Asymptotic Strategy-proofness of the Plurality and the Run-off Rules

Arkadii Slinko

Abstract: In this paper we prove that the plurality rule and the run-off procedure are asymptotically strategy-proof for any number of alternatives and that the ratio of the number of all manipulable profiles upon the total number of profiles in both cases is it the order of $O\left(1/\sqrt{n}\right)$.

1. Introduction

The well-known impossibility theorem of Gibbard and Satterthwaite states that every non-dictatorial social choice function is manipulable [3, 8, 6]. This result is also valid for social choice rules, i.e., for those social choice functions for which the choice set is not necessary singleton-valued (see [2] and the literature there). Therefore we have to accept a certain degree of manipulability while trying to minimize it as much as possible. In this respect, the social choice rules for which the probability of possibility to manipulate tends to zero as the number of agents grows, are especially attractive and must be used each time when the number of agents is large. We call such rules asymptotically non-manipulable or asymptotically strategy-proof.

Since all non-dictatorial social choice rules are manipulable, it would be useful to know which social choice rules are manipulable to a lesser extent. To this end, an experimental approach to the study of the degree of manipulability of social choice rules was initiated in [4] and continued in [1], where 26 different rules were investigated by means of computer experiments. Another approach for evaluating the degree of manipulability is theoretical. The

author in [9] proved the asymptotic strategy-proofness of the plurality rule ($m = 3, 4$ alternatives) and the run-off procedure ($m = 3$ alternatives) and proved the asymptotic strategy-proofness of the majoritarian compromise [10]. In all cases the asymptotics for the ratio of the number of manipulable profiles upon the total number of profiles appeared to be $O\left(1/\sqrt{n}\right)$.

These results, together with the results of the aforementioned experiments give hope that all "natural" rules are asymptotically strategy-proof, where the appropriate content for the concept of being "natural" is yet to be defined. The same speed of convergence to zero of the probability to obtain a manipulable profile for all these rules is a rather convincing evidence that there must be a common reason for them to be asymptotically strategy-proof. Our results in this paper also support this vaguely stated conjecture.

The concept of asymptotic strategy-proofness stipulates that, as the number of agents goes to infinity, then, with probability approaching 1, no one agent can gain an advantage by misrepresenting her true preferences. The stronger condition, that we are going to introduce, stipulates that, as the number of agents goes to infinity, then, with probability approaching 1, a profile, chosen at random, will be stable in the sense that no one agent can change the result of elections at all, neither to her advantage nor to her disadvantage. We call such rules asymptotically fool-proof. A fool-proof social choice rule is not affected by a random mistake that an agent might do casting her ballot.

In this paper we prove that the plurality rule and the run-off procedure are asymptotically fool-proof for any number of alternatives and that the ratio of the number of all manipulable profiles upon the total number of profiles in both cases is in the order of $O\left(1/\sqrt{n}\right)$. For the plurality this asymptotics is exact.

It should be noted that manipulability by coalitions of voters is a separate problem (see, for example [5]), and we do not touch it in this paper.

2. Basic Concepts

Let $\mathbb{N}$ stand for the set of all positive integers and let $I_m = \{1, 2, \ldots, m\}$. The elements of I_m are called alternatives. Let $\mathcal{N}$ be a finite society of n agents confronted with the choice among these alternatives. By $\mathcal{L}(I_m)$ we denote the set of all linear orders on I_m; they represent the preferences of agents over I_m. If $R_i \in \mathcal{L}(I_m)$ represents the preferences of the i-th agent, then by aR_ib, where $a, b \in I_m$, we denote that this agent prefers a to b. The

elements of the cartesian product

$$\mathcal{L}(I_m)^n = \mathcal{L}(I_m) \times \ldots \times \mathcal{L}(I_m) \quad (n \text{ times})$$

are called n-profiles or simply profiles. They represent the collection of preferences of all agents in $\mathcal{N}$.

A family of correspondences $F = \{F_n\}$, $n \in \mathbb{N}$,

$$F_n \colon \mathcal{L}(I_m)^n \to \mathcal{P}(I_m),$$

where $\mathcal{P}(I_m)$ is the power set of I_m, we will call a social choice rule. Normally it is assumed that F represents a certain algorithm which, on accepting a positive integer n and an n-profile $R \in \mathcal{L}(I_m)^n$, outputs a subset $F_n(R)$ of I_m.

In this paper, first, we deal with the plurality rule, which, in our understanding, on accepting $n \in \mathbb{N}$ and an n-profile R, outputs the set of those alternatives which received a maximal number of agents' first preferences. The number of first preferences that an alternative $a \in I_m$ received will be called the score of this alternative and will be denoted by $score(a)$. Thus, in this terminology we may say that the plurality winners are the alternatives with the maximal score.

Definition 1 *Let $R = (R_1, \ldots, R_n)$ be a profile. We say that the profile R is manipulable if there exists a linear order $R_i^{'}$ such that for a profile $R^{'} = (R_1, \ldots, R_i^{'}, \ldots, R_n)$, where $R_i^{'}$ replaces R_i, we have one of the two possibilities:*

1. *For some $a \in F_n(R^{'})$ it is true that aR_ib for all $b \in F_n(R)$;*
2. *The best element of $F(R^{'})$ relative to R_i is the same as the best element of $F_n(R)$ relative to R_i but $F_n(R^{'})$ is strictly contained in $F_n(R)$.*

The rational behind such definition is as follows. Suppose that the alternatives chosen, if we have more than one of them, will further participate in a lottery with equal chances to win. Then every outcome can be viewed as a probability vector. Introducing lexicographic ordering on probability vectors, we assume that an agent will prefer one outcome to another if the probability vector of the first outcome is lexicographically earlier than that of the second. This definition was implicitely suggested in [3].

Definition 2 *Let $R = (R_1, \ldots, R_n)$ be a profile. We say that the profile R is unstable if there exists a linear order $R_i^{'}$ such that for a profile $R^{'} = (R_1, \ldots, R_i^{'}, \ldots, R_n)$, where $R_i^{'}$ replaces R_i, we have $F_n(R^{'}) \neq F_n(R)$.*

Clearly every manipulable profile is unstable but the reverse is not always true.

The Kelly's index of manipulability of F, as suggested in [4], is

$$K_F(n,m) = \frac{d_F(n,m)}{(m!)^n}, \tag{1}$$

where $d_F(n,m)$ is the total number of all manipulable profiles. Let us also define the index of instability of F by the formula

$$L_F(n,m) = \frac{e_F(n,m)}{(m!)^n}, \tag{2}$$

where $e_F(n,m)$ is the total number of all unstable profiles. If we choose profiles at random from the uniform distribution, then, of course, $K_F(n,m)$ and $L_F(n,m)$ have meanings of the probabilities to obtain a manipulable or unstable profile, respectively.

Definition 3 *We say that a social choice rule F is asymptotically strategy-proof for m alternatives if $K_F(n,m) \to 0$ as $n \to \infty$ and asymptotically fool-proof for m alternatives if $L_F(n,m) \to 0$ as $n \to \infty$.*

Of course, $K_F(n,m) \leq L_F(n,m)$, and asymptotic fool-proofness implies asymptotic strategy-proofness. The two concepts, as we will show later, coincide for the plurality rule for $m \geq 3$ alternatives. But for the plurality it is much easier to count unstable profiles, that is why this concept is so important.

3. The Plurality Rule: Manipulable Versus Unstable Profiles.

We say that a profile $R \in \mathcal{L}(I_m)^n$ has the type $[k_1, \ldots, k_m]$, where $k_1 \geq k_2 \geq \ldots \geq k_m$, if there is a permutation $a_1, \ldots, a_m$ of the alternatives from I_m such that the score of a_i is k_i.

Proposition 1 *A profile is unstable for the plurality rule if and only if it is of the type* $[k_1, \ldots, k_m]$*, where* $k_1 \leq k_2 + 2$*.*

Proof: If $k_1 \geq k_2 + 3$, i.e., the alternative a_1 has the score which is greater by at least 3 than the score of the alternative a_2, then no one agent can make a_2 a winner or make the scores of a_1 and a_2 equal. On the other hand, if the difference in scores of a_1 and a_2 is less than or equal to 2, the result of these elections may be changed by any agent who had a_1 as her first preference. □

We will refer to the profiles specified in Proposition 1 as to the profiles of class A, when $k_1 = k_2$; to the profiles of class B, when $k_1 = k_2 + 1$; and to the profiles of class C, when $k_1 = k_2 + 2$.

Proposition 2 *A profile of the type* $[k_1, \ldots, k_m]$ *of class A is always manipulable for the plurality rule unless for some* $2 \leq s \leq m$ *we have* $k_1 = \ldots = k_s = n/s$*.*

Proof: Clearly the condition $k_1 = \ldots = k_s = n/s$, in case $s < m$, implies $k_{s+1} = \ldots = k_m = 0$. It is easy to check that the profiles satisfying the above condition are not manipulable. Suppose that for a profile R this condition does not hold. Then for some $2 \leq s \leq m$ we have $k_1 = \ldots = k_s > k_{s+1} \neq 0$. Then the alternatives $a_1, \ldots, a_s$ get equal score which is maximal. Thus the set of alternatives $\{a_1, \ldots, a_s\}$ will be chosen. Let us consider an agent whose first preference is a_{s+1}. Suppose that a_1 is her best alternative among the chosen ones. Then she can manipulate by pretending that a_1 is her first preference, swapping it with a_{s+1}. This will make a_1 the sole winner to her advantage. □

Proposition 3 *Let* R *be a profile of the type* $[k_1, \ldots, k_m]$ *of class B and* $k_1 > k_2 = \ldots = k_s > k_{s+1}$*. Suppose that the score of the alternative* a_i *is* k_i*. Then* R *is manipulable for the plurality rule if and only if there exist two different numbers* $i > 1$ *and* $s \geq j > 1$ *such* $k_i \neq 0$ *and* $a_j R_i a_1$*.*

Proof: The agents whose first preference is a_1 cannot manipulate. An agent, whose first preference is a_i, $i > 1$, is in a position to manipulate if $a_j R_i a_1$ for some $j \neq i$ such that $1 < j \leq s$. She might declare that her top alternative is a_j instead of a_i in which case it will also be chosen together with a_1. This is to her advantage. □

Proposition 4 *Let R be a profile of the type $[k_1, \ldots, k_m]$ of class B. Then it is manipulable for the plurality rule with probability not smaller then $1 - (m-1)2^{-t}$, where $t = k_3 + \ldots + k_m$.*

Proof: Assuming $k_1 > k_2 = \ldots = k_s > k_{s+1}$, suppose that nobody can manipulate in such a way that a_2 is selected together with a_1. Then the profile must satisfy the following property: each agent, whose first preference is different from a_1 and a_2, prefers a_1 to a_2. This occurs with probability $1/2^t$, where $t = k_3 + \ldots + k_m$. The probability that nobody can manipulate in such a way that a_j is selected for $2 < j \leq s$ is not greater than $1/2^t$ for each j. Since $s \leq m - 1$, the proposition is proved. □

Proposition 5 *A profile of class C is never manipulable.*

Proof: If the difference in scores of a_1 and a_2 is exactly 2, then the only way in which the result may be changed is that an agent who voted for a_1 makes the scores of a_1 and a_2 equal by placing a_2 first. Clearly this is not to her advantage and hence it is not a manipulation. □

Lemma 1 *There exists a positive real number $\alpha > 0$ such that the probability for a random profile R to be of the type $[k_1, \ldots, k_m]$, with $k_m \leq \alpha n$, exponentially decreases as $n \to \infty$.*

Proof: The number of all possible allocations of first preferences is m^n. The number of all allocations of first preferences in such a way that at least one alternative receives less than or equal to $s = \alpha n$ first preferences can be estimated as being less than or equal to

$$m\left[(m-1)^n + \binom{n}{1}(m-1)^{n-1} + \cdots + \binom{n}{s}(m-1)^{n-s}\right].$$

Thus the probability to get such a profile is bounded above by the number

$$m\left(1 - \frac{1}{m}\right)^n \left[1 + \binom{n}{1}\frac{1}{m-1} + \cdots + \binom{n}{s}\frac{1}{(m-1)^s}\right] \leq$$

$$m\left(1 - \frac{1}{m}\right)^n \left[1 + \binom{n}{1} + \cdots + \binom{n}{s}\right] <$$

$$m\left(1-\frac{1}{m}\right)^n 2^{nh(\alpha)} = m\left[\left(1-\frac{1}{m}\right)2^{h(\alpha)}\right]^n,$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the enthropy function. Here we are using the following well-known formula (see any textbook on error-correcting codes, e.g., [7])

$$\sum_{i=0}^{\alpha n}\binom{n}{i} = (1-\alpha)^{1/2}(1-2\alpha)^{-1}(2\pi\alpha n)^{-1/2}2^{nh(\alpha)}\left(1+o(1)\right). \tag{3}$$

It implies that for some constant C, which depends only on α,

$$\sum_{i=0}^{\alpha n}\binom{n}{i} \sim \frac{C}{\sqrt{n}}2^{nh(\alpha)}, \tag{4}$$

where the symbol $\sim$ as usual means that the ratio of the two quantities approaches unity. This gives us the inequality

$$\sum_{i=0}^{\alpha n}\binom{n}{i} < 2^{nh(\alpha)},$$

which is true for sufficiently large n. Now we have to choose α so that

$$\left(1-\frac{1}{m}\right)2^{h(\alpha)} < 1.$$

This is possible since $h(\alpha) \to 0$ as $\alpha \to 0$. The lemma is proved. □

It follows now that in our considerations we can discard those profiles $[k_1, \ldots, k_m]$, where k_m grows slower than a linear function of n. This makes all profiles of class B manipulable with probability approaching 1. As to profiles of class A, all nonmanipulable types of profiles, but one, are now discarded. But this only exception is not negligible at all. Among all profiles, the profiles of the type $[k, k, \ldots, k]$, where $k = n/m$ (of course, this happens only when n is a multiple of m), are the most numerous. The total number of such profiles will be

$$M(n,m) = \frac{n!}{((n/m)!)^m}.$$

By the Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+o(1)\right) \tag{5}$$

we estimate the ratio of the number of such profiles upon the total number of profiles to be

$$\frac{M(n,m)}{m^n} = \frac{n!}{m^n((n/m)!)^m} \sim m^{m/2}(2\pi n)^{-(m-1)/2} = O\left(n^{-(m-1)/2}\right). \qquad (6)$$

Let us compute the number $N(n,m)$ of profiles of the type

$$[k{+}1, k{+}1, k, \ldots, k, k{-}2], \qquad k = n/m. \qquad (7)$$

We obtain

$$N(n,m) = \frac{n!}{((k{+}1)!)^2(k!)^{m-3}(k{-}2)!} = \frac{k(k{-}1)}{(k{+}1)^2}M(n,m) =$$

$$(1 - 3x + o(x))M(n,m),$$

where $x = 1/k = m/n$. In a similar way we find that the number of profiles of the type

$$[k{+}2, k{+}2, k, \ldots, k, k{-}4] \qquad (8)$$

will be $K(n,m) = (1 - 12x + o(x))M(n,m)$. Since $K(n,m) + N(n,m) > M(n,m)$, we have more profiles of the types (7) and (8) altogether than that of the type $[k, k, \ldots, k]$. Thus we proved that, even in case when n is a multiple of m, there are more manipulable profiles of type A than non-manipulable ones.

Let us also consider the type of profiles $[k_1, \ldots, k_m]$ of class C different from the type $[k_1, \ldots, k_m]$ for which $k_2 = \ldots = k_m = k_1 - 2$. We put in a correspondence to it the type $[\ell_1, \ldots, \ell_m]$ such that 1) $\ell_1 = k_1{-}1$; 2) $\ell_s = k_s{+}1$, where s is the smallest number greater than 2 such that $k_{s-1} > k_s$; 3) $\ell_i = k_i$ for $i \neq 1$ and $i \neq s$. The new profile will be of class B. We note that the latter type is more numerous since the distribution is more uniform.

Now consider the type $[k_1, \ldots, k_m]$ for which $k_2 = \ldots = k_m = k_1 - 2$. It exists only when $n \equiv 2 \pmod{m}$, for which case the type $[k, k, \ldots, k]$ does not exist and all profiles of class A are manipulable. Thus we may put in a correspondence to it the class $[k_1, \ldots, k_m]$ with $k_3 = \ldots = k_m = k_1 - 1 = k_2 - 1$, which is of class A. Thus we proved that we have more manipulable profiles than non-manipulable. Hence we have proved:

Theorem 1 *For the plurality rule, when $m \geq 3$ and $n \to \infty$, at least half of all unstable profiles are manipulable. For the plurality rule asymptotic strategy-proofness and asymptotic fool-proofness are equivalent.* □

4. The Plurality Rule: Main Results.

In this section we prove:

Theorem 2 *For any number of alternatives $m \geq 2$ the plurality rule P is asymptotically fool-proof and asymptotically strategy-proof. Moreover, there are constants d_m and D_m, which depend only on m but not on n, such that*

$$\frac{d_m}{\sqrt{n}} \leq K_P(n,m) < L_P(n,m) \leq \frac{D_m}{\sqrt{n}} \tag{9}$$

with $d_m > 0$ for $m \geq 3$.

Proof: We will prove that for $m \geq 2$ there are constants c_m and C_m, which depend only on m but not on n, such that

$$\frac{c_m}{\sqrt{n}} \leq L_P(n,m) \leq \frac{C_m}{\sqrt{n}}\,; \tag{10}$$

the full statement will then follow from Theorem 1 with $d_m = c_m/2$ and $D_m = C_m$.

¿From what has been said in the previous section, it follows that in order to classify a profile as unstable we need only the information about its top line, i.e., the information about the first preferences of agents. Thus, for the purpose of this section, it is convenient to reduce a profile to the string of its first preferences. Then the formulae for the index of manipulability and the index of instability must be amended accordingly. Since the total number of profiles will now be m^n, this number should replace $(m!)^n$ in (1) and (2).

The case of $m = 2$ alternatives is always a special case, when we deal with manipulability. In this case, as it follows from Propositions 2, 3 and 5, for the plurality rule we do not have any manipulable profiles at all (and so $d_2 = 0$). Unstable profiles do not have any irregularities for $m = 2$. For an even n they will be of the types $[n/2, n/2]$ and $[n/2+1, n/2-1]$; and for an odd n they will be of the type $[[n/2]+1, [n/2]]$, where $[x]$ denotes the integer part of x. Thus

$$e(n,2) = \begin{cases} \binom{n}{n/2} + 2\binom{n}{n/2+1}, & n \text{ even}, \\ 2\binom{n}{[n/2]}, & n \text{ odd} \end{cases}$$

which, according to the Stirling formula (3), gives

$$\frac{2^n}{\sqrt{n}} \leq e(n,2) \leq 3\frac{2^n}{\sqrt{n}}\,, \qquad \frac{1}{\sqrt{n}} \leq L(n,2) \leq \frac{3}{\sqrt{n}}. \tag{11}$$

This will be the basis for an inductive proof of (10). Suppose that for $m \geq 3$ the constants c_{m-1} and C_{m-1} exist such that

$$\frac{c_{m-1}}{\sqrt{n}} \leq L_P(n, m{-}1) \leq \frac{C_{m-1}}{\sqrt{n}}\,. \tag{12}$$

The idea is that for an unstable profile removing one of the alternatives with the lowest score, which is less than or equal to n/m, must give us an unstable profile on the remaining $m{-}1$ alternatives.

Let $a \in I_m$ be an alternative. We note that for $k < n$

$$P_k = Prob\,\{score(a){\leq}k\} = \sum_{i=0}^{k} \binom{n}{k} \frac{(m{-}1)^{n-k}}{m^n} < 1. \tag{13}$$

We also note that, since at least one alternative has its score lower than or equal to n/m, it is true that $P_{n/m} \geq 1/m$.

Since the lowest score must be smaller than n/m, by (12) and (13) we have

$$L_P(n,m) \leq \sum_{a=1}^{m} \sum_{k=0}^{n/m} Prob\,\{score(a){=}k\}\, L_P(n{-}k, m{-}1)$$

$$\leq mC_{m-1} \sum_{k=0}^{n/m} Prob\,\{score(a){=}k\}\, \frac{1}{\sqrt{n-k}} \leq \frac{m^{3/2}}{\sqrt{m-1}} C_{m-1} P_{n/m}\, \frac{1}{\sqrt{n}}$$

$$\leq \frac{m^{3/2}}{\sqrt{m-1}} C_{m-1}\, \frac{1}{\sqrt{n}} = \frac{C_m}{\sqrt{n}}.$$

To prove the lower bound, we single out one alternative, say a, and notice that, if $score(a) \leq n/m$, then a profile is unstable if and only if it remains unstable after removing a. Thus

$$e_P(n,m) \geq \sum_{k=0}^{n/m} \binom{n}{k} e_P(n{-}k, m{-}1),$$

and hence by (12)

$$L_P(n,m) \geq c_{m-1} \sum_{k=0}^{n/m} \binom{n}{k} \frac{(m-1)^{n-k}}{m^n \sqrt{n-k}} \geq c_{m-1} P_{n/m} \frac{1}{\sqrt{n}} \geq \frac{c_{m-1}}{m} \frac{1}{\sqrt{n}} = \frac{c_m}{\sqrt{n}}.$$

The induction is therefore completed. □

5. The Run-off Procedure

To determine the winner according to the run-off procedure, first the simple majority rule is used, i.e., an alternative which score is greater than $n/2$ will be chosen. If such an alternative does not exist, the two alternatives with best scores are taken (if the scores of the second and the third best alternatives are equal, then a certain tie-breaking procedure must be used). Thus in the first round two alternatives are chosen and only they participate in the second and final round. Assuming that the preferences of the agents in relation to these two alternatives do not change, the simple majority rule is applied to determine the winner among the two remaining alternatives (again it could be neccessary to break a tie). It is normally assumed that the tie-breaking procedure works as follows: if there is a choice between two or several alternatives of I_m with equal scores, then the preferences of the first voter are followed. We will not specify which tie-breaking procedure is used because, no matter what it is, we are unable to describe all manipulable profiles as we did for the plurality. Instead, we will describe a class of profiles which contains all manipulable profiles under all possible tie-breaking arrangements.

We start with some general comments. As usual, when we talk about a profile of the type $[k_1, \ldots, k_m]$, we mean that the alternative a_i gets k_i first preferences. We identify the following three situations which could lead to instability or even to manipulability of a profile:

1. For an even n, the profile belongs to a type $[k_1, \ldots, k_m]$, where $k_1 = n/2$; in this case any agent, whose first preference is not a_1, can prevent elections going to the second round (where some even nastier candidate will win) by changing her first preference in favour of a_1;

2. The profile is of the type $[k_1, \ldots, k_m]$, where $k_2 = k_3$, or $k_2 = k_3 + 1$, or $k_2 = k_3 + 2$. In this case there is a potential possibility to manipulate with the decision which alternatives will proceed to the second round;

3. The alternatives a_1 and a_2 win the first round and in the second round they have equal scores or their scores differ by just one vote or just by two votes.

Let us note that there can be no manipulation in the second round, thus we have only the two first situations that could lead to manipulability of a profile. It is interesting to note that in the second case even the situation, when $k_2 = k_3 + 2$ can now be manipulable. Also, no more than $\binom{m}{2}$ pairs of alternatives may appear in the second round and the same number of retricted profiles may occur. Since each of them is unstable with probability, which is $O(1/\sqrt{n})$, this gives us $O(1/\sqrt{n})$ of unstable profiles which fall under the third situation. Hence this source of instability is benign.

The main working concept will be now introduced.

Definition 4 *Any profile which falls under the situations 1 or 2 is called potentially manipulable.*

The good news is that, the probability of the first situation to occur is negligible. This will be a consequence of the following lemma in which we will prove even stronger result which will be need further.

Lemma 2 *There exists a real number $0 < \gamma < 1/2$ such that the probability for a random profile to be of the type $[k_1, \ldots, k_m]$ with $k_1 \geq \gamma n$ is exponentially small, when $n \to \infty$.*

Proof: In other words we have to prove that the probability that the best score is greater than or equal to γn is exponentially small. Since

$$\binom{n}{n/2} < \frac{2^n}{\sqrt{n}},$$

the probability for a random profile to be of the type $[k_1, \ldots, k_m]$ with $k_1 \geq \gamma n$ can be estimated as follows:

$$\sum_{a=1}^{m} \sum_{k=\gamma n}^{n} Prob\{score(a)=k\} \leq m \sum_{k=\gamma n}^{n} \binom{n}{k} \frac{(m-1)^{n-k}}{m^n} <$$

$$mn(1-\gamma)\binom{n}{n/2} \frac{(m-1)^{n(1-\gamma)}}{m^n} < C\sqrt{n} \frac{2^n (m-1)^{n(1-\gamma)}}{m^n}$$

$$= C\sqrt{n}\left(\frac{2(m-1)^{(1-\gamma)}}{m}\right)^n.$$

The function

$$u(x) = \frac{2(m-1)^{1-x}}{m}.$$

is continuous and decreasing. For $m \geq 3$ we have $u(1/2) = 2\sqrt{m-1}/m < 1$. Hence the number $\gamma < 1/2$ exists such that $u(\gamma) < 1$. This gives us the number sought for. □

By $K'_R(n,m)$ we denote the total number of all potentially manipulable profiles for the run-off procedure with m alternatives and n voters.

Lemma 3 *For any number of alternatives $m \geq 3$ there are constants $0 < c_m \leq C_m$, which depend only on m but not on n, such that the number of potentially manipulable profiles for the run-off procedure satisfies the inequality*

$$\frac{c_m}{\sqrt{n}} \leq K'_R(n,m) \leq \frac{C_m}{\sqrt{n}}. \tag{14}$$

Proof: As in the proof of Theorem 2 we will use induction on m. Suppose $m = 3$. The highest score must be greater than $n/3$ but, due to Lemma 2, it can be considered as smaller than γn for some $\gamma < 1/2$. Thus, by (13) and (14) we have

$$K'_R(n,3) \leq \sum_{a=1}^{3} \sum_{k=n/3}^{\gamma n} Prob\{score(a)=k\}\, L_P(n-k,2),$$

where $L_P(n,2)$ is the number of unstable profiles for the plurality in case of two alternatives. Thus due to (11)

$$K'_R(n,3) \leq 3 \sum_{k=n/3}^{\gamma n} Prob\{score(a)=k\}\, \frac{1}{\sqrt{n-k}}$$

$$< 3\frac{(P_{\gamma n} - P_{1/3n})}{\sqrt{n-\gamma n}} \leq \frac{3}{\sqrt{1-\gamma}}\frac{1}{\sqrt{n}} = \frac{C_3}{\sqrt{n}}.$$

Also, if we fix a certain alternative a and consider those potentially manipulable profiles where a has the largest score, we will get the lower bound:

$$K'_R(n,3) \geq \sum_{k=n/3}^{n} Prob\{score(a)=k\}\, L_P(n-k,2)$$

$$> \frac{1}{\sqrt{n}} \left(\sum_{k=n/3}^{n} Prob\{score(a)=k\} \right) \geq \frac{1}{3\sqrt{n}} = \frac{c_3}{\sqrt{n}}.$$

The sum in the bracket on the right-hand-side is greater than 1/3 because one of the alternatives always gets a score greater than or equal to $n/3$.

The induction now can be completed as in the proof of Theorem 2. □

Theorem 3 *For any number of alternatives $m \geq 3$ the run-off procedure is asymptotically fool-proof and asymptotically strategy-proof with the number of all unstable profiles $L_R(n,m)$ and the number of all manipulable profiles $K_R(n,m)$ being in the order of $O\left(1/\sqrt{n}\right)$.*

Proof: The case $m = 2$ is the same as for the plurality. For $m \geq 3$ all unstable profiles are either potentially manipulable or belong to the group of profiles which fall under the third situation. The number of profiles in each group is in the order of $O\left(1/\sqrt{n}\right)$ so the same is true for the number of unstable profiles. Thus $K_R(n,m) \leq L_R(n,m) = O\left(1/\sqrt{n}\right)$.□

Finally we will suggest an example showing that not all rules are asymptotically strategy-proof.

Example 1 *Let us define the following "odd" rule. For a profile R, let us define $F_n(R)$ to be the set of all alternatives with the highest odd score. For this rule all profiles are unstable and the rule is not asymptotically strategy-proof.*

Sketch of the proof: For half of the profiles the highest score is even. Let us consider one of such profiles. Let a_1 be an alternative with the highest score which is even. Any agent who has a_1 as her first preference can manipulate by pretending that a_1 is not her first preference but the alternative whith the lowest score instead. By doing this she will achieve that a_1 is included in the choice set. This works for the profiles of all types except the type $[n/k, n/k, \ldots, n/k]$. But the probability of such profiles go to zero. □

6. Conclusion.

The asymptotic strategy-proofness of the plurality and the run-off procedure is a strong indication that all "natural" social choice rules are asymptotically strategy-proof. Nevertheless what "natural" means is yet to be

established. It is also quite clear that the slow speed of convergence, in the order of $1/\sqrt{n}$ for the plurality and the run-off (and also for the majoritarian compromise [10]) might be a consequence of some unknown property which defines the class of "natural" rules.

The concepts of asymptotic strategy-proofness and asymptotic fool-proofness are likely to be equivalent for most, if not all, rules but asymptotic fool-proofness seems to be a useful tool in investigating asymptotic strategy-proofness since it is much easier in some cases to count unstable profiles rather than manipulable ones.

7. Aknowledgements.

This paper was written when the author was a visiting Professor of the Department of Economics and a guest of the Centre for Economic Design at Boğaziçi University in Turkey. I am greatful to Prof. Semih Koray for the organisation of this visit. I thank all the members of the Centre, especially Prof. Murat Sertel, and all the members of the Economics Department for their hospitality. Especially I am grateful to Prof. Dr. Fuad Aleskerov who attracted my attention to the topic of manipulability and his encouragements. I thank Hervé Moulin for the interest to this work and the reference [5]. A partial support from TÜBİTAK NATO PC-B Programme is gratefully acknowledged.

References

[1] Aleskerov F and Kurbanov E (1999) Degree of Manipulability of Social Choice Procedures. In: Current Trends in Economics. Eds: Alkan, A., Aliprantis, C.D. and Yannelis, N.C. Springer, 13–27.

[2] Ching S and Zhou L (1999) Multi-Valued Strategy-Proof Social Choice Rules, to appear.

[3] Gibbard A (1973) Manipulation of Voting Schemes: A General Result: Econometrica 41:587–601.

[4] Kelly J (1993) Almost All Social Choice Rules are Highly Manipulable, but Few aren't: Social Choice and Welfare 10:161–175.

[5] Kim KH and Roush FW (1996) Statistical Manipulability of Social Choice Functions: Group Decision and Negotiation 5:263–282.

[6] Pattanaik PK (1976) Voting and Collective Choice. Cambridge University Press, Cambridge.

[7] Peterson WW and Weldon EJ (1972) Error-Correcting Codes, 2nd ed., Cambridge, MIT Press.

[8] Satterthwaite MA (1975) Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions: Journal of Economic Theory 10:187–217.

[9] Slinko A (1999) On Asymptotic Strategy-proofness of the Plurality and Run-off Rules. Preprint ISS/EC-99-01. Department of Economics. Boğaziçi University. Istanbul.

[10] Slinko A (1999) The Majoritarian Compromise Is Asymptotically Strategy-Proof, to appear.