The Majoritarian Compromise Is Asymptotically Strategy-Proof

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Abstract: In this paper we investigate the social choice rule known as majoritarian compromise. We prove that it is asymptotically strategy-proof for $m \geq 3$ alternatives and that the ratio of the number of all manipulable profiles upon the total number of profiles is in the order of $O\left(1/\sqrt{n}\right)$.

1. Introduction

Let $\mathbb{N}$ stand for the set of all positive integers and let $I_m = \{1, 2, \ldots, m\}$. The elements of I_m will be called alternatives. By $\mathcal{L}(I_m)$ we denote the set of all linear orders on I_m; they represent the preferences of agents over I_m. The elements of the cartesian product

$$\mathcal{L}(I_m)^n = \mathcal{L}(I_m) \times \ldots \times \mathcal{L}(I_m) \quad (n \text{ times})$$

are called n-profiles or simply profiles. They represent the collection of preferences of an n-element society of agents $\mathcal{N} = \{1, 2, \ldots, n\}$. If a linear order $R_i \in \mathcal{L}(I_m)$ represents the preferences of the i-th agent, then by aR_ib, where $a, b \in I_m$, we denote that this agent prefers a to b.

A family of correspondences $F = \{F_n\}$, $n \in \mathbb{N}$,

$$F_n \colon \mathcal{L}(I_m)^n \to \mathcal{P}(I_m),$$

where $\mathcal{P}(I_m)$ is the power set of I_m, we will call a social choice rule (SCR). Normally it is assumed that F represents a certain algorithm which, on accepting a positive integer n and an n-profile $R \in \mathcal{L}(I_m)^n$, outputs a subset $F_n(R)$ of I_m.

One of the SCRs is the majoritarian compromise procedure, which was suggesteded by Sertel circa 1986 and has been widely discussed recently; details can be found in [4, 9]. In this paper we study majoritarian compromise from the following point of view.

The well-known impossibility theorem of Gibbard and Satterthwaite states that every non-dictatorial singleton-valued SCR is manipulable [3, 6, 8]. This result is also valid for arbitrary social choice rules with an appropriate concept of manipulability (see [2] and the literature there, and also [11]). Thus it is clear that for large groups of voters only those social choice rules must be used, for which the probability of possibility to manipulate tends to zero as the number of agents grows. We call such SCRs asymptotically nonmanipulable or asymptotically strategy-proof.

Since all non-dictatorial SCRs are manipulable, it would be useful to know which SCRs are manipulable to a lesser extent. To this end, an experimental approach to the study of the degree of manipulability of SCRs was initiated in [5] and continued in [1], where 26 different rules were investigated by means of computer experiments. Another approach for evaluating the degree of manipulability is theoretical. The author in [10, 11] proved the asymptotic strategy-proofness of the plurality rule and the run-off procedure and obtained that the asymptotics of the ratio of the number of manipulable profiles upon the total number of profiles is in the order of $O\left(1/\sqrt{n}\right)$. In the case of the plurality this asymptotics is exact.

The main result of this paper is that the majoritarian compromise is asymptotically strategy-proof and that the speed of convergence to zero of the probability to obtain a manipulable profile is also in the order of $O\left(1/\sqrt{n}\right)$.

2. Asymptotic Strategy-Proofness

The classical concept of manipulability was defined for singleton-valued SCRs which are often called social choice functions [3, 8]. There are several different definitions of manipulability for arbitrary SCRs. In [10, 11] the author used the following one.

Definition 1 *Let F be a SCR and let $R = (R_1, \ldots, R_n)$ be a profile. We say that the profile R is manipulable for F if there exists a linear order $R_i^{'}$ such that for a profile $R^{'} = (R_1, \ldots, R_i^{'}, \ldots, R_n)$, where $R_i^{'}$ replaces R_i, we have one of the two possibilities:*

1. *For some $a \in F_n(R^{'})$ it is true that $a \, R_i \, b$ for all $b \in F_n(R)$;*

2. *The best element of $F(R^{'})$ relative to R_i is the same as the best element of $F_n(R)$ relative to R_i but $F_n(R^{'})$ is strictly contained in $F_n(R)$.*

The rational behind such definition is as follows. Suppose that the winners, if we have more than one of them, will further participate in a lottery with equal chances to win. Then every outcome can be viewed as a probability vector. Introducing lexicographic ordering on probability vectors, we assume that an agent will prefer one outcome to another if the probability vector of the first outcome is lexicographically earlier than that of the second. This definition was implicitely suggested in [3]. Another definition can be found, for example, in [2].

The difficulty which causes these disagreements is clear. For any definition of manipulability one has to know what changes of $F_n(R)$ are advantageous for the manipulating agent and hence one has to rank all subsets of alternatives one way or another. The latter can be done in many different ways. To avoid this difficulty we give the following definition.

Definition 2 *Let F be a SCR and let $R = (R_1, \ldots, R_n)$ be a profile. We say that the profile R is unstable for F if there exists a linear order $R_i^{'}$ such that for a profile $R^{'} = (R_1, \ldots, R_i^{'}, \ldots, R_n)$, where $R_i^{'}$ replaces R_i, we have $F_n(R^{'}) \neq F_n(R)$.*

Clearly every manipulable profile, no matter how the manipulability is defined, is unstable. The reverse is, of course, not always true. Speaking about manipulability we will have Definition 1 in mind.

The Kelly's index of manipulability of F, as suggested in [5], is

$$K_F(n,m) = \frac{d_F(n,m)}{(m!)^n}, \tag{1}$$

where $d_F(n,m)$ is the total number of all manipulable profiles for the set of alternatives of cardinality m. Let us also define the index of instability of F by the formula

$$L_F(n,m) = \frac{e_F(n,m)}{(m!)^n}, \tag{2}$$

where $e_F(n,m)$ is the total number of all unstable profiles.

Definition 3 *We say that a SCR F is asymptotically strategy-proof for m alternatives if $K_F(n,m) \to 0$ as $n \to \infty$ and asymptotically fool-proof for m alternatives if $L_F(n,m) \to 0$ as $n \to \infty$.*

Of course, $K_F(n,m) \leq L_F(n,m)$ and asymptotic fool-proofness implies asymptotic strategy-proofness.

3. The Majoritarian Compromise Procedure

Let us briefly review the majoritarian compromise procedure. According to it, to determine the winner(s), first, a simple majority rule is used, i.e., an alternative which was top-ranked by at least half of the voters will be chosen and the procedure terminates. (Note that two of them might be chosen.) If such an alternative does not exist, the first and the second preferences are taken together and again any alternative which was ranked first or second by at least half of the voters will be chosen. If this choice is nonempty, the procedure stops. If such an alternative does not exist, we include the third preferences and so on. In the k-th round every alternative which is ranked no worse that k-th best at least by half of the agents will be chosen and the procedure stops (if such an alternative exists).

If m is even, then at most $k = m/2$ rounds will be needed to determine the winner(s). Let n be arbitrary and let $m = 2k-1$ be odd. Then it is easy to see that $k-1$ rounds may not be sufficient to determine the winner(s). What then the last round must be? In [4, 9] this was not emphasised but we need to make this clear because, as n approaches infinity, the procedure will terminate in the last round with probability approaching 1.

In the sequel we will always consider that a profile is writen down in a rectangular table with m rows and n columns in which the i-th column contains the ranking of the i-th voter, her best alternative at the top, her second best right below, etc. Since the upper k rows of the table contain $kn > m \cdot \frac{n}{2}$ positions, then the average number of votes for each alternative in the k-th round must be $\frac{k}{m}n$. As we will show below, if we maintain using the $1/2$ majority rule in this last round, then, as n gets large, with probability 1 all alternatives will get elected and everything will depend on the tie-breaking rule (if it is employed). Therefore we should consider that an alternative is elected if it gained at least the average number of votes, i.e., more than or equal to $\frac{k}{m}n$. For example, for $m = 3$ the $2/3$ majority rule must be used, and for $m = 5$ we should use the $3/5$ majority rule.

We will need the following well-known in coding theory formula (see, for example, [7])

$$\binom{n}{\alpha n} = (2\pi\alpha(1-\alpha)n)^{-1/2} 2^{nh(\alpha)} \left(1 + o(1)\right), \tag{3}$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the enthropy function, and the inequality

$$\binom{n}{n/2} < \frac{2^n}{\sqrt{n}}. \tag{4}$$

Lemma 1 *Suppose $\ell < m/2$. Then the probability that an alternative $a \in I_m$ is found in the upper ℓ rows of the table at least $n/2 - 1$ times is in the order of $O(\alpha^n)$, for some $0 < \alpha < 1$, when $n \to \infty$.*

Proof: We consider that n is even; when n is odd, some minor and obvious changes should be made. Suppose, first, that an alternative $a \in I_m$ is found in the upper ℓ rows of the table exactly $s \geq n/2-1$ times. Let $i = m/2-\ell \geq 1/2$. Then the probability ν_s that an alternative $a \in I_m$ is found s times in the upper ℓ rows of the table is equal to

$$\nu_s = \frac{((m-1)!)^n}{((m!)^n}\binom{n}{s}(m/2-i)^s(m/2+i)^{n-s} = \frac{1}{m^n}\binom{n}{s}(m/2-i)^s(m/2+i)^{n-s}.$$

When $s \geq n/2$, then $n - s \leq s$, and we get

$$\nu_s = \frac{1}{2^n}\binom{n}{s}\left(1-\frac{2i}{m}\right)^s\left(1+\frac{2i}{m}\right)^{n-s} \leq \frac{1}{2^n}\binom{n}{s}\left(1-\frac{4i^2}{m^2}\right)^{n-s}\left(1-\frac{2i}{m}\right)^{2s-n}$$

$$\leq \frac{1}{2^n}\binom{n}{s}\left(1-\frac{1}{m^2}\right)^{n/2} = \frac{1}{2^n}\binom{n}{s}\alpha^n,$$

where $\alpha = \sqrt{(1-1/m^2)}$. When $s = n/2-1$, then $n-s = n/2+1$, and

$$\nu_s = \frac{1}{2^n}\binom{n}{s}\left(1-\frac{2i}{m}\right)^s\left(1+\frac{2i}{m}\right)^{n-s} \leq \frac{1}{2^n}\binom{n}{s}\left(1-\frac{4i^2}{m^2}\right)^s\left(1+\frac{2i}{m}\right)^2$$

$$\leq \frac{C}{2^n}\binom{n}{s}\left(1-\frac{1}{m^2}\right)^{n/2} = \frac{C}{2^n}\binom{n}{s}\alpha^n$$

for some $C > 1$. The probability $\nu = \nu_{n/2-1} + \nu_{n/2} + \cdots + \nu_n$ in question, then, can be estimated as

$$\nu \leq \frac{C}{2^n} \sum_{s=n/2-1}^{n} \binom{n}{s} \alpha^n \leq C\alpha^n.$$

The lemma is proved.

Corollary 1 *As n approaches infinity, then the probability that the majoritarian compromise procedure terminates in the ℓ-th round, for some $\ell < m/2$, is in the order of $O(\alpha^n)$, for some $0 < \alpha < 1$.*

Corollary 2 *Let R be a profile such that one of the agents can manipulate with the majoritarian compromise procedure either causing it to terminate in the ℓ-th round, for some $\ell < m/2$, when it was not the case for R, or causing it to continue to the $(\ell+1)$-th round, while the procedure was to terminate in the ℓ-th round, for some $\ell < m/2$. Then, as n approaches infinity, the probability of drawing such a profile from the uniform distribution is in the order of $O(\alpha^n)$, for some $0 < \alpha < 1$.*

Now we can prove the claim which was made earlier.

Proposition 1 *If $m = 2k-1$ is odd and the $1/2$ majority rule is used in the last k-th round, then, as n approaches infinity, with probability approaching 1 all m alternatives will be chosen by the majoritarian compromise procedure.*

Proof: By Corollary 1 it follows that the probability to get a profile for which the majoritarian compromise procedure terminates earlier than in the k-th round is negligible. Suppose that a certain alternative a is not chosen in the k-th round. Then we have more than $n/2$ a's in the lower $k-1$ rows of the table. By Lemma 1, applied to the table reflected in the k-th row, this probability is also negligible. The proposition is proved.

We are ready to prove our main result.

Theorem 1 *The majoritarian compromise is asymptotically fool-proof for $m \geq 2$ alternatives with the probability of an unstable profile being in the order of $O(1/\sqrt{n})$.*

Proof: By Lemma 1 and Corollaries we may consider only those profiles for which the procedure terminates in the last round. In general we have to consider four cases depending on the parities of n and m. The parity of n is not really important although the formulae will differ slightly depending on the parity of n. We will assume that n is even. It will be absolutely clear what changes should be made for the other case. The parity of m is more important since the last round is different for even and odd m. Here we have to consider two cases.

a) $m = 2k$ is even. Then a profile is unstable if and only if, for some alternative a, there are either $n/2$ or $n/2-1$ entries of a in the upper half of the table. Thus by (4)

$$L(n,m) \le m\frac{\binom{n}{n/2}(m/2)^{n/2}(m/2)^{n/2}\,((m-1)!)^n}{(m!)^n} +$$

$$m\frac{\binom{n}{n/2-1}(m/2)^{n/2-1}(m/2)^{n/2+1}\,((m-1)!)^n}{(m!)^n} \le 2m\binom{n}{n/2}\frac{1}{2^n} < \frac{2m}{\sqrt{n}}.$$

b) $m = 2k-1$ is odd. Then a profile is unstable if and only if, for some alternative a, there are either $\frac{(m+1)n}{2m}$ or $\frac{(m+1)n}{2m} - 1$ entries of a in the upper k rows of the table. Thus, using (3)

$$L(n,m) \le m\frac{((m-1)!)^n}{(m!)^n}\binom{n}{\frac{(m+1)n}{2m}}\left(\frac{m+1}{2}\right)^{\frac{(m+1)n}{2m}}\left(\frac{m-1}{2}\right)^{\frac{(m-1)n}{2m}} +$$

$$m\frac{((m-1)!)^n}{(m!)^n}\binom{n}{\frac{(m+1)n}{2m}-1}\left(\frac{m+1}{2}\right)^{\frac{(m+1)n}{2m}-1}\left(\frac{m-1}{2}\right)^{\frac{(m-1)n}{2m}+1}$$

$$\le \frac{C_1}{m^n}\binom{n}{\frac{(m+1)n}{2m}}\left(\frac{m+1}{2}\right)^{\frac{(m+1)n}{2m}}\left(\frac{m-1}{2}\right)^{\frac{(m-1)n}{2m}}$$

$$\sim \frac{C_2}{\sqrt{n}}2^{n\left(h\left(\frac{(m+1)}{2m}\right)-1\right)}\left[\left(1+\frac{1}{m}\right)^{\frac{m+1}{2m}}\left(1-\frac{1}{m}\right)^{\frac{m-1}{2m}}\right]^n.$$

A simple calculation shows that

$$\log_2\left[\left(1+\frac{1}{m}\right)^{\frac{m+1}{2m}}\left(1-\frac{1}{m}\right)^{\frac{m-1}{2m}}\right] = 1 - h\left(\frac{m-1}{2m}\right).$$

Since $h\left(\frac{m-1}{2m}\right) = h\left(\frac{m+1}{2m}\right)$ we obtain that $L(n,m) = O\left(1/\sqrt{n}\right)$ as required. $\square$

3. Conclusion.

There is a strong indication that all "natural" social choice rules are asymptotically strategy-proof but what "natural" means is yet to be established. It is also quite clear that the slow speed of convergence, in the order of $1/\sqrt{n}$ is quite typical for "natural" rules such as plurality and run-off [11] and might be a consequence of some common property. We see that the majoritarian compromise follows the same pattern and hence must be also "natural."

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