process. In Section 4 under the same assumption the exact expression for the analogue of the Black-Scholes formulas for averaged values of European options and corresponding variances are presented.

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This paper is dedicated to Professor S. Albeverio on occasion of his sixtieth birthday.

2 Operators of Averaged Evolution

If the variable interest rate r is a bounded measurable function of time, then the capital gain from the investment of x_0 dollars under this variable interest rate can be calculated via solution of the differential equation

$$\frac{dx}{dt} = r(t)x,$$

$$x(0) = x_0,$$

which gives

$$x(T) - x_0 = x_0 \left[e^{\int_0^T r(t)dt} - 1 \right].$$

We assume now that $r(t)$ is a Markov process governed by the Fokker-Plank equation for transition probabilities w:

$$\frac{dw}{dt} + Mw = 0, \tag{2}$$

where M is a bounded operator acting in a Hilbert space of functions $f(r)$ on the corresponding stochastic space - the set of all values of $r(t)$. The operator M is submitted to the condition that the Markov property for transition probabilities holds. For the purpose of pricing of financial derivatives under the floating interest rate it is enough to consider the case, where is a finite or countable set. Therefore we will assume that is realized as $\mathbf{C}_N, N =$ card $< \infty$, with the euclidean norm or as the space $\mathbf{l}_2$ in countable case, respectively. We will assume also that a finite or infinite matrix M has form

$$M = \lambda(I - W), \tag{3}$$

where $\lambda > 0$ and W is a stochastic matrix. Actually (3) means that possible changes of $r(t)$ are only instantaneous random jumps between points of with the matrix of the transition probabilities W, which may happen at random moments of time distributed according to the Poisson law, that is the probability of occurance of n such moments in the interval $(0, t)$ is equal to

$$e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

In other words 3 means that $r(t)$ is a pseudo-Poisson process [14]. For such process the result of averaging of the present values of financial derivatives over all anticipated trajectories $r(t), t > 0$, which begin at $r(0) = r_{in}$, may be calculated in general on the base of the following special statement, Theorem 1 below, which is a result of application of the general ideas of the theory of quantum evolution on the Markov background [8, 9, 10, 11, 12] to our special case.

Let us consider operators $H(r), r \in$, which acts in a Hilbert space and

- are closed and have r-independent dense domain ;
- $$\Re(H(r)\varphi, \varphi) \geq \gamma \|\varphi\|^2, \varphi \in , \tag{4}$$

  where $\gamma > -\infty$ is also r-independent;
- there exists $C > 0$ and $\alpha > 0$ such that the evolution operators

$$Q(t; r) = \exp(-tH(r)), t > 0, \tag{5}$$

satisfy the following uniform estimate for arbitrary $r, r' \in$ and $t \in (0,T)$, $0 < T < \infty$:

$$\|Q(t;r) - Q(t;r')\| \leq Ct^{\alpha}. \tag{6}$$

The transformation of the Cauchy data $\psi(0) \in$ of the equation

$$\frac{\partial}{\partial t}\psi = -H(r(t))\psi, \tag{7}$$

along the selected trajectory $r(t)$ is formally produced by $\mathbf{T}$ - product

$$\psi(t) := U(t;r)\psi(0) = \mathbf{T}\exp\left(-\int_0^t H(r(s))ds\right)\psi(0). \tag{8}$$

Under our assumptions $r(t)$ and hence $H(r(t))$ with the probability equal to one are piecewise constant functions having only a finite number of jumps on each interval $[0,t]$.

Let $t_1 < t_2 < ... < t_s$ be the set of all possible jumps on the time interval $(0,t)$ for the fixed trajectory $r(t)$. The evolution operator along this trajectory of the Markov process

$$U(t;r) = Q(t - t_s; r(t_s + 0)) \cdot Q(t_s - t_{s-1}; r(t_{s-1} + 0)) \cdot ... \cdot Q(t_1; r(0)) \tag{9}$$

generally do not form a semigroup. Nevertheless averaging over trajectories generates a semigroup of bounded operators which may be used to calculate the evolution operator (9) averaged over all trajectories starting and finishing at the fixed points $r(0) = r_{in}$ and $r(t) = r_{out}$:

$$\langle U(t;r)\rangle\,|_{r_{out},r_{in}}$$

as described below.

Let $\mathrm{diag}_r H$ be the linear operator in the tensor product $\times$ of the Hilbert spaces and , which acts on vectors of the form $x \otimes f(r)$, where $x \in$, $f(\cdot) \in$, as follows

$$\mathrm{diag}_r H\,(x \otimes f(\cdot))\,(r) = H(r)x \otimes f(r).$$

It is easy to see that $\Re\,\mathrm{diag}_r H$ is a densely defined closed operator in $\times$ and $\Re\,\mathrm{diag}_r H \geq \alpha > -\infty$. Let $\widehat{M}$ be the bounded operator in $\times$ defined as $I \times M$, where I is the identity operator in , $Ie = e$, $e \in$. Under our assumptions each vector $\mathbf{h} \in \times$ can be represented as vector functions $h(r), r \in$, taking values in such that

$$\|\mathbf{h}\|^2 = \sum_{r\in} \|h(r)\|^2 .$$

We denote by $P_r, r \in$, operators from $\times$ onto , which acts on the vector $\mathbf{h}$ in $\times$ as

$$P_r\mathbf{h} = h(r).$$

Observe that $(P_r^* x)(r') = x \otimes \delta_{r,r'}, x \in$.

Theorem 1 *Let operators $H(r)$ and the process $r(t)$ satisfy the above assumptions. Then the evolution operator (9) averaged over all trajectories of the process with fixed ends $r(0) = r_{in}, r(t) = r_{out}$, is given by the expression*

$$\langle U(t;r)\rangle \mid_{r_{out},r_{in}} = P_{r_{out}} \mathbf{G}(t) P^*_{r_{in}}, \tag{10}$$

where $\mathbf{G}(t), t > 0$, is the semigroup of bounded operators in $\times$ generated by $\mathrm{diag}_r H + \widehat{M}$, i.e.

$$\mathbf{G}(t) = \exp\left(-t\left(\mathrm{diag}_r H + \widehat{M}\right)\right), t > 0. \tag{11}$$

Proof. Let us consider the evolution operator $U(t;r)$ for the equation (9) on a fixed trajectory $r(t)$ which has s jumps in $(0,t)$. This operator we approximately represent in the form of a $\mathbf{T}$-product

$$\begin{aligned} U_n(t;r) &= \prod_{k=1}^{n} \exp\left(-\Delta_N H(r_k + 0)\right), \\ \Delta_n &= \frac{t}{N}, \, r_k = r((k-1)\Delta_n + 0). \end{aligned} \tag{12}$$

According to (4), (6) and (7, 8) for the uniform error of this approximation we have the estimate

$$\|U(t;r) - U_n(t;r)\| \leq \frac{s\ Ct^\alpha e^{-\gamma t}}{n^\alpha}. \tag{13}$$

To calculate the average of the evolution over the set of all trajectories with the fixed ends r_{in}, r_{out} we form first the $\mathbf{T}$-product (12) over all trajectories passing the prescribed gates r_k, $k = 1, ..., N-1$ at the fixed moments leaving at $t_k = k\Delta_n$. By (2) the emergency probability of such trajectory is given by the product

$$\begin{aligned} w(r_1, ..., r_{n-1} \quad | \quad r_{in}(= r_0), r_{out}(= r_n)) &= \prod_{k=1}^{n} G(r_k, \, r_{k-1}, \Delta_n), \\ G(r_k, \, r_{k-1}, \Delta_n) \quad &= \quad \left(\exp(-M\Delta_n)\right)_{|_{r_k, \, r_{k-1}}}. \end{aligned} \tag{14}$$

Multiplying the evolution operator $U(t;r)_{|_{r_{out}, r_{in}}}$ along a certain trajectory by the corresponding emergency probability (14) and forming a sum of the obtained expressions over all trajectories fitting the at the moments t_k to all prescribed $''gates''\mathrm{r}_k, 1 \leq k \leq N-1$ we receive $P_{r_{out}} \langle U(t;r)\rangle P^*_{r_{in}}$. Applying the same procedure to $U(t;r)_n$ yields the expression

$$\begin{aligned} P_{r_{out}} \langle U(t;r)_n\rangle P^*_{r_{in}} &= \sum_{r_k \in} \prod_{k=1}^{n} G(r_k, \, r_{k-1}, \Delta_n) \exp\left(-\Delta_n H(r_{k-1})\right) = \\ P_{r_{out}} &\prod_{k=1}^{n} \left(\exp(-M\Delta_n) \exp\left(-\mathrm{diag}_r H \Delta_n\right)\right) P^*_{r_{in}} = \\ P_{r_{out}} &\left(\exp(-\widehat{M}\frac{t}{n}) \exp\left(-\mathrm{diag}_r H \frac{t}{n}\right)\right)^n P^*_{r_{in}}. \end{aligned} \tag{15}$$

By our assumptions the probability of some trajectory $r(t)$ to have exactly s jumps on the interval $(0,t)$ is

$$e^{-\lambda t}\frac{(\lambda t)^s}{s!}.$$

Therefore by (13)

$$\begin{aligned}&\|\langle U(t;r)\rangle\mid_{r_{out},r_{in}}-\langle U(t;r)\rangle_n\mid_{r_{out},r_{in}}\|\le\\ &\le\langle\|U(t;r)\mid_{r_{out},r_{in}}-U(t;r)_n\mid_{r_{out},r_{in}}\|\rangle\le\\ &\le\langle\|U(t;r)-U(t;r)_n\|\rangle\quad\le\quad\frac{\lambda t^{1+\alpha}}{n^\alpha}e^{-\gamma t}.\end{aligned}\tag{16}$$

Applying to (15) the Trotter formula [15], which is valid under our assumptions, we obtain the following formula for the strong limit of the products 15

$$s-\lim_{n\to\infty}\left(\exp(-\widehat{M}\frac{t}{n})\exp\left(-\mathrm{diag}_r H\frac{t}{n}\right)\right)^n=\exp\left(-t\left(\mathrm{diag}_r H+\widehat{M}\right)\right),\tag{17}$$

Combining (17) and (16) yields the assertion of the theorem.

Corollary *Let operators* $H(r)$ and *the Markov process* $r(t)$ *satisfy the above assumptions and* $\psi_{in}(t), t>0$, *be a random vector function defined by the expression*

$$\psi_{in}(t):=U(t;r)\psi_{in}=\mathbf{T}\exp\left(-\int_0^t H(r(s))ds\right)\psi_{in},\ \psi_{in}\in .$$

For arbitrary $\psi_{out}\in$ *the value of the functional* $\Psi_{out,in}(t):=(\psi_{in}(t),\psi_{out})$ *averaged over all trajectories* $r(t)$ *beginning at* r_{in} *and ending at* r_{out} *is equal respectively to*

$$\langle\Psi_{out,in}(t)\rangle_{r_{in}}=\sum_{r_{out}}(P_{r_{out}}\mathbf{G}(t)P^*_{r_{in}}\psi_{in},\psi_{out})\tag{18}$$

or

$$\langle\Psi_{out,in}(t)\rangle_{r_{out}}=\sum_{r_{in}}(P_{r_{out}}\mathbf{G}(t)P^*_{r_{in}}\psi_{in},\psi_{out}),.\tag{19}$$

We use the avaraging techniques to compute the variance of the random evolution. We assume in addition now that

- the linear operator

  $$\mathrm{diag}_r H\times I\times I+I\times\mathrm{diag}_r H\times I$$

  in the tensor product $\times\times$, which is defined on the linear set $\times\times$ and acts on vectors of the form $x_1\otimes x_2\otimes f(r)$, where $x_1,x_2\in$, $f(\cdot)\in$, as follows

  $$\begin{aligned}&(\mathrm{diag}_r H\times I\times I+I\times\mathrm{diag}_r H\times I))\,(x_1\otimes x_2\otimes f(\cdot))\,(r)\\ =\ &H(r)x_1\otimes x_2\otimes f(r)+x_1\otimes H(r)x_2\otimes f(r)\end{aligned}$$

  is closable and its closure $\mathrm{diag}_r W$ has the property

  $$\Re\mathrm{diag}_r W\ge 2\gamma>-\infty.$$

- the semigroups generated by the operators $H(r), \mathrm{diag}_r H, \mathrm{diag}_r W$ are *real* i.e. the congugation operator J,
$$\mathrm{J}^2 = I, \ (\mathrm{J}x, y) = \overline{(x, y)}$$
defined on the whole space , commutes with operators of all above semigroups in corresponding Hilbert spaces.

Let $\widehat{\widehat{M}}$ be the bounded operator in $\times$ defined as $I \times I \times M$. Observe that M and $\widehat{\widehat{M}}$ are real operators. As above we represent each vector $\mathbf{h} \in \ \times \ \times$ as vector functions $h_{\times 2}(r), r \in$,with values in $\times$ and denote by $P_r^{\times 2}, r \in$, operators from $\times \ \times$ onto , which acts on $\mathbf{h} \in \ \times \ \times$ as
$$P_r^{\times 2}\mathbf{h} = h_{\times 2}(r).$$

Applying the same arguments as in proof of Theorem 1 we obtain in addition to (18) a tool for calculation of the variance
$$\mathrm{Var}[\Psi_{out,in}(t)] := \left\langle |\Psi_{out,in}(t)|^2 \right\rangle_{r_{in}} - \left| \langle \Psi_{out,in}(t) \rangle_{r_{in}} \right|^2. \tag{20}$$

Theorem 2 *Let operators $H(r)$ and the Markov process $r(t)$ satisfy the all above assumptions. Then the value of functional $|\Psi_{out,in}(t)|^2$ averaged over all trajectories $r(t)$ inceptive at r_{in} or ending at r_{out} equals*
$$\left\langle |\Psi_{out,in}(t)|^2 \right\rangle_{r_{in}} = \sum_{r_{out}} (P^{\times 2}_{r_{out}} \mathbf{G}_{\times 2}(t) P^{\times 2*}_{rin} \psi_{in} \otimes \mathrm{J}\psi_{in}, \psi_{out} \otimes \mathrm{J}\psi_{out})_\times \tag{21}$$
or
$$\left\langle |\Psi_{out,in}(t)|^2 \right\rangle_{r_{out}} = \sum_{r_{in}} (P^{\times 2}_{r_{out}} \mathbf{G}_{\times 2}(t) P^{\times 2*}_{r_{in}} \psi_{in} \otimes \mathrm{J}\psi_{in}, \psi_{out} \otimes \mathrm{J}\psi_{out})_\times , \tag{22}$$
where $\mathbf{G}_{\times 2}(t), t > 0$, is the semigroup of bounded operators in $\times \times$ generated by $\mathrm{diag}_r W + \widehat{\widehat{M}}$, i.e.
$$\mathbf{G}_{\times 2}(t) = \exp\left(-t \left(\mathrm{diag}_r W + \widehat{\widehat{M}} \right) \right), t > 0. \tag{23}$$

Proof. Under our assumption for a certain trajectory $r(t)$ with arbitrary end r_{in} the operator $U(t; r)$ is real. Therefore
$$\begin{aligned} |\Psi_{out,in}(t)|^2 &= (U(t;r)\psi_{in}, \psi_{out})\overline{(U(t;r)\psi_{in}, \psi_{out})} = \\ &= (U(t;r)\psi_{in}, \psi_{out})(U(t;r)\mathrm{J}\psi_{in}, \mathrm{J}\psi_{out}) = \\ &= (U_{\times 2}(t;r)(\psi_{in} \otimes \mathrm{J}\psi_{in}), \psi_{out} \otimes \mathrm{J}\psi_{out})_\times , \end{aligned} \tag{24}$$
where $U_{\times 2}(t; r) = U(t; r) \otimes U(t; r)$ is the resolving operator for the Cauchy problem for the equation
$$\frac{\partial}{\partial t}\psi = \overline{(H(r(t)) \times I + I \times H(r(t)))}\psi$$
in $\times$. Now using (24) and repeating word for word the reasonings given as a proof of the Theorem 1 we obtain (21) with $\mathbf{G}_{\times 2}(t)$ defined by (23).

3 Bond valuation

Recall that a pure discount bond, or zero coupon bond, makes no intermediate payments between its issue date and its maturity date T. It promises only to pay a certain amount at its maturity, which is called its par value, or face value. The difference at any intermediate t between the par value Z and the lower selling price $V(t,T)$ is the bond discount. If the varying interest rate is deterministic, then

$$V(t,T) = \exp\left(-\int_t^T r(s)ds\right) Z. \tag{25}$$

If values of interest rates $r(s)$ form the Markov process governed by (2), $r(t) = r_{in}$ and there are no arbitrage opportunities, then

$$V(t,T) = \left\langle \exp\left(-\int_t^T r(s)ds\right) Z \right\rangle_{r_{in}}, \tag{26}$$

where as before the brackets with subscript r_n denote the ensemble averaging over all trajectories of the process inceptive at r_{in}.

Let us assume that $r(s)$ takes only $2N+1$ or $2N$ equally spaced values

$$r_m = r_0 + m\delta,\ m = 0, \pm 1, ..., \pm N,\ \delta > 0, \tag{27}$$

or

$$r_m = r_0 + m\delta,\ m = \pm\frac{1}{2}, ..., \pm\frac{2N+1}{2},\ \delta > 0, \tag{28}$$

respectively. Then for the problem of calculation of $V(t,T)$ the operator M in (2) can be considered as a $(2N+1)\times(2N+1)$ or $2N\times 2N$ matrix ensuring the Markov property and the conservation law for transition probabilities. If the space of states for each value r_m is one-dimensional then the operator $H(r_m)$ is simply the multiplication by r_m and hence $\mathrm{diag}_r H$ is the diagonal matrix $r_0 I + R$ with elements $r_m \delta_{m,m'}$, where $\delta_{m,m'}$ is the Kronecker symbol, I is the unity matrix.Applying Theorem 1 and Corollary 2 we obtain the expression for the averaged selling price of the Bond in form:

$$V(t,T) = \sum_m \left(e^{-(R+M)(T-t)}\right)_{m,m_{in}} \cdot e^{-r_0(T-t)} Z. \tag{29}$$

under present value of the interest rate $r(t) = r_0 + m_{in}\delta$.

¿From now on we shall consider the special case, where for r taking $N' = 2N+1$ or $N' = 2N$ values (27) or (28), respectively we have

$$M = \lambda(I-P), P = \frac{1}{N'}\begin{pmatrix} 1, & 1, & ... & 1 \\ 1, & 1, & ... & 1 \\ ... & ... & ... & ... \\ 1, & 1, & ... & 1 \end{pmatrix}, \lambda > 0. \tag{30}$$

Here I is the identity matrix; P is the rank one matrix, which represents the projection onto the onedimensional subspace of column vectors with the all elements equal to each other. Observe that such a choice of M provides the property

$$e^{-(T-t)M} = P + (I-P)e^{-\lambda(T-t)}_{T\to\infty},$$

$$\lim_{T\to\infty} e^{-(T-t)M} = P = \frac{1}{N'}\begin{pmatrix} 1, & \dots & 1 \\ \dots & \dots & \dots \\ 1, & \dots & 1 \end{pmatrix}.$$

Using the notation

$$\mathbf{e} = \frac{1}{\sqrt{N'}}\begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}.$$

we get the representation for P in form

$$P = (*, \mathbf{e})_{\mathbf{C}_{N'}}\mathbf{e}.$$

and for the generator of the averaged evolution

$$R + M = \lambda I + R - \lambda(., \mathbf{e})_{\mathbf{C}_{N'}}\mathbf{e}.$$

Basing on the Theorem 1 we calculate now the averaged evolution of the investment via the Riesz integral of the resolvent of $R+M$. Let us consider the resolvent of the "nonperturbed' diagonal operator R:

$$G_w = (R-w)^{-1}.$$

Then the resolvent of $R+M$ is given by the Krein formula

$$(R+M-w)^{-1} = G_{w-\lambda} + \lambda\Theta(w-\lambda)^{-1}(*, G_{\overline{w}-\lambda}\mathbf{e})_{\mathbf{C}_{N'}} G_{w-\lambda}\mathbf{e}$$

involving the perturbation denominator

$$\Theta(w) = 1 - \lambda(G_w\mathbf{e}, \mathbf{e}) = 1 - \frac{\lambda}{N'}\sum_m \frac{1}{m\delta - w}.$$

For $N' = 2N+1$ and $N' = 2N$ we have

$$\begin{aligned}\Theta_{2N+1}(w) &= 1 + \frac{\lambda}{2N+1}\left(\frac{1}{w} + \sum_{n=1}^{N}\frac{2w}{w^2 - n^2\delta^2}\right), \\ \Theta_{2N}(w) &= 1 + \frac{\lambda}{N}\sum_{n=1}^{N}\frac{w}{w^2 - (n-\frac{1}{2})^2\delta^2}.\end{aligned}$$

Considering the Riesz Integral over some simple contour Λ encircling all eigenvalues of the operators R and $R+M$ we may represent the evolution operator as

$$e^{-(R+M)(T-t)} = -\frac{1}{2\pi i}\oint_\Lambda e^{-(T-t)w}(R+M-w)^{-1}dw =$$

$$-\frac{1}{2\pi i}\oint_{\Lambda} e^{-(T-t)w}\left(G_{w-\lambda}+\lambda\Theta(w-\lambda)^{-1}(*,G_{\overline{w}-\lambda}\mathbf{e})_{\mathbf{C}_{N'}}G_{w-\lambda}\mathbf{e}\right)dw. \tag{31}$$

Since

$$\begin{aligned}&\sum_{m}\left(G_{w-\lambda}+\lambda\Theta(w-\lambda)^{-1}(*,G_{\overline{w}-\lambda}\mathbf{e})_{\mathbf{C}_{N'}}G_{w-\lambda}\mathbf{e}\right)_{m,m_{in}}\\ = \;&\Theta(w-\lambda)^{-1}\frac{1}{m_{in}\delta+\lambda-w}\end{aligned}$$

Then for the present value of the interest rate $r_0+m_i\delta$ from (29) and (31)we obtain the sum of residues:

$$V(t,T)=e^{-(T-t)(r_0+\lambda)}\sum_{w_n}\frac{1}{\Theta'(w_n)(w_n-m_{in}\delta)}e^{-(T-t)w_n}, \tag{32}$$

where $w_n$runs over the set of all zeros of the perturbation determinant $\Theta(w)$.

In the simplest case when the interest rate randomly jumps between two values $r_0\pm\frac{\delta}{2}$, applying (32) yields

$$\begin{aligned}V(t,T)&=e^{-(r_0+\frac{\lambda}{2})(T-t)}\{\cosh\frac{1}{2}\sqrt{\lambda^2+\delta^2}(T-t)\\ &+\frac{\lambda\pm\delta}{\sqrt{\lambda^2+\delta^2}}\sinh\frac{1}{2}\sqrt{\lambda^2+\delta^2}(T-t)\}Z,\ r(t)=r_0\pm\frac{\delta}{2}.\end{aligned} \tag{33}$$

Since in the case of bond valuation $\dim=\dim\times K=1$ Theorem 2 appers to be trivial for calculation of the mean-square deviation of the bond price from the anticipated value of it

$$\mathfrak{D}_{m_{in}}(V):=\left[\left\langle\exp\left(-2\int_t^T r(s)ds\right)\right\rangle_{r_{in}}-V^2(t,T)\right]^{\frac{1}{2}} \tag{34}$$

Indeed, by (34) the average of the distribution of squares of the bond prices may be found by the same formulas (32), (33), in which only $r_0$and δ must be redoubled.

For the model (33) under the condition $\delta\ll\lambda$, $\lambda(T-t)\ll 1$ we have

$$\frac{\mathfrak{D}_{m_{in}}(V)}{V(T,t)}=O\left(\sqrt{\lambda\delta^2}(T-t)^{\frac{3}{2}}\right).$$

Note that the above obtained expressions may be used for specification of time values of a series of payments, future values etc. with account of random jumps of interest rates.

4 The averaged formula for the Black-Scholes option pricing model

¿From the formal point of view calculation of averaged values of financial derivatives in the frameworks of the Black-Scholes model is reduced to determination of the averaged resolving operator of the Cauchy problem for the equation

$$\frac{\partial v}{\partial\tau}=\frac{\partial^2 v}{\partial x^2}+(q(\tau)-1)\frac{\partial v}{\partial x}-q(\tau)v,\ q(\tau)=\frac{2r(\tau)}{\sigma^2},-\infty<x<\infty, \tag{35}$$

Here $r(\tau) \geq 0, 0 < \tau < \infty$,is the stochastic process with transition probabilities governed by the equation

$$\frac{dw}{d\tau} = -\frac{2}{\sigma^2} M w, \ \tau = \frac{\sigma^2}{2}(T-t). \tag{36}$$

Under our assumption the set of values of $r(\tau)$ is finite or countable. Equation (35) has form (1) with densely defined closed operator

$$H_{BS}(r) = -\frac{\partial^2}{\partial x^2} - (q(\tau) - 1)\frac{\partial}{\partial x} + q(\tau)$$

in the Hilbert space $= \mathbf{L}^2(-\infty, \infty)$, which is defined on the r-independent linear set of all absolutely continuous functions with absolutely continuous first derivative and square integrable second derivative.

Proposition 3 *Operators $H(r)$ satisfy conditions (4) and (6) with $\alpha = \frac{1}{2}$*

Proof The standard unitary Fourier transform in $\mathbf{L}^2(-\infty, \infty)$

$$(\mathfrak{F} f)(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx, \ f \in \mathbf{L}^2(-\infty, \infty),$$

grades $H_{BS}(r)$ into the multiplication operator by the function

$$\widehat{H}_{BS}(r; k) := k^2 + q - ik(q-1).$$

We see that (4) holds since

$$\Re \widehat{H}_{BS}(r; k) = k^2 + q \geq 0.$$

The image of the semigroup of operators $Q(t; r)$ generated by $H_{BS}(r)$ under the Fourier transformation is the semigroup of the multiplication operators

$$\widehat{Q}(t; r)(k) = \exp\left(-\widehat{H}_{BS}(r; k)t\right), \ t > 0.$$

Taking into account that for different values r_1, r_2 satisfying the condition

$$\left|e^{-r_1 t} - e^{-r_2 t}\right| \leq t$$

holds the inequality

$$\begin{aligned} 4e^{-2k^2 t} \sin^2\left(\frac{r_1 - r_2}{2} kt\right) &= \frac{1}{2} t (r_1 - r_2)^2 \left(2k^2 t e^{-2k^2 t}\right) \frac{4\sin^2\left(\frac{r_1 - r_2}{2} kt\right)}{(r_1 - r_2)^2 k^2 t^2} \\ &\leq \frac{1}{2e}(r_1 - r_2)^2 t, \end{aligned}$$

we obtain for any $f \in \mathbf{L}^2(-\infty, \infty)$ the estimate

$$\int_{-\infty}^{\infty} \left|Q(t; r_1)(k) - Q(t; r_2)(k)\right|^2 |f(k)|^2 \, dk$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left| e^{-(ik+1)r_1 t} - e^{-(ik+1)r_2 t} \right|^2 e^{-2k^2 t} \left| f(k) \right|^2 dk \\
&\leq \left| e^{-r_1 t} - e^{-r_2 t} \right|^2 \|f\|^2 + 4 \int_{-\infty}^{\infty} e^{-2k^2 t} \sin^2 \left(\frac{r_1 - r_2}{2} kt \right) \left| f(k) \right|^2 dk \\
&\leq \left(t^2 + \frac{1}{2e} t \right) (r_1 - r_2)^2 \|f\|^2 .
\end{aligned}
$$

Therefore

$$
\|Q(t;r_1) - Q(t;r_2)\|_{t \to 0} = O(\sqrt{t}).
$$

Due to Proposition 3 operators $H(r)$ satisfy all conditions of the Theorem 1. Hence the averaged resolving operator for the Cauchy problem for the equation (35) can be calculated via the Theorem 1 and Corollary 2 using as a tool the related semigroup $\mathbf{G}(t)$ in the space $\mathbf{L}^2(-\infty,\infty) \times \mathcal{L}$, which may be considered as the Hilbert space $\mathbf{L}^2(-\infty,\infty;\mathcal{L})$ of vector functions $f(x)$ on the real axis with values in $\mathcal{L}$ and the norm

$$
\|f\|^2 = \int_{-\infty}^{\infty} \|f(x)\|_{\mathcal{L}}^2 \, dx.
$$

Under the unitary Fourier Transform the semigroup $\mathbf{G}(t)$ grades into the semigroup of operators acting as "multiplication" by the operator functions

$$
\widehat{\mathbf{G}}(t)(k) = \exp \left(-(\widehat{H}_{BS}(r;k) + \frac{2}{\sigma^2} M)\tau \right).
$$

We assume now that $r(\tau)$ takes only a finite number of different equally spaced values like in (27), (28) and M has form (30) as in Section 3 and keep the notations r_0, r_m introduced there:

$$
\begin{aligned}
q_0(k) &= \frac{2r_0}{\sigma^2} - ik\left(\frac{2r_0}{\sigma^2} - 1\right) + k^2; \\
\epsilon(k) &= (1 - ik)\frac{2\delta}{\sigma^2}, \ \eta = \frac{2\lambda}{\sigma^2}.
\end{aligned}
$$

Then for each $k \in (-\infty, \infty)$ we see that

$$
\mathrm{diag}_r \widehat{H}(r;k) = q_0(k) \cdot I + R(k)
$$

is the diagonal matrix with elements $(q_0(k) + \epsilon(k)m)\delta_{m,m'}$. With all this notations

$$
\widehat{\mathbf{G}}(t)(k) = e^{-q_0(k)\tau} \exp\left(-(R(k) + \eta M)\tau\right). \tag{37}
$$

If M is defined by (30), then introducing as above the function

$$
\Theta(z,k) = 1 - \eta((R(k) - z)^{-1}\mathbf{e}, \mathbf{e}) = 1 - \frac{\eta}{N'} \sum_m \frac{1}{m\epsilon(k) - z}
$$

and repeating arguments resulted in (32) we obtain the following expression for the Fourier transform of the resolving operator of the Black - Scholes equation (35) averaged over all trajectories of r with $r(t) = r_{in}$. For the problem with given data at time $T > t$ this coincides with the multiplication by the function

$$G(k,T-t) = \exp\left(-\frac{\sigma^2}{2}q_0(k)(T-t)\right) \times$$

$$\sum_{w_n} \frac{1}{\Theta'(w_{n;r},k)(w_{n;k} - m_{in}\epsilon(k))} e^{-(T-t)w_{n;k}}, \tag{38}$$

where $w_{n;k}$ runs the set of all zeros of $\Theta(w,k)$.

Observe that the factor

$$G_0(k,T-t) = \exp\left(-\frac{\sigma^2}{2}q_0(k)(T-t)\right)$$

in (38) is just the Fourier transform of the Green function for the Black-Scholes equation with the constant interest rate $r_0$equal to the simple mean of those over the stochastic space and a given value of a sought function at the moment T.

Let us introduce the Fourier transform of (38)

$$D(x,T-t;r_{in}) := \sum_{w_n} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\Theta'(w_{n;r},k)(w_{n;k} - m_{in}\epsilon(k))} e^{-(T-t)w_{n;k}+ikx}$$

and denote by $C(S,T-t;E,r_0,\sigma)$ the price of simple European call option on some asset with the present value $S = Ee^x$ calculated via the Black-Scholes formula[1] for the given strike price E, volatility σ and the interest rate r_0. The following assertion is an interpretation of the expression (38).

Theorem 4 *Let the random $r(\tau)$ takes only a finite number of different equally spaced values and the matrix of transition probabilities M for the stochastic process has form (30). Then the present value of the simple European call option averaged over all trajectories of the process $r(t')$ inceptive at $t' = t$ at r_{in} is given by the formula*

$$\langle C(S,T-t;E,r,\sigma)\rangle_{r_{in}} = \int_0^{\infty} \frac{dS'}{S'} D(\ln\frac{S}{S'},T-t;r_{in}) C(S',T-t;E,r_0,\sigma).$$

Now on the base of Theorem 2 let us calculate $\left\langle C^2(S,T-t;E,r,\sigma)\right\rangle_{r_{in}}$. In the part of operator W appearing in this theorem now acts the properly defined closed normal operator

$$H_{\times 2BS}(r) = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - (q(\tau)-1)\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) + 2q(\tau)$$

in the Hilbert space

$$= \mathbf{L}^2(-\infty,\infty) \otimes \mathbf{L}^2(-\infty,\infty) = \mathbf{L}^2(\mathbf{E}_2).$$

Using the two dimensional Fourier transformation we can prove as above that operators $H_{\times 2BS}(r)$ satisfy conditions (4), (6) with $\alpha = \frac{1}{2}$. Therefore the averaged resolving operator for the Cauchy problem for the equation

$$\frac{\partial v}{\partial \tau} = H_{\times 2BS}(r)v, \ -\infty < x_1, x_2 < \infty,$$

can also be calculated on basis of Theorem 1 and Corollary 2 . The image of resulted semigroup $\mathbf{G}_{\times 2}(t)$ under two-dimensional Fourier transformation acts as the multiplication operator in $\mathbf{L}^2(\mathbf{E}_2) \otimes$ on the matrix function

$$\widehat{\mathbf{G}}_{\times 2}(t)(k) = e^{-q_0(k_1)\tau} e^{-q_0(k_2)\tau} \exp\left(-(R(k_1 + k_2) + \eta M)\tau\right).$$

This yields the following assertion:

Theorem 5 *Let the random $r(\tau)$ is such as in Theorem 4. Then of the square of the present value of the simple European call option averaged over all trajectories of the process $r(t')$ inceptive at $t' = t$ at r_{in} is given by the formula*

$$\left\langle C^2(S, T-t; E, r, \sigma)\right\rangle_{r_{in}} = \int\limits_0^\infty \frac{dS'}{S'} D(\ln \frac{S}{S'}, T-t; r_{in}) C^2(S', T-t; E, r_0, \sigma).$$

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