Covering Properties and Metrisation of Manifolds 2*

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Abstract

There are many conditions equivalent to metrisability for a topological manifold which are not equivalent to metrisability for topological spaces in general. What are the weakest such? We show that a number of weak covering properties which are equivalent to metrisability for a manifold, for example metaLindelöf, may be further weakened by considering only covers of cardinality the first uncountable ordinal. Extensions to higher cardinals are discussed.

1 Introduction and Definitions.

By a topological manifold we mean a connected Hausdorff space each point of which has a neighbourhood homeomorphic to euclidean space. In [2] there is a list of over 50 conditions which are equivalent to metrisability for a manifold but not for a topological space in general. As one might expect, some of these conditions are strictly stronger than metrisability and some are strictly weaker than metrisability in a general space. In this paper we investigate just how weak covering properties can be made while still being equivalent to metrisability for a manifold.

All cardinals are assumed infinite. We denote the cardinality of a set X by $|X|$. If $x \in X$ and $\mathcal{F}$ is a family of subsets of X then $\mathrm{ord}(x, \mathcal{F})$ is the *order* of $\mathcal{F}$ at x, ie $|\{F \in \mathcal{F} \ / \ x \in F\}|$. When X is a topological space, we denote by $\chi(x, X)$ the *character* of x in X, ie the least infinite cardinality of a local basis at x. A good reference for the set theory used in this paper is [7].

The following properties are studied in [1] where Theorem 4.1 states that every locally metrisable, linearly Lindelöf space is hereditarily Lindelöf. They observe that their proof may be modified to show that every locally metrisable

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ω_1-Lindelöf space is hereditarily Lindelöf. (As noted in [1] and in Proposition 14 below, every linearly Lindelöf space is ω_1-Lindelöf.) Setting $\kappa = \omega_1$ in Proposition 12 shows that local metrisability can be replaced by local hereditary Lindelöfness.

Definition 1 *A space X is* linearly Lindelöf *provided that every open cover of X which is a chain has a countable subcover. A family $\mathcal{F}$ of subsets of a set X is a* chain *provided that $\forall F, G \in \mathcal{F}$ either $F \subset G$ or $G \subset F$.*

A space X is ω_1-Lindelöf *provided that every open cover of X of cardinality ω_1 has a countable subcover.*

Recall also the following definition.

Definition 2 *Let θ and κ be two cardinal numbers. A topological space X is* $[\theta, \kappa]$-compact, [8], *if and only if every open cover of X of cardinality at most κ has a subcover of cardinality less than θ.*

If $\theta = \omega$ then $[\theta, \kappa]$-compact is also called initially κ-compact. *If $\kappa \geq |X|$ then $[\theta, \kappa]$-compact is also called* finally θ-compact.

Motivated by these definitions we formulate the following definitions, where θ and κ are two cardinal numbers:

Definition 3 *A space X is* linearly $[\theta, \kappa]$-compact *provided that every open cover $\mathcal{U}$ of X which is a chain and satisfies $|\mathcal{U}| \leq \kappa$ has a subcover $\mathcal{V}$ with $|\mathcal{V}| < \theta$.*

A space X is (linearly) $[\theta, \kappa]$-metacompact *provided that every open cover $\mathcal{U}$ of X which (is a chain and) satisfies $|\mathcal{U}| \leq \kappa$ has an open refinement $\mathcal{V}$ such that* $\mathrm{ord}(x, \mathcal{V}) < \theta$ *for each $x \in X$.*

A space is nearly (linearly) $[\theta, \kappa]$-metacompact *if we merely demand that* $\mathrm{ord}(x, \mathcal{V}) < \theta$ *for each point x in some dense subset of X.*

An $[\omega_1, \omega_1]$-metacompact space may also be called an *ω_1-metaLindelöf* space, and is a weak form of metaLindelöfness as it requires point-countability of a refinement only for open covers of cardinality ω_1. Theorem 13 tells us that under appropriate conditions, which all manifolds satisfy, an ω_1-metaLindelöf space is in fact metaLindelöf. *(Nearly) linearly metaLindelöf* and *(nearly) ω_1-metaLindelöf* are defined analogously.

Given a set X and a collection $\mathcal{S}$ of subsets of X, a *choice function* is a function $f : \mathcal{S} \to X$ such that $f(S) \in S$ for each $S \in \mathcal{S}$.

Definition 4 *A space X has* property (ω_1)pp, *[4], provided that each open cover $\mathcal{U}$ of X (with $|\mathcal{U}| = \omega_1$) has an open refinement $\mathcal{V}$ such that for each choice function $f : \mathcal{V} \to X$ with $f(V) \in V$ for each $V \in \mathcal{V}$ the set $f(\mathcal{V})$ is closed and discrete in X.*

The main result in this paper is the following.

Theorem 5 *Let M be a manifold. Then the following are equivalent:*

(a) M is metrisable;

(b) M is linearly metaLindelöf;

(c) M is ω_1-metaLindelöf;

(d) M is nearly linearly metaLindelöf;

(e) M is nearly ω_1-metaLindelöf;

(f) for every open cover $\mathcal{U}$ of M with $|\mathcal{U}| = \omega_1$ there is an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed and discrete;

(g) for every open cover $\mathcal{U}$ of M with $|\mathcal{U}| = \omega_1$ there is an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed;

(h) for every open cover $\mathcal{U}$ of M with $|\mathcal{U}| = \omega_1$ there is an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is discrete.

Of course with the Continuum Hypothesis this tells us no more than what we already know from [2], ie that every (nearly) meta-Lindelöf manifold (equivalently, manifold with property pp) is metrisable, as every manifold has the cardinality of the continuum, by [6, Theorem 2.9].

2 Preliminary Results.

Recall that the *character* of a space X is the least cardinal κ for which every point of X has a local base of cardinality at most κ.

We say that a sequence $\langle V_\alpha \rangle$ of subsets of a space is *strongly increasing* provided that $\overline{V_\alpha} \subset V_{\alpha+1}$ for each α.

Lemma 6 *Let κ be a regular cardinal. Suppose that X is a space such that $\chi(x, X) < \kappa$ for each $x \in X$ and $\langle V_\alpha \rangle$ is a strongly increasing κ-sequence of subsets of X. Then $\cup_{\alpha<\kappa} V_\alpha$ is closed in X.*

Proof. Suppose that $x \in \overline{\cup_{\alpha<\kappa} V_\alpha}$. Let $\{U_\beta \ / \ \beta \leq \theta\}$ be a neighbourhood base at x, where $\theta < \kappa$. For each β we have $U_\beta \cap (\cup_{\alpha<\kappa} V_\alpha) \neq \varnothing$ so $U_\beta \cap V_{\alpha_\beta} \neq \varnothing$ for some $\alpha_\beta < \kappa$. Let $\alpha = \sup\{\alpha_\beta \ / \ \beta \leq \theta\}$. Then $\alpha < \kappa$ and $U_\beta \cap V_\alpha \neq \varnothing$ for all β, and hence $x \in \bar{V}_\alpha \subset V_{\alpha+1}$. Thus $\overline{\cup_{\alpha<\kappa} V_\alpha} \subset \cup_{\alpha<\kappa} V_\alpha$. ■

Lemma 7 *Let κ be a regular cardinal. Suppose that X is a connected space and that $\mathcal{V}$ is an open cover of X such that $\mathrm{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$ and each member of $\mathcal{V}$ has density $< \kappa$. Then $|\mathcal{V}| < \kappa$.*

Proof. We may assume that $\varnothing \notin \mathcal{V}$.

Pick any $V_0 \in \mathcal{V}$ and set $\mathcal{V}_0 = \{V_0\}$. Assuming that $\mathcal{V}_i \subset \mathcal{V}$ has been defined, let $V_i = \cup \mathcal{V}_i$ and set $\mathcal{V}_{i+1} = \{V \in \mathcal{V} \ / \ V \cap V_i \neq \varnothing\}$. It suffices to show that $|\mathcal{V}_i| < \kappa$ and that $\mathcal{V} = \cup_{i=0}^{\infty} \mathcal{V}_i$.

(i) We show that $|\mathcal{V}_i| < \kappa$ by induction on i, the result being trivial when $i = 0$. Suppose that $|\mathcal{V}_i| < \kappa$. Then because κ is regular, V_i has a dense subset, say D_i, with $|D_i| < \kappa$. For each $V \in \mathcal{V}_{i+1}$ we have $V \cap V_i \neq \varnothing$ so $V \cap D_i \neq \varnothing$. Again because κ is regular, $\mathcal{V}_{i+1} = \cup_{d \in D_i}\{V \in \mathcal{V} \ / \ d \in V\}$ has cardinality less than κ since $\mathrm{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$.

(ii) A space is connected if and only if for every open cover of the space and for every pair of points of the space there is a finite sequence of members of the open cover such that the first member of the sequence contains the first point, the last member of the sequence contains the last point and each member of the sequence has non-empty intersection with its neighbouring members of the sequence. Let $V \in \mathcal{V}$. Pick any $x \in V_0$ and $y \in V$. Let $\langle W_i \rangle$ be a finite sequence of members of $\mathcal{V}$ such that $x \in W_0$, $y \in W_n$ and $W_{i-1} \cap W_i \neq \varnothing$ for each $i = 1, \ldots n$. We may assume that $W_0 = V_0$ and $W_n = V$. Then for each i, $W_i \in \mathcal{V}_i$. In particular $V \in \mathcal{V}_n$. ∎

Lemma 8 *A point $x \in X$ is a limit point of X if and only if for each collection $\mathcal{V}$ of open sets containing x, with $|\mathcal{V}| \geq \chi(x, X)$, there exists a choice function $f : \mathcal{V} \to X$, such that $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$.*

Proof. $\Rightarrow$: Suppose that $\mathcal{V}$ is a collection of open sets containing x with $|\mathcal{V}| \geq \chi(x, X)$, say $\{V_\alpha \ / \ \alpha < \chi(x, X)\} \subset \mathcal{V}$ satisfies $V_\alpha \neq V_\beta$ whenever $\alpha \neq \beta$. Let $\{W_\alpha \ /\alpha < \chi(x, X)\}$ be a neighbourhood basis at x. Then we may define $f : \mathcal{V} \to X$ so that $f(V) \in V - \{x\}$ if $V \neq V_\alpha$ for any $\alpha < \chi(x, X)$ and $f(V_\alpha) \in V_\alpha \cap W_\alpha - \{x\}$. Then $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$.

$\Leftarrow$: Let U be any neighbourhood of x and take $\mathcal{V}$ to be a collection of open neighbourhhods of x forming a neighbourhood basis at x. Then $|\mathcal{V}| \geq \chi(x, X)$. Let $f : \mathcal{V} \to X$ be a choice function such that $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$. Then $f(U) \in U - \{x\}$, so x is a limit point of X. ∎

3 Finally κ-metacompact Spaces.

Proposition 9 *Let κ be a regular cardinal. Every connected, $[\kappa, \kappa]$-metacompact space of density $< \kappa$ is $[\kappa, \kappa]$-compact.*

Proof. Suppose that X is a connected, $[\kappa, \kappa]$-metacompact space of density $< \kappa$ and let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| = \kappa$. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ such that $\mathrm{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$. As an open subset of a space of density $< \kappa$, each member of $\mathcal{V}$ has density $< \kappa$. By Lemma 7, $|\mathcal{V}| < \kappa$ and hence $\mathcal{U}$ has a subcover of cardinality less than κ. ∎

Let X be a topological space and A a non-empty subset of X. A point $x \in X$ is a *point of complete accumulation* of A if and only if for every neighbourhood N of x we have $|A \cap N| = |A|$.

Proposition 10 *Let κ be a regular cardinal. A space X is $[\kappa, \kappa]$-compact if and only if every $A \subset X$ such that $|A| = \kappa$ has a point of complete accumulation.*

Proof. ($\Rightarrow$). Suppose that $|A| = \kappa$ but for each $x \in X$ there is open $U_x \subset X$ such that $x \in U_x$ and $|U_x \cap A| < \kappa$. Write $A = \{a_\alpha \ / \ \alpha < \kappa\}$. Then for each $x \in X$ there is $\alpha_x < \kappa$ such that $U_x \cap A \subset \{a_\alpha \ / \ \alpha < \alpha_x\}$.

For each $\alpha < \kappa$ let $U_\alpha = \cup\{U_x \ / \ \alpha_x < \alpha\}$. Then $\{U_\alpha \ / \ \alpha < \kappa\}$ is an open cover of X of cardinality κ so has a subcover of cardinality $< \kappa$. As $U_\alpha \subset U_\beta$ whenever $\alpha < \beta$ and κ is regular it follows that $U_\alpha = X$ for some $\alpha < \kappa$. Consider $a_\alpha \in X$. There is $x \in X$ such that $\alpha_x < \alpha$ and $a_\alpha \in U_x$. Then $a_\alpha \in U_x \cap A \subset \{a_\beta \ / \ \beta < \alpha\}$, a contradiction.

($\Leftarrow$). Suppose $\mathcal{U}$ is an open cover of X of cardinality κ, say $\mathcal{U} = \{U_\alpha \ / \ \alpha < \kappa\}$. For each $\alpha < \kappa$ let $V_\alpha = \cup_{\beta<\alpha} U_\beta$. Then $\mathcal{V}$ is also an open cover of X and $\mathcal{V}$ has a subcover of cardinality $< \kappa$ if and only if $\mathcal{U}$ has.

Suppose that $\mathcal{V}$ does not have a subcover of cardinality $< \kappa$. Because κ is regular we may assume that $V_\alpha \neq V_\beta$ whenever $\alpha \neq \beta$; otherwise replace $\mathcal{V}$ by a subfamily indexed by a cofinal subset of κ. Thus for each $\alpha < \kappa$ we may choose $a_\alpha \in V_{\alpha+1} - V_\alpha$ and let $A = \{a_\alpha \ / \ \alpha < \kappa\}$. The set A has cardinality κ so has a point of complete accumulation in X, say x. On the one hand for each open $U \subset X$ for which $x \in U$ we have $|U \cap A| = |A|$. On the other hand we may choose $\alpha < \kappa$ such that $x \in V_\alpha$, in which case V_α is an open set containing x but $|V_\alpha \cap A| = |\alpha| < \kappa = |A|$, a contradiction. Thus $\mathcal{V}$, and hence $\mathcal{U}$, has a subcover of cardinality $< \kappa$. ■

Proposition 11 *Let κ be a regular cardinal. Let X be a space which is not hereditarily finally κ-compact. Then there is a subspace $Y \subset X$ such that $|Y| = \kappa$ and that no subset $Z \subset Y$ of cardinality κ is finally κ-compact.*

Proof. Replacing X by a subspace which is not finally κ-compact if necessary, we may assume that X is not finally κ-compact. Let $\mathcal{U}$ be an open cover of X having no subcover of cardinality $< \kappa$. We may assume that if $|\mathcal{U}| = \lambda$ then no subfamily of $\mathcal{U}$ of cardinality less than λ covers X. We will exhibit a λ-sequence $\langle x_\alpha \rangle$ of distinct points of X such that setting $Y_\alpha = \{x_\beta \ / \ \beta < \alpha\}$ then for every regular cardinal $\mu \leq \lambda$ the subspace $Y_\mu \subset X$ has the properties that $|Y_\mu| = \mu$ and that for every subset $Z \subset Y_\mu$ of cardinality μ there is an open cover of Z so that no subcover has cardinality less than μ. Write $\mathcal{U} = \{U_\alpha \ / \ \alpha < \lambda\}$; we may assume that $\varnothing \notin \mathcal{U}$.

We define increasing $f : \lambda \to \lambda$ such that $U_{f(\alpha)} \not\subset \cup_{\beta<f(\alpha)} U_\beta$ and $\cup_{\beta\leq f(\alpha)} U_\beta = \cup_{\beta\leq\alpha} U_{f(\beta)}$ for each $\alpha < \lambda$.

Set $f(0) = 0$. Now suppose that $\alpha < \lambda$ is such that $f(\beta)$ has been defined for each $\beta < \alpha$. Then

$$\cup_{\beta<\alpha} \cup_{\gamma\leq f(\beta)} U_\gamma = \cup_{\beta<\alpha} \cup_{\gamma\leq\beta} U_{f(\gamma)} = \cup_{\beta<\alpha} U_{f(\beta)}.$$

so cannot be all of X as $\alpha < \lambda$. Thus $\{\delta < \lambda \ / \ U_\delta \not\subset \cup_{\beta<\alpha} \cup_{\gamma\leq f(\beta)} U_\gamma\} \neq \varnothing$; let $f(\alpha)$ be its least member.

We verify that the inductive hypotheses are satisfied. If $\delta < f(\alpha)$ is such that $\delta > f(\beta)$ for all $\beta < \alpha$, then $U_\delta \subset \cup_{\beta<\alpha} \cup_{\gamma\leq f(\beta)} U_\gamma$. Thus

$$\begin{aligned} \cup_{\delta<f(\alpha)} U_\delta &= \cup\{U_\delta \ / \ f(\beta) < \delta < f(\alpha) \text{ for all } \beta < \alpha\} \cup (\cup_{\beta<\alpha} \cup_{\gamma\leq f(\beta)} U_\gamma) \\ &= \cup_{\beta<\alpha} \cup_{\gamma\leq f(\beta)} U_\gamma \end{aligned}$$

and hence $U_{f(\alpha)} \not\subset \cup_{\beta<f(\alpha)}U_\beta$. Furthermore,

$$\begin{aligned}\cup_{\beta\le f(\alpha)}U_\beta &= U_{f(\alpha)} \cup (\cup_{\gamma<\alpha} \cup_{\beta\le f(\gamma)} U_\beta)\\ &= U_{f(\alpha)} \cup (\cup_{\gamma<\alpha} \cup_{\beta\le\gamma} U_{f(\beta)})\\ &= \cup_{\beta\le\alpha}U_{f(\beta)}.\end{aligned}$$

Now for each $\alpha<\lambda$ choose any $x_\alpha \in U_{f(\alpha)} - \cup_{\beta<\alpha}U_{f(\beta)}$ and let $Y_\alpha = \{x_\beta \;/\; \beta < \alpha\}$. Then $|Y_\mu| = \mu$ for any cardinal $\mu \le \lambda$. If in addition μ is a regular cardinal and $Z \subset Y_\mu$ is of cardinality μ then $\{U_{f(\alpha)} \;/\; x_\alpha \in Z\}$ is an open cover of Z having no subcover of cardinality less than μ. ∎

Proposition 12 (cf [1, theorem 4.1]) *Let κ be a regular cardinal. Every locally hereditarily finally κ-compact, $[\kappa,\kappa]$-compact space is hereditarily finally κ-compact.*

Proof. Suppose that X is a locally hereditarily finally κ-compact, $[\kappa,\kappa]$-compact space which is not hereditarily finally κ-compact. Then by Proposition 11 there is a subspace $Y \subset X$ such that $|Y| = \kappa$ and each subset of Y of cardinality $<\kappa$ is not finally κ-compact.

As X is $[\kappa,\kappa]$-compact by Proposition 10 there is a point $a \in X$ of complete accumulation of Y. Let U be a hereditarily finally κ-compact neighbourhood a. Because $U \cap Y$ has cardinality κ it cannot be finally κ-compact by choice of Y. On the other hand $U \cap Y$ is a subset of the hereditarily finally κ-compact space U so must be finally κ-compact. ∎

Theorem 13 *Let κ be a regular cardinal. Suppose that X is a space which is of character $<\kappa$, is locally connected, locally hereditarily finally κ-compact and locally hereditarily of density $<\kappa$. If X is $[\kappa,\kappa]$-metacompact then X is finally κ-metacompact.*

Proof. As X is locally connected, every component is open so by looking at each component separately if necessary we may assume that X is connected also. We construct a strongly increasing κ-sequence $\langle V_\alpha\rangle$ of non-empty, connected, open and finally κ-compact subsets of X.

Because X is locally connected and locally hereditarily finally κ-compact we may begin by choosing any non-empty, connected, open, finally κ-compact subset $V_0 \subset X$. For any other limit ordinal α, if V_β has already been constructed for all $\beta<\alpha$, let $V_\alpha = \cup_{\beta<\alpha}V_\beta$.

Suppose that V_α has been constructed. Because V_α is finally κ-compact it also has a dense subset of cardinality $<\kappa$. Thus $\bar{V}_\alpha$ has a dense subset of cardinality $<\kappa$. $\bar{V}_\alpha$ is also connected as V_α is. Furthermore, as a closed subset of a $[\kappa,\kappa]$-metacompact space $\bar{V}_\alpha$ is also $[\kappa,\kappa]$-metacompact. Thus by Proposition 9 $\bar{V}_\alpha$ is $[\kappa,\kappa]$-compact. It now follows from Proposition 12 that $\bar{V}_\alpha$ is finally κ-compact. For each $x \in \bar{V}_\alpha - V_\alpha$ choose $U_x \subset X$ open and finally κ-compact such that $x \in U_x$. Then $\{U_x \;/\; x \in \bar{V}_\alpha - V_\alpha\}$ is an open cover of the finally κ-compact subset $\bar{V}_\alpha - V_\alpha$ so has a subcover of cardinality $<\kappa$. The collection consisting of this subcover together with V_α is a collection of fewer than κ many open finally

κ-compact subsets of X so their union is also open and finally κ-compact and contains $\bar{V}_\alpha$. Let $V_{\alpha+1}$ be the component of this union containing V_α.

Suppose that $\mathcal{U}$ is an open cover of X. Then for each $\alpha < \kappa, \mathcal{U}$ is also an open cover of the finally κ-compact set V_α: let $\mathcal{U}_\alpha$ be a subcover of cardinality $< \kappa$. Then $\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ is a subfamily of $\mathcal{U}$ of cardinality at most κ which covers $\cup_{\alpha<\kappa}V_\alpha$, hence the connected space X, by Lemma 6 because this union is non-empty, open and closed. As X is $[\kappa,\kappa]$-metacompact it follows that this subfamily has an open refinement whose order at each point is less than κ and hence so does $\mathcal{U}$. Thus X is finally κ-metacompact. ∎

Remark. The three local properties 'of character $< \kappa$, locally hereditarily finally κ-compact and locally hereditarily of density $< \kappa$' of Theorem 13 are all implied by the single local property: locally of weight $< \kappa$. In the case where $\kappa = \omega_1$ these four properties are, respectively, first countable, locally hereditarily Lindelöf, locally hereditarily separable and locally second countable.

Proof of the equivalence of Theorem 5 (a)-(e)
Every manifold is T_3, connected, first countable, locally connected and locally hereditarily Lindelöf so we may conclude (c)$\Rightarrow$(a) from Theorem 13.

As every metrisable space is paracompact, it is also linearly metaLindelöf so (a)$\Rightarrow$(b) in Theorem 5. Proposition 14 concludes the proof of equivalence of (a), (b) and (c) in Theorem 5 while Proposition 15 shows the equivalence of these properties with (d) and (e). ∎

Proposition 14 (cf [1]) *Every linearly (meta)Lindelöf space is ω_1-(meta)Lindelöf.*

Proof. We will just consider the Lindelöf case. Let $\mathcal{U}$ be an open cover of the linearly Lindelöf space X such that $|\mathcal{U}| = \omega_1$. Then we can write $\mathcal{U} = \{U_\alpha \ / \ \alpha < \omega_1\}$. For each $\alpha < \omega_1$ let $V_\alpha = \cup\{U_\beta \ / \ \beta < \alpha\}$. Then $\{V_\alpha \ / \ \alpha < \omega_1\}$ is an open cover of X which is a chain. Thus as X is linearly Lindelöf it follows that there is a countable subcover, hence there is $\alpha < \omega_1$ such that $V_\alpha = X$. Then $\{U_\beta \ / \ \beta < \alpha\}$ is a countable subcover of $\mathcal{U}$. ∎

Proposition 15 [3, Lemma 3.2] *A locally hereditarily separable space is (linearly) [ω_1-] metaLindelöf if and only if it is nearly (linearly) [ω_1-]metaLindelöf.*

Proof. We need only show the reverse implication. Suppose that $\mathcal{U}$ is an open cover of X (which is a chain) [with $|\mathcal{U}| \leq \omega_1$]. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ and $D \subset X$ a dense subset such that $\mathcal{V}$ is point countable at each point of D. We show that $\mathcal{V}$ is point countable.

Let $x \in X$ be arbitrary. Choose a hereditarily separable open neighbourhood $O \subset X$ of x. Let E be a countable subset of $D \cap O$ which is still dense in O. Choose a function $\varphi : \mathcal{V}_x = \{V \in \mathcal{V} \ / \ x \in V\} \to E$ as follows: for each $V \in \mathcal{V}_x$ we have $E \cap V \cap O \neq \varnothing$; let $\varphi(V) = e$ for some $e \in E \cap V \cap O$. Because $\mathcal{V}_e$ is countable it follows that $\varphi^{-1}(e)$ is countable for each $e \in E$. Thus $\mathcal{V}_x = \varphi^{-1}(E)$ is also countable, i.e. $\mathcal{V}$ is point countable. ∎

4 Spaces with Property pp.

Lemma 16 *Let $\mathcal{V}$ be an open cover of a T_1 space X. Then the following are equivalent:*

(a) For every choice function $f : \mathcal{V} \to X$, the set $f(\mathcal{V})$ is closed and discrete;

(b) For every choice function $f : \mathcal{V} \to X$, the set $f(\mathcal{V})$ is closed;

(c) For every choice function $f : \mathcal{V} \to X$, the set $f(\mathcal{V})$ is discrete.

Proof. It suffices to show that (b) and (c) are equivalent.
(b)$\Rightarrow$(c). Suppose that $f : \mathcal{V} \to X$ is a choice function but $f(\mathcal{V})$ is not discrete. Then there is $x \in f(\mathcal{V})$ every neighbourhood of which meets $f(\mathcal{V})$ in some point other than x. Define $g : \mathcal{V} \to X$ by $g(V) = f(V)$ if $f(V) \neq x$ and $g(V) \in V - \{x\}$ if $f(V) = x$. Then $x \in \overline{g(\mathcal{V})} - g(\mathcal{V})$ so $g(\mathcal{V})$ is not closed.
(c)$\Rightarrow$(b). Suppose that $f : \mathcal{V} \to X$ is a choice function but $f(\mathcal{V})$ is not closed, say $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$. Pick $V_x \in \mathcal{V}$ such that $x \in V_x$. Define $g : \mathcal{V} \to X$ by $g(V) = f(V)$ unless $V = V_x$ and let $g(V_x) = x$. Because X is T_1 it follows that every neighbourhood of x meets $g(\mathcal{V})$ in some point other than x so $g(\mathcal{V})$ is not discrete. ∎

Proposition 17 *Let κ be a cardinal. Suppose that X has character at most κ and has no isolated points, and that every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = \kappa^+$ has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to X$ the set $f(\mathcal{V})$ is closed. Then X is $[\kappa^+, \kappa^+]$-metacompact.*

Proof. Let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| = \kappa^+$. Apply Lemma 8 to the open refinement $\mathcal{V}$ given by hypothesis: then $\mathrm{ord}(x, \mathcal{V}) < \kappa < \kappa^+$ for each $x \in X$. ∎

We can now complete the proof of Theorem 5.

By Lemma 16 (f), (g) and (h) are equivalent. By Proposition 17 with $\kappa = \omega$, (g) implies (c). Finally every metrisable manifold is pp and hence satisfies (f).

5 Some Questions.

Are there even weaker covering conditions which are equivalent to metrisability for a manifold?

In [2] we find that the following conditions are each equivalent to metrisability for a manifold:

- M is normal and θ-refinable;
- M is normal and subparacompact.

Let X be a space.

X is *θ-refinable* ([9]) if every open cover can be refined to an open θ-cover, ie a cover $\mathcal{U}$ which can be expressed as $\cup_{n\in\omega}\mathcal{U}_n$ where each $\mathcal{U}_n$ covers X and for each $x \in X$ there is n such that $ord(x, \mathcal{U}_n) < \omega$.

X is *subparacompact*, [5] (where it is called F_σ-screenable), if every open cover has a σ-discrete closed refinement.

Question 18 *Must a manifold be metrisable if it is normal and every open cover of cardinality at most ω_1 has an open θ-refinement?*

Question 19 *Must a manifold be metrisable if it is normal and every open cover of cardinality at most ω_1 has a σ-discrete closed refinement?*

Our theme suggests the following definition.

Definition 20 *Say that X is* $[\omega_1, \omega_1]$-θ-refinable *if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = \omega_1$ has a θ-refinement.*

Clearly every ω_1-metaLindelöf space is $[\omega_1, \omega_1]$-θ-refinable as we have $\mathrm{ord}(x, \mathcal{V}) < \omega_1$ for each $x \in X$ if $\mathcal{V}$ is a point-countable open cover of X.

Question 21 *Is every $[\omega_1, \omega_1]$-θ-refinable manifold θ-refinable?*

A positive answer to this question would mean that a manifold is metrisable if and only if it is normal and $[\omega_1, \omega_1]$-θ-refinable.

One way to prove that a metaLindelöf manifold is metrisable is to use the fact that every connected, locally separable, metaLindelöf space is Lindelöf using Lemma 7. It is a simple matter to improve this to the following.

Proposition 22 *Let κ be a regular cardinal. Then any connected and finally κ-metacompact space which is locally of density $< \kappa$ is finally κ-compact.*

Comparing this with Proposition 9 leads to the following question.

Question 23 *Let κ be a regular cardinal. Must every connected and $[\kappa, \kappa]$-metacompact space which is locally of density $< \kappa$ be $[\kappa, \kappa]$-compact?*

Note that in Proposition 17 we have only concluded that X is $[\kappa^+, \kappa^+]$-metacompact rather than $[\kappa, \kappa^+]$-metacompact even though the open cover of size κ^+ has been refined to an open cover of order less than κ: we did not carry out a similar reduction of an open cover of cardinality κ because we did not need to. This raises the following question.

Question 24 *Is there a space X with character at most κ and having no isolated points such that every open cover of size κ^+ has an open refinement $\mathcal{V}$ whose order at each point is less than κ but X is not $[\kappa, \kappa^+]$-metacompact?*

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