Developable Spaces and Problems of Fletcher and Lindgren*

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Abstract

In this paper, we answer two questions of P. Fletcher and W. Lindgren [1]. We prove that a space X is $w\Delta$ and has a quasi–G_δ^*–diagonal if and only if it is developable, a space X is β–space with a quasi–G_δ^*–diagonal if and only if it is semi–stratifiable, a space X is β, quasi–γ–space and has a quasi–G_δ^*–diagonal if and only if X is developable and a space X is metrizable if and only if it is paracompact β–space with a quasi–G_δ–diagonal.

1 Introduction

In this brief note we present some conditions which imply developability, and consequently we give full positive answers for two questions of Fletcher and Lindgren [1]: is every quasi–developable space c–semi–stratifiable and is every quasi–developable β–space developable?

In [12], the author makes it possible to factorize quasi–developability into two parts: a space X is quasi developable if and only if it is a quasi–$w\Delta$–space with a quasi–G_δ^*–diagonal. This result plays an important role in getting the results in this paper.

A COC-map (= countable open covering map) for a topological space X is a function from $N \times X$ into the topology of X such that for every $x \in X$ and $n \in \mathbb{N}, x \in g(n,x)$ and $g(n+1,x) \subseteq g(n,x)$. It is well known that many important classes of generalized metrizable spaces can be characterized in

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terms of a COC-map. In particular, X is developable [4]($w\Delta$–space) if and only if X has a COC-map g such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all n, then $\langle x_n \rangle$ converges to p (then $\langle x_n \rangle$ has a cluster point).

A space X is called quasi–γ [9] if and only if X has a COC-map g such that if $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, and the sequence $\langle y_n \rangle$ converges in X, then the sequence $\langle x_n \rangle$ has a cluster point; a space X is called semi–stratifiable [6] (β–space [5]) if and only if X has a COC-map g such that if for each $x \in g(n, x_n)$ for each $n \in \mathbb{N}$ then x is a cluster point of $\langle x_n \rangle$ ($\langle x_n \rangle$ has a cluster point).

Let $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a sequence of open families of X. Define $c(x) = \{n : x \in \mathcal{G}_n^* = \bigcup\{G : G \in \mathcal{G}_n\}\}$. A space X has a quasi–G_δ^*–diagonal [12] if there is such a sequence $\mathcal{G}$ such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}$; a space X is called a quasi–$w\Delta$–space [12] if X has such a sequence $\mathcal{G}$ such that

(1) for all x, $c(x)$ is infinite.

(2) if $\langle x_n \rangle$ is a sequence with $x_n \in st(x, \mathcal{G}_n)$ for all $n \in c(x)$ then $\langle x_n \rangle$ has a cluster point.

A space X is called c–semi–stratifiable [9] if there a sequence $\langle g(n, x) \rangle$ of open neighborhoods of x such that for each compact set $K \subset X$, if $g(n, K) = \bigcup\{g(n, x) : x \in K\}$, then $\bigcap\{g(n, K) : n \geq 1\} = K$. The COC–map $g : \mathbb{N} \times X \to \tau$ is called a c–semi–stratification of X. All spaces will be regular, unless we state otherwise.

2 Main Results

Theorem 2.1 *A space X is β–space with a quasi–G_δ^*–diagonal if and only if it is semi–stratifiable.*

Proof. The necessity of the condition is obvious because every regular semi–stratifiable space is β and has a G_δ^*–diagonal. To prove the sufficiency of the condition, let $\langle \mathcal{V}_n : n \in \mathbb{N} \rangle$ be a quasi–G_δ^*–diagonal sequence of X and let $g : \mathbb{N} \times X \to \tau$ be a β–map of X. Set $c_{\mathcal{V}}(x) = \{n : st(x, \mathcal{V}_n) \neq \emptyset\}$. Then $\bigcap_{n\in c_{\mathcal{V}}(x)} \overline{st(x, \mathcal{V}_n)} = \{x\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$, and for $F \in \mathcal{F}$ put

$$\mathcal{G}_F = \{\bigcap\nolimits_{i\in F} V_i : V_i \in \mathcal{V}_i\}.$$

For $n \in \mathbb{N}$ and $F \in \mathcal{F}$, set $F_n(x) = c_{\mathcal{V}}(x) \cap \{1, 2, ..., n\}$. Then $st(x, \mathcal{G}_{F_n}) \subseteq st(x, \mathcal{V}_m)$ for each $n \in \mathbb{N}$, each $F \in \mathcal{F}$ and each $m \in F$. Put $d(x) = \{F_n(x) :$

$n \in \mathbb{N}\}$. Note that $d(x) \subseteq c_{\mathcal{G}}(x)$. Since $c_{\mathcal{V}}(x)$ is infinite, $d(x)$ is infinite. Becuse $F_n \subseteq F_m$ for $m \geq n$, $st(x, \mathcal{G}_{F_m}) \subseteq st(x, \mathcal{G}_{F_n})$ for $m \geq n$.

Define a map $h : \mathbb{N} \times X \to \tau$ by

$$h(n,x) = \begin{cases} g(n,x) \cap st(x, \mathcal{G}_{F_n}) & \text{if } x \in \mathcal{G}^*_{F_n}. \\ g(n,x) & \text{if } x \notin \mathcal{G}^*_{F_n} \end{cases}.$$

We prove that $h(n,x)$ is a semistratifiable–map. Let $x \in h(n, x_n)$. It is clear that h is a β–map, so, $\langle x_n \rangle$ has a cluster point, say p. Suppose that $x \neq p$. Choose k large enough that $x \in \overline{st(x, \mathcal{V}_k)}$ but $p \notin \overline{st(x, \mathcal{V}_k)}$.

For each $n \geq k$, we have $k \in F_n$ so

$$x_n \in h(n,x) \subset st(x, \mathcal{G}_{F_n}) \subseteq st(x, \mathcal{V}_k).$$

Thus the open neighborhood $X - \overline{st(x, \mathcal{V}_k)}$ of p contains at most $k-1$ members of the sequence $\langle x_n : n \in \mathbb{N} \rangle$, which contradicts the fact that p is a cluster point of $\langle x_n \rangle$. ■

Corollary 2.2 *Every quasi–developable β–space is developable.*

Proof. This follows from [12, Theorem 3.1 (a space is a quasi–developable if and only if it is a quasi–$w\Delta$–space with quasi–G^*_δ–diagonal)], Theorem 2.1 and since every semistratifiable space is perfect the proof done. ■

Corollary 2.3 *A space X is $w\Delta$ and has a quasi–G^*_δ–diagonal if and only if it is developable.*

Proof. This follows from [12, Theorem 3.1], Corollary 2.2 and since every $w\Delta$–space is β–space. ■

From [12, Corollary 3.4 (Every paracompact $w\Delta$–space with quasi–G_δ-diagonal is metrizable)], and Theorem 2.1 we have the following metrization result:

Corollary 2.4 *A space X is metrizable if and only if it is paracompact β–space with a quasi–G_δ–diagonal.*

Lemma 2.5 *Let X be a space with a quasi–G^*_δ–diagonal sequence. Then X has a quasi–G^*_δ–diagonal sequence $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ such that for each $x \in X$ there is an infinite subset $d(x) \subseteq c_{\mathcal{G}}(x)$ such that if $x_n \in st(x, \mathcal{G}_n)$ for each $n \in d(x)$ then $\langle x_n \rangle$ either clusters at x or it does not cluster at all.*

Proof. Let $\langle \mathcal{H}_n : n \in \mathbb{N} \rangle$ be a quasi–G_δ^*–diagonal sequence of X. Without loss of generality we may assume that $c_{\mathcal{H}}(x)$ is infinite for each $x \in X$ and $\mathcal{H}_1 = \{X\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ set

$$\mathcal{G}_F = \{\bigcap\nolimits_{i \in F} H_i : H_i \in \mathcal{H}_i\}.$$

For $n \in \mathbb{N}$ and $x \in X$, set $F_n(x) = c_{\mathcal{H}}(x) \cap \{1, 2, ..., n\}$. Put $d(x) = \{F_n(x) : n \in \mathbb{N}\}$. Note that $d(x) \subseteq c_{\mathcal{G}}(x)$. Since $c_{\mathcal{H}}(x)$ is infinite, $d(x)$ is infinite. Because $F_n(x) \subseteq F_m(x)$ for $m \geq n$, $st(x, \mathcal{G}_{F_m(x)}) \subseteq st(x, \mathcal{G}_{F_n(x)})$ for $m \geq n$.

For each $n \in \mathbb{N}$ suppose that $x_n \in st(x, \mathcal{G}_{F_n(x)})$. Then for $m \geq n$ we have

$$x_m \in st(x, \mathcal{G}_{F_m(x)}) \subset st(x, \mathcal{G}_{F_n(x)}),$$

so
$$\overline{\{x_m \ / \ m \geq n\}} \subset \overline{st(x, \mathcal{G}_{F_n(x)})}.$$

Since $\cap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_{F_n(x)})} = \{x\}$ it follows that either $\langle x_n \rangle$ clusters at x or does not cluster at all. ■

Remark 2.6 *Let X be a space and $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ a countable family of collections of open subsets of a space X, such that for all x, $c(x) = \{n \in \mathbb{N} : x \in \mathcal{G}_n^*\}$ is infinite. Consider the following condition on $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$: if $\langle x_n : n \in \mathbb{N} \rangle$ is a sequence with $x_n \in st(x, \mathcal{G}_n)$ for all $n \in c(x)$ then x is a cluster point of $\langle x_n : n \in \mathbb{N} \rangle$.*

For all spaces, this condition is equivalent to the following condition: for each point $x \in X$ the set $st(x, \mathcal{G}_n)$ is nonempty for infinitely many n and the nonempty sets of the form $st(x, \mathcal{G}_n)$ form a local base at x for all $x \in X$. Thus the condition (2) *is a characterization of a quasi–developable space.*

Theorem 2.7 *Every quasi–developable space is a c–semi–stratifiable space.*

Proof. Let $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ be a quasi–development sequence in a space X. Define

$$g(n, x) = \begin{cases} st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^*. \end{cases}$$

Let $h(n, x) = \bigcap_{i=1}^n g(i, x)$. We prove that $h(n, x)$ is a c–semi–stratifiable–map. We claim that $C = \cap_{n \in \mathbb{N}} h(n, C)$ for any compact $C \subset X$, where $h(n, C) = \bigcup_{c \in C} h(n, c)$. As $\mathcal{G}_1 = \{X\}$ it follows readily that $C \subset \cap_{n \in \mathbb{N}} h(n, C)$ so it is appropriate concentrate on the reverse inclusion. To prove that, let $y \in \bigcap h(n, C)$, so $y \in h(n, c_n)$ for some $c_n \in C$. Then $y \in st(c_n, \mathcal{G}_n)$ for infinitely many $n \in \mathbb{N}$. It follows that $c_n \in st(y, \mathcal{G}_n)$ for infinitely many $n \in \mathbb{N}$. From Remark 2.6, $\langle c_n \rangle$ clusters at y. Hence, $y \in C$. ■

Theorem 2.8 *A space X is developable if and only if it is β, quasi–γ and has a quasi–G_δ^*–diagonal.*

Proof. The necessity of the conditions is obvious. To prove the sufficiency of the conditions, let f be a β–map and g a quasi–γ–map of X. Define $h(n,x) = f(n,x) \cap g(n,x)$. It is clear that h is a β and quasi–γ–map of X. We prove that h is a $w\Delta$–map of X. Let $\{x, x_n\} \subset h(n, y_n)$. By the β–condition, $\langle y_n \rangle$ converges and so by the quasi–γ–condition $\langle x_n \rangle$ has a cluster point. Thus h is $w\Delta$–map of X. From Corollary 2.3, X is a developable space. ■

Now, it is natural to ask:

Question 2.9 *Is every quasi–$w\Delta$–space with G_δ^*–diagonal developable?*

We answer this question in negative manner.

Example 2.10 *There is a p–adic analytic manifold which is separable, submetrizable, quasi–developable, but not perfect ([11, Example 7.4.7]). This example also can serve as a quasi–semi–stratifiable space (see [7] for the definition) which has a G_δ^*–diagonal but which is not semi–stratifiable.* ■

Example 2.11 *There is a quasi–developable manifold which has a G_δ–diagonal but not a G_δ^*–diagonal (see [3, Example 2.2]) This example also can serve as a quasi–$w\Delta$ manifold which is not $w\Delta$. (It is not even a β–manifold).* ■

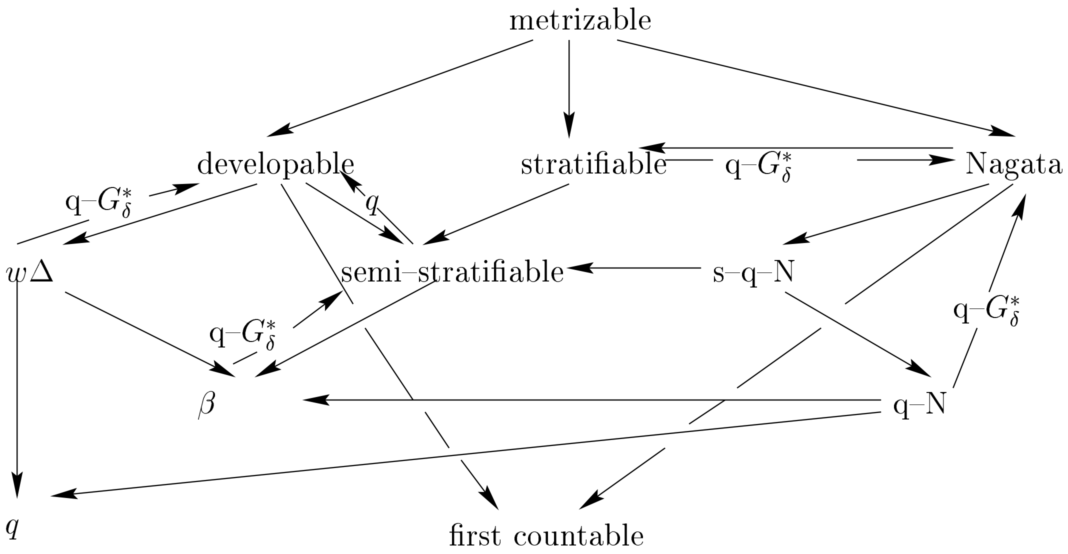

Figure 1: Relationships between some generalized metric spaces and quasi–G_δ^*–diagonal.

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