Metrizability of Manifolds by Diagonal Properties*

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Abstract

This paper investigates metrization theory of manifolds. We show that diagonal properties play a central role in developing metrizability of manifolds.

1 Introduction

A significant question in topology is that of deciding when a topological space is metrizable, there being many criteria which have now been developed to answer the question. Among the most natural is the following: a topological space is metrizable if and only if it is paracompact, Hausdorff and locally metrizable. Note that manifolds are always Hausdorff and locally metrizable so this criterion gives a criterion for the metrizability of a manifold, viz that a manifold is metrizable if and only if it is paracompact. Many other metrization criteria have been discovered for manifolds, as seen by Theorem 2 [10], which lists criteria which require at least some of the extra properties possessed by manifolds.

The question whether perfect normality is equivalent to metrizability for a manifold is an old one, dating back to [2]. It was shown in [21] that under MA+$\neg$CH the two conditions are equivalent. On the other hand in [23] there is constructed an example of a perfectly normal non-metrizable manifold under CH. The same situation prevails when we consider strong hereditary separability. In [14] it is shown that under MA+$\neg$CH every strongly hereditarily separable space is Lindelöf. On the other hand even when we combine the two notions the resulting manifold need not be metrizable in general; in [11] there is constructed under CH a manifold which is strongly hereditarily separable and perfectly normal but not metrizable.

In this paper we study diagonal properties in manifolds to arrive at developability and metrizability. We show that the following conditions are each equivalent to a manifold being metrizable:

- it is perfectly normal and has a quasi-G^*_δ-diagonal;
- it is separable and has an S_2–diagonal with property $(*)$;

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- it is separable, hereditarily normal and has an S_2–diagonal;
- it is separable and has a point finite S_2–diagonal;
- it is separable and has a point countable quasi–G^*_δ–diagonal;
- it is R–perfect with a quasi–G^*_δ–diagonal;
- it has a quasi–regular–G_δ-diagonal.

Let $\mathcal{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be a countable family of collections of subsets of a space X. Consider the following conditions on $\mathcal{G}$:

(a) for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a collection of open sets in X;

(b) each $\mathcal{G}_n$ is a cover of X;

(c) for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in st(x, \mathcal{G}_n) \subset X - \{y\};$$

(d) for each $x \in X$ and $n \in \mathbb{N}$, $st(x, \mathcal{G}_n)$ is an open subset of X;

(e) for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\};$$

(f) for each $x \in X$, $\{st(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in \mathcal{G}^*_n\}$ is a local base at x;

Definition 1.1 *X has a* **quasi–G^*_δ–diagonal** *if there exists a family $\mathcal{G}$ satisfying (a) and (e). Recall that X has a* **quasi–G_δ–diagonal** *if there exists a family $\mathcal{G}$ satisfying (a)and (c); X has a* **G_δ–diagonal** *if there exists a family $\mathcal{G}$ satisfying (a), (b) and (c); X has a* **G^*_δ–diagonal** *if there exists a family $\mathcal{G}$ satisfying (a), (b) and (e); X has an* **S_1–diagonal** *if there exists a family $\mathcal{G}$ satisfying (b), (c) and (d); X has an* **S_2–diagonal** *if there exists a family $\mathcal{G}$ satisfying (b), (d) and (e).*

Definition 1.2 *A space X is* **developable** *[***quasi–developable***] if there exists a family $\mathcal{G}$ satisfying (a), (b) and (f) [(a) and (f)]. A space X is* **o–semi–developable** *if there exists a family $\mathcal{G}$ satisfying (b), (d) and (f). A space X is* **semi–developable** *if there exists a family $\mathcal{G}$ satisfying (b) and (f).*

Let X be a space, and (A, B) a pair of subsets of X. A collection $\mathcal{U}$ of (open) (closed) subsets of X is **separating** [respectively, **strongly separating**] (open) (closed) for (A, B) if, given distinct points $x_0 \in A$ and $x_1 \in B$, there is $U \in \mathcal{U}$ such that $x_0 \in U$ but $x_1 \notin U$ [respectively, there is $U \in \mathcal{U}$ so that $x_0 \in U$ while $x_1 \notin \overline{U}$]. A subset, C, of a space X, will be called an **S_δ–subset** (respectively, **R_δ–subset**) if there is a countable open separating (respectively, strongly separating) family for the pair $(C, X - C)$.

A space, X, has an **S_δ–diagonal** (respectively, an **R_δ–diagonal**) if the diagonal in X^2 is an S_δ subset (respectively, an R_δ subset).

Recall that a subset H of the space X is a **regular G_δ–set** if there is a sequence $\langle U_n : n \in \mathbb{N}\rangle$ of open sets in X such that $H = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U}_n$. We say that X has a **regular–G_δ–diagonal** [12] if $\Delta = \{(x, x) : x \in X\}$ is a regular G_δ–set in X^2 .

2 Quasi–Developable Manifolds

Theorem 2.1 *Let X be locally compact and locally connected. If X has a quasi–G^*_δ–diagonal, then X is quasi–developable.*

Proof.
Let $\langle \mathcal{H}_n : n \in \mathbb{N}\rangle$ be a quasi–G^*_δ diagonal sequence for X.

Set $c_{\mathcal{H}}(x) = \{n : st(x, \mathcal{H}_n) \neq \emptyset\}$. Then $\bigcap_{n \in c_{\mathcal{H}}(x)} \overline{st(x, \mathcal{H}_n)} = \{x\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ put

$$\mathcal{G}_F = \{\bigcap_{i \in F} H_i : H_i \in \mathcal{H}_i\}.$$

Set $\mathcal{G}_F{}' = \{U : U$ is a component of G for some $G \in \mathcal{G}_F\}$. By local connectedness, all the sets in $\mathcal{G}_F{}'$ are open. We show that $\{\mathcal{G}_F{}' : F \in \mathcal{F}\}$ is a quasi–development of X. Suppose $x \in X$ and U is open such that $x \in U$ and $\overline{U}$ is compact. For each $n \in \mathbb{N}$ put $F_n = c_{\mathcal{H}}(x) \cap \{1, 2, ..., m\}$, where $m \geq n$ is minimal such that $\{1, 2, ..., m\} \cap c_{\mathcal{H}}(x) \neq \emptyset$. We claim that $\overline{st(x, \mathcal{G}_{F_n}{}')} \subseteq U$ for some n. Otherwise, since each $\overline{st(x, \mathcal{G}_{F_n}{}')}$ is connected, $\overline{st(x, \mathcal{G}_{F_n}{}')} \cap \partial U \neq \emptyset$ for each $n \in \mathbb{N}$. But ∂U is compact and $\overline{st(x, \mathcal{G}_{F_n}{}')} \cap \partial U$ is a decreasing sequence of non–empty closed sets, so $\partial U \cap \bigcap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_{F_n}{}')} \neq \emptyset$. But $\bigcap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_{F_n}{}')} \subseteq \bigcap_{n \in c_{\mathcal{H}}(x)} \overline{st(x, \mathcal{H}_n)} = \{x\}$. Hence $x \in \partial U$, a contradiction. ∎

From Theorem 2.1 and [4, Theorem 1.8] we get the following result:

Corollary 2.2 *Every locally compact, locally connected, R–perfect space with a quasi–G^*_δ–diagonal is metrizable.*

In 1935 Alexandroff, then in 1949 Wilder, asked questions which led topologists to investigate whether every perfectly normal manifold is metrizable. Rudin and Zenor [21], [23] showed that the answer depends on the set theory. We show that addition of the condition that the manifold has a quasi-G^*_δ-diagonal suffices to ensure metrizability.

Theorem 2.3 *A manifold is metrizable if and only if it is perfectly normal and has a quasi-G^*_δ-diagonal.*

Proof. Every metrizable space is perfectly normal and has a quasi-G^*_δ-diagonal. The converse follows from Reed and Zenor's metrization theorem [19], [12, Theorem 2.15] and Theorem 2.1. ∎

There are manifolds which are normal but not metrizable, for example the long ray. There are also manifolds which are separable but not metrizable, for example the Prüfer manifold (see Example 5.1 for a description of a version of the Prüfer manifold). There are even manifolds which are both normal and separable but not metrizable, [22].

3 Separable Manifolds With S_2–Diagonals

Definition 3.1 *A space X has S_2–diagonal with property $(*)$ if X has an S_2-sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ satisfying the following additional condition: for each $x, y \in X, y \in \overline{st(x, \mathcal{G}_n)}$ if and only if $x \in \overline{st(y, \mathcal{G}_n)}$, for all $n \in \mathbb{N}$.*

Balogh and Bennett [4] proved that a locally compact, locally connected space with a point countable strongly separating open cover is metrizable. We give here an application of this result.

Theorem 3.2 *A separable, locally compact, locally connected space with S_2–diagonal with property $(*)$ is metrizable.*

Proof. By Balogh and Bennett's result it suffices to show that if X is a separable space having an S_2–diagonal with property $(*)$ then X has a point–countable strongly separating open cover.

Let D be a countable dense subset of X and let $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ be an S_2–sequence with property $(*)$. Let $\mathcal{C} = \{\bigcap_{i=1}^{n} st(d, \mathcal{G}_i) : n \in \mathbb{N} \text{ and } d \in D\}$. Then $\mathcal{C}$ is a countable open cover of X. It is claimed that $\mathcal{C}$ is strongly separating.

Suppose $x, y \in X$ with $x \neq y$. Then there exist $l, m \in \mathbb{N}$ such that $x \notin \overline{st(y, \mathcal{G}_l)}$ and $y \notin \overline{st(x, \mathcal{G}_m)}$. Let $n = \max\{l, m\}$. Consider

$$U = \bigcap_{i=1}^{n} st(x, \mathcal{G}_i) - \overline{st(y, \mathcal{G}_l)}.$$

Then U is a non–empty open subset of X so we may choose $d \in D \cap U$. Then $d \in \bigcap_{i=1}^{n} st(x, \mathcal{G}_i)$ so $x \in \bigcap_{i=1}^{n} st(d, \mathcal{G}_i)$. If $y \in \overline{\bigcap_{i=1}^{n} st(d, \mathcal{G}_i)}$ then $y \in \overline{st(d, \mathcal{G}_l)}$ and hence by property $(*)$, $d \in \overline{st(y, \mathcal{G}_l)}$ contradicting $d \in U$. Thus $\bigcap_{i=1}^{n} st(d, \mathcal{G}_i)$ is a member of $\mathcal{C}$ containing x but whose closure does not contain y and this completes the proof. ■

Corollary 3.3 *A manifold is metrizable if and only if it is separable and has S_2–diagonal with property $(*)$.*

Remark 3.4 *In 1984 P.Nyikos [16] constructed an example of a separable Moore manifold which is not metrizable.*

By using the same technique as in Theorem 3.2 and applying [4, Corollary 1.5], we have the following theorem.

Theorem 3.5 *A manifold is metrizable if and only if it is separable and hereditarily normal and has S_2–diagonal.*

In 1993 D. Gauld [11] constructed in CH a strongly hereditarily separable, nonmetrizable manifold. We show that by adding the condition that the manifold has a point finite S_2–diagonal to separability suffices to ensure metrizability.

Definition 3.6 *A point finite sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of covers of space X is called a point finite S_2–diagonal (semi–development) if and only if the sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ is S_2–diagonal (semi–development). A space with a point finite semi–development is called a point finite semi–developable space.*

Definition 3.7 *A point countable sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of collections of open subsets of space X is called a point countable quasi–G_δ^*–diagonal (quasi–development) if and only if the sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ is quasi–G_δ^*–diagonal (quasi–development). A space with a point countable quasi–development is called a point countable quasi–developable space.*

Lemma 3.8 *If $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ is an S_2-sequence for a space X and $x_n \in st(x, \mathcal{G}_n)$ for each n and fixed $x \in X$, then either x is a cluster point of $\langle x_n : n \in \mathbb{N} \rangle$ or $\langle x_n : n \in \mathbb{N} \rangle$ does not cluster at all.*

Proof. There is no loss of generality if we assume that $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ is a decreasing sequence (i.e. $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for all $n \in \mathbb{N}$) . Then, $\overline{\{x_m : m \geq n\}} \subset \overline{st(x, \mathcal{G}_n)}$. Since $\bigcap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_n)} = \{x\}$, either x is a cluster point of $\langle x_n : n \in \mathbb{N} \rangle$ or $\langle x_n : n \in \mathbb{N} \rangle$ does not cluster at all. ■

In the following theorem we use the same technique as in [12, Theorem 2.15].

Theorem 3.9 *A manifold is hereditarily separable and metrizable if it is separable and has a point finite S_2–diagonal.*

Proof. Let $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ be a point finite S_2–sequence for X. We show that $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ is a point finite semi-development. We may assume that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. By passing to components, we may assume that for each $x \in X$ and each $n \in \mathbb{N}, st(x, \mathcal{G}_n)$ is connected. Suppose that $x_n \in st(x, \mathcal{G}_n)$ for each n and fixed $x \in X$. By lemma 3.8 x is a cluster point of $\langle x_n : n \in \mathbb{N} \rangle$ or $\langle x_n : n \in \mathbb{N} \rangle$ does not cluster at all. Suppose it does not cluster at all. So there is a compact neighborhood U of x and $U \cap \overline{\{x_n\}} = \emptyset$, so $st(x, \mathcal{G}_n)$ is not subset of U for each n. Then since $st(x, \mathcal{G}_n)$ is connected, $\overline{st(x, \mathcal{G}_n)} \cap \partial U \neq \emptyset$ for each n. Since ∂U is compact, $\partial U \cap \bigcap_n \overline{st(x, \mathcal{G}_n)} \neq \emptyset$, a contradiction. Since every separable regular space with a point finite semi-development is metrizable [1, Theorem 1.7, Proposition 1.12], the proof is done. ■

We can prove the following theorem using the same technique.

Theorem 3.10 *Every manifold with S_2–diagonal is o-semi-developable.*

Bennett [6] proved that in a quasi–developable space, hereditarily $\aleph_1$–compact, hereditarily Lindelöf and hereditarily seprability are equivalent; and each of these conditions implies metrizability of the space if it is regular. We prove the following lemma by analogous method to the [1, Proposition 1.12].

Lemma 3.11 *A separable space with a point–countable quasi–development is hereditarily separable.*

Proof. Let X be a space with a point–countable quasi–development $\mathcal{G} = \langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ and a countable dense subset D. Let $Y \subset X$. Define $\mathcal{D} = \{G : G \in \mathcal{G}_n$ for some $n, G \cap D \neq \emptyset \neq G \cap Y\}$. Then, since $\mathcal{G}$ is point–countable and D is countable, $\mathcal{D}$ is countable. For each $G \in \mathcal{D}$ choose $p \in G \cap Y$ and let P be the set of all points so chosen. Then P is a countable subset of Y. We prove P is dense in Y. Let U be an open set in Y containing the point $y \in Y$. Let $\hat{\mathcal{G}_n} = \{G \cap Y : G \in \mathcal{G}_n\}$ for each $n \in \mathbb{N}$. Then $\hat{\mathcal{G}} = \langle \hat{\mathcal{G}_n} : n \in \mathbb{N} \rangle$ is a quasi–development for Y, and therefore, there $n \in \mathbb{N}$ such that $y \in st(y, \hat{\mathcal{G}_n}) \subset U$. Since D is dense in X, there is some $G \in \mathcal{G}_n$ such that $G \cap D \neq \emptyset$ and $y \in G$. Then $G \in \mathcal{D}$, and hence, there is a point $p \in P$ contained in G, by definition of P. Therefore $p \in G \cap Y \subset st(y, \hat{\mathcal{G}_n}) \subset U$, i.e., $U \cap P \neq \emptyset$. Hence, $y \in \bar{P}^Y$. ■

Corollary 3.12 *A regular separable point–countable quasi–developable space is metrizable.*

From Theorem 2.1 and Corollary 3.12 we have the following result:

Corollary 3.13 *Every locally compact, locally connected, regular separable space with a point–countable quasi–G^*_δ–diagonal is metrizable.*

The following open problems will be formulated for the manifold case, but are also relevant in the context of locally connected, locally compact spaces.

Question 3.14 *Is every normal manifold with S_2–diagonal metrizable?*

Question 3.15 *Is every hereditarily normal manifold with S_2–diagonal metrizable?*

Question 3.16 *Is every separable normal manifold with S_2–diagonal metrizable?*

Question 3.17 *Is every hereditarily normal manifold with a quasi-G^*_δ-diagonal metrizable?*

By Theorem 2.1 and [3, Theorem 2.3], the answer to Question 3.16 is yes if $2^{\omega_1} > 2^\omega$ holds.

4 Quasi–Regular–G_δ–Diagonals

Lemma 4.1 *A space X has a quasi–G_δ–diagonal if and only if there is a countable sequence $\langle U_n : n \in \mathbb{N}\rangle$ of open subsets in X^2, such that for all $(x,y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x,x) \in U_n$ but $(x,y) \notin U_n$.*

Proof. Let $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ be a quasi–G_δ–diagonal sequence for X.

Define $U_n = \bigcup\{G \times G : G \in \mathcal{G}_n\}$. Then the each U_n is open in X^2. Further, if $(x,y) \in X^2$ such that $x \neq y$, then there is $n \in \mathbb{N}$, such that $x \in st(x,\mathcal{G}_n)$ and $y \notin st(x,\mathcal{G}_n)$. Then $(x,x) \in U_n$ but $(x,y) \notin U_n$.

Conversely suppose we have a sequence $\langle U_n : n \in \mathbb{N}\rangle$ as in the statement of the Lemma. Define $\mathcal{G}_n = \{G : G \text{ is open and } G \times G \subseteq U_n\}$. Suppose distinct x and y are in X. Pick $n \in \mathbb{N}$ so that $(x,x) \in U_n$ but $(x,y) \notin U_n$. Then $x \in st(x,\mathcal{G}_n)$ while $y \notin st(x,\mathcal{G}_n)$. ∎

Corollary 4.2 *Every space with an S_δ–diagonal has a quasi–G_δ diagonal.*

Prompted by the above corollary, and by analogy with 'regular G_δ–diagonal', we make the following definition. A space X has a quasi–regular–G_δ diagonal if and only if there is a countable sequence $\langle U_n : n \in \mathbb{N}\rangle$ of open subsets in X^2, such that for all $(x,y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x,x) \in U_n$ but $(x,y) \notin \overline{U_n}$.

Lemma 4.3 *Every space with an R_δ–diagonal has a quasi–regular–G_δ–diagonal.*

Lemma 4.4 *Every space with a quasi–regular–G_δ–diagonal has a quasi–G^*_δ–diagonal.*

Proof. Suppose we have a sequence $\langle U_n : n \in \mathbb{N}\rangle$ as in the definition of quasi–regular–G_δ–diagonal. Define $\mathcal{G}_n = \{G : G \text{ is open and } G \times G \subseteq U_n\}$. Suppose distinct x and y are in X. Pick $n \in \mathbb{N}$ so that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. Then $x \in st(x, \mathcal{G}_n)$ while $y \notin \overline{st(x, \mathcal{G}_n)}$. ■

A sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of families of open sets of a space X is called:

1. a **strong quasi–development** for X if for every $x \in X$ and any open neighborhood U of x there exist an open neighborhood V of x and a natural number i such that $x \in st(V, \mathcal{G}_i) \subset U$;

2. a **quasi–development of order 2** for X if $\{st^2(x, \mathcal{G}_n)\}_{n\in\mathbb{N}} - \{\emptyset\}$ is a local base at x for each $x \in X$.

X is called **hereditarily screenable** if every subspace of X is screenable. A space X is **screenable** if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ which can be decomposed as $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$ such that each $\mathcal{V}_n$ is disjoint.

The proof of our next result relies on a theorem of Costantini, Fedeli and Pelant [7].

Theorem 4.5 *For every space X the following conditions are equivalent:*

1. *X has a σ–disjoint base;*

2. *X has a strong quasi–development;*

3. *X has a quasi–development of order 2;*

4. *X is quasi–developable and hereditarily screenable.*

Theorem 4.6 *Let X be a locally compact, locally connected space. If X has a quasi–regular–G_δ–diagonal, then X is metrizable.*

Proof. By Theorem 4.5 and well–known result 'a locally compact, locally connected space, is hereditarily screenable if and only if it is metrizable', we only need to show that X has a quasi–development $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ such that, for each $x \in X, \{st^2(x, \mathcal{G}_n)\}_{n\in\mathbb{N}} - \{\emptyset\}$ is a local base at x.

Let $\langle U_n : n \in \mathbb{N}\rangle$ be as in the definition of quasi–regular G_δ diagonal. So, the sets U_n are open in X^2 and for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U}_n$. Put $\mathcal{H}_n = \{H : H \text{ is open }, H \times H \subseteq U_n\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$, and for $F \in \mathcal{F}$ put

$$\mathcal{G}_F = \{\bigcap\nolimits_{i\in F} H_i : H_i \in \mathcal{H}_i\}$$

and $\mathcal{G}_F{}' = \{U : U \text{ is a component of } G \text{ for some } G \in \mathcal{G}_F\}$.

We show that for each $x \in X, \{st^2(x, \mathcal{G}_F{}')\}_{F\in\mathcal{F}} - \{\emptyset\}$ is a local base at x. Take any $x \in X$ and suppose $x \in V$ is open and $\overline{V}$ is compact. For each $n \in \mathbb{N}$ put $F_n = c_{\mathcal{H}}(x) \cap \{1, 2, ..., m\}$, where $m \geq n$ is minimal such that $\{1, 2, ..., m\} \cap c_{\mathcal{H}}(x) \neq \emptyset$.

Suppose, for a contradiction, for all $n \in \mathbb{N}, st^2(x, \mathcal{G}_{F_n}{}') \cap (X - V) \neq \emptyset$. Since each $\overline{st^2(x, \mathcal{G}_{F_n}{}')}$ is connected, $\overline{st^2(x, \mathcal{G}_{F_n}{}')} \cap \partial V \neq \emptyset$ for each $n \in \mathbb{N}$. But ∂V is compact and $\overline{st^2(x, \mathcal{G}_{F_n}{}')} \cap \partial V$ is a decreasing sequence of non–empty closed sets, so $\partial V \cap \bigcap_{n\in\mathbb{N}}\overline{st^2(x, \mathcal{G}_{F_n}{}')} \neq \emptyset$, hence there is $y \in \bigcap_{n\in\mathbb{N}}\overline{st^2(x, \mathcal{G}_{F_n}{}')} \cap \partial V$.

Of course $x \neq y$. So by the definition of quasi–regular–G_δ–diagonal, there is n such that $(x,x) \in U_n$ but $(x,y) \notin \overline{U}_n$.

By the same argument as in Theorem 2.1, we know that $\{\mathcal{G}_{F}{}' : F \in \mathcal{F}\}$ is a quasi–development of X. Therefore there exists I and $J \in \mathcal{F}$ such that

$$(x,y) \in st(x, \mathcal{G}_{I}{}') \times st(y, \mathcal{G}_{J}{}') \subseteq X^2 - U_n.$$

Choose $m \geq \max\{I, n\}$, so that $I \subseteq F_m$. It follows that $y \in \overline{st^2(x, \mathcal{G}_{F_m}{}')}$, so $st^2(x, \mathcal{G}_{F_m}{}') \cap st(y, \mathcal{G}_{J}{}') \neq \emptyset$. Then there exists, $G_1, G_2 \in \mathcal{G}_{F_m}{}'$ and $G_3 \in \mathcal{G}_{J}{}'$ such that $y \in G_3$, $x \in G_1, G_1 \cap G_2 \neq \emptyset$ and $G_2 \cap G_3 \neq \emptyset$. Let $z_1 \in G_1 \cap G_2$ and $z_2 \in G_2 \cap G_3$. Then $(z_1, z_2) \in (G_1 \times G_3) \cap (G_2 \times G_2)$. Now, $G_1 \in \mathcal{G}_{F_m}{}', G_3 \in \mathcal{G}_{J}{}'$, so $G_1 \times G_3 \subseteq st(x, \mathcal{G}_{F_m}{}') \times st(y, \mathcal{G}_{J}{}')$. Also, $G_2 \in \mathcal{G}_{F_m}{}'$ and $n \in F_m$, so $G_2 \subseteq H$ for some $H \in \mathcal{H}_n$. Therefore $G_2 \times G_2 \subseteq H \times H \subseteq U_n$, so $(z_1, z_2) \in U_n$.

In other words, $(z_1, z_2) \in G_2 \times G_3 \subseteq (st(x, \mathcal{G}_{F_m}{}') \times st(y, \mathcal{G}_{J}{}')) \cap U_n$, and this is a contradiction. ∎

From Theorem 4.5 and and well–known result 'a locally compact, locally connected space, is hereditarily screenable if and only if it is metrizable', we derive an alternative proof of the following result:

Theorem 4.7 *A manifold is metrizable if and only if it has a σ–disjoint base.*

5 Examples

The implications between diagonal properties of manifolds is shown in the following diagram:

$$\begin{tikzcd}
\boxed{\text{quasi regular–}G_\delta} \arrow[d] & \boxed{\text{regular–}G_\delta} \arrow[l] \arrow[d] \\
\boxed{\text{quasi–}G_\delta^*} \arrow[d] & \boxed{G_\delta^*} \arrow[l] \arrow[d] \\
\boxed{\text{quasi–}G_\delta} & \boxed{G_\delta} \arrow[l]
\end{tikzcd}$$

Figure 1: Relationships between diagonal properties.

During his visit to Auckland, P. Nyikos informed the second author that Rudin's first manifold in [22] is hereditarily normal with a G_δ–diagonal but is not metrizable, so this example can serve as a manifold with a G_δ–diagonal but not a G_δ^*–diagonal and a manifold with a quasi–G_δ–diagonal but not a quasi–G_δ^*–diagonal.

The Prüfer manifold (a description of a version of Prüfer manifold is given in Example 5.1) is an example of a manifold with G_δ^*–diagonal but not regular–G_δ–diagonal.

The Bennett–Balogh manifold [5] is an example of a manifold with quasi–G_δ–diagonal but without a G_δ–diagonal.

Example 5.2 has been given in [9, Example 2.2]. It answers a problem of Nyikos in the negative that there is a quasi–developable 2–manifold with a G_δ–diagonal, but which is not developable. This example is a manifold with a quasi–G^*_δ–diagonal but not a quasi– regular–G_δ–diagonal and also can serve as a manifold with a quasi–G^*_δ–diagonal but not a G^*_δ–diagonal.

The presentation of [9, Example 2.2] is concise. The authors feel the techniques used in its construction might have wider application, so we outline here a general method of creating quasi–developable 2–manifolds with G_δ–diagonal.

Example 5.1 *A class of quasi–developable manifolds with G_δ–diagonal.*

Construction. We start with a description of a version of the Prüfer manifold. The underlying set is $P = H^+ \cup G$ where $H^+ = \mathbb{R} \times (0, +\infty)$ and $G = \mathbb{R} \times (-1, 0]$. The topology on P is defined as follows. Points of H^+ have the usual Euclidean topology, so the nth neighbourhood of $(x, y) \in H^+$ can be taken to be $U(x, y; n) =$ (disc about (x, y) of radius $1/n$) $\cap H^+$. A point (x, a), where $a \in (-1, 0]$ has as its nth neighbourhood, $V(x, a; n) = ((\{x\} \times (a - 1/n, a + 1/n)) \cap G) \cup$ (set of points in H^+ within the disc of radius $1/n$ about $(x, 0)$ and between either the lines of slope $1/(a-1/n)$ and $1/(a+1/n)$ or the lines of slope $1/(-a - 1/n)$ and $1/(-a + 1/n)$) (illustrated in Figure 2, with picture proof, $V(x, a; n)$ is homeomorphic to $\mathbb{R}^2$).

Suppose B and C are two disjoint subset of $\mathbb{R}$ and for each element $c \in C$, there are one or two sequences $\langle l_{c,n} : n \in \mathbb{N}\rangle$ and $\langle r_{c,n} : n \in \mathbb{N}\rangle$ so that $l_{c,n}, r_{c,n} \in B$, $l_{c,m} < l_{c,n} < c$ if $m < n$, $c < r_{c,n} < r_{c,m}$ if $m < n$, and both $|c - l_{c,n}|$ and $|c - r_{c,n}|$ are strictly less than $1/n^2$. Thus, $(l_{c,n})_{n\in\mathbb{N}}$ converges to c from below, and $(r_{c,n})_{n\in\mathbb{N}}$ converges to c from above in the real line with its usual topology so that the distance of $l_{c,n}$ to c and $r_{c,n}$ to c strictly decreases with n.

We will construct a manifold $M = M(B, C)$ having underlying set $H^+ \cup \hat{B} \cup \hat{C}$, where $\hat{B} = B \times (-1, 0]$, and $\hat{C} = C \times (-1, 0]$ and a space $X = X(B, C)$ with underlying set $B \cup C$, and topology to be described shortly.

Give $H^+ \cup \hat{B}$, the subspace topology from the Prüfer manifold. Then it is an open submanifold. We will define the topology at points of $\hat{C}$ such that

1. M is a manifold
2. M is quasi–developable
3. M has a G_δ diagonal
4. the identity map $\hat{B} \cup \hat{C} \to X \times (-1, 0]$ is a homeomorphism.

Let $c \in C$ and set $R = \{r_{c,n} : n \in \mathbb{N}\}$ and $L = \{l_{c,n} : n \in \mathbb{N}\}$ whichever is defined. Three cases arise depending whether only R or only L or both are defined.

We define the nth neighborhood of the point $(c, a) \in \hat{C}$ (illustrated in Figure 3) to be:

Case (1): Both R and L are defined.

$$\begin{aligned} W(c, a; n) \quad = \quad & (V(c, a; n) \cap (\{c\} \times \mathbb{R})) \\ & \cup (\bigcup_{k\geq n+1} (V(l_{c,k}, a; n) \cap ((\frac{l_{c,k-1} + l_{c,k}}{2}, \frac{l_{c,k} + l_{c,k+1}}{2}] \times \mathbb{R}))) \\ & \cup (\bigcup_{k\geq n+1} (V(r_{c,k}, a; n) \cap ([\frac{r_{c,k+1} + r_{c,k}}{2}, \frac{r_{c,k} + r_{c,k-1}}{2}) \times \mathbb{R}))). \end{aligned}$$

Case (2): R is undefined.

$$\begin{aligned} W(c,a;n) &= (V(c,a;n)\cap([c,+\infty)\times\mathbb{R})) \\ &\quad \cup(\bigcup_{k\geq n+1}(V(l_{c,k},a;n)\cap((\frac{l_{c,k-1}+l_{c,k}}{2},\frac{l_{c,k}+l_{c,k+1}}{2}]\times\mathbb{R}))). \end{aligned}$$

Case (3): L is undefined.

$$\begin{aligned} W(c,a;n) &= (V(c,a;n)\cap((-\infty,c]\times\mathbb{R})) \\ &\quad \cup(\bigcup_{k\geq n+1}(V(r_{c,k},a;n)\cap([\frac{r_{c,k+1}+r_{c,k}}{2},\frac{r_{c,k}+r_{c,k-1}}{2}]\times\mathbb{R}))). \end{aligned}$$

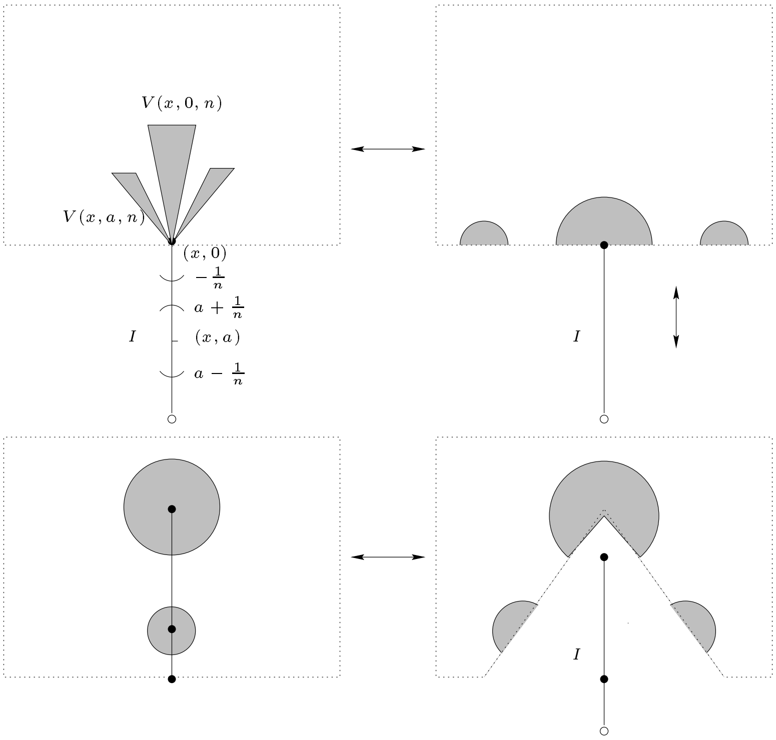

Figure 2: $V(x,a;n)$ is homeomorphic to $\mathbb{R}^2$.

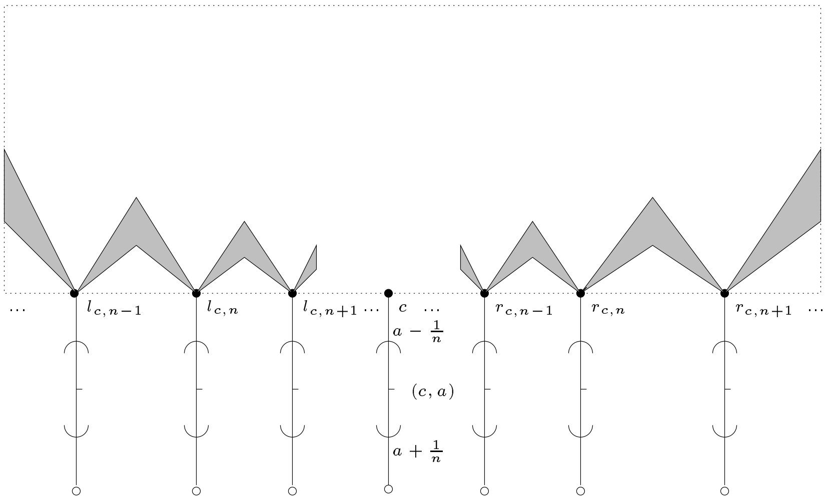

Figure 3: $W(c, a; n)$ is homeomorphic to $\mathbb{R}^2$.

X is topologised by isolating points of B and letting basic neighbourhhood of a point $c \in C$ consist of c together with tails of the corresponding sequences $\langle l_{c,n}\rangle$ and $\langle r_{c,n}\rangle$ (or just one of those sequences when only one is defined).

Define $\mathcal{G}_n = \{U(x, y; n) : x \in \mathbb{R}, y > 0\}$, $\mathcal{H}_n = \{V(x, a; n) : x \in B, a \in (-1, 0]\}$ and $\mathcal{I}_n = \{W(c, a; n) : c \in C \text{ and } a \in (-1, 0]\}$.

One can easily check that M is Hausdorff.

Claim: $W(c, a; n)$ is homeomorphic to $\mathbb{R}^2$.

Proof. Let ϕ be the natural identification of $\mathbb{R} \times (0, \infty)$ with $(\mathbb{R} \times (0, \infty) \setminus S^+) \cup S^-$ where $S^+ = \bigcup_{k=1}^{\infty}(\{l_{c,k}\} \times (0, 1] \cup \{c\} \times (0, 1] \cup \{r_{c,k}\} \times (0, 1])$ and $S^- = \bigcup_{k=1}^{\infty}(\{l_{c,k}\} \times (-1, 0] \cup \{c\} \times (-1, 0] \cup \{r_{c,k}\} \times (-1, 0])$.

Observe that $(\mathbb{R} \times (0, \infty) \setminus S^+)$ is homeomorphic to $\mathbb{R} \times (0, \infty)$. Let $\psi : (\mathbb{R} \times (0, \infty) \setminus S^+) \cup S^- \to \mathbb{R} \times (0, \infty) \cup S^-$ be a function which is the identity on S^- and is one of those homeomorphisms 'pushing out' S^+ from $\mathbb{R} \times (0, \infty)$.

Now $\psi.\phi$ is a bijection between $\mathbb{R} \times (0, \infty)$ and $(\mathbb{R} \times (0, \infty)) \cup S^-$. Give this second set the induced topology. (So $(\mathbb{R} \times (0, \infty)) \cup S^-$ is homeomorphic to $\mathbb{R} \times (0, \infty)$ which in turn is homeomorphic to $\mathbb{R}^2$.)

Consider the effect of $\psi.\phi$ on a basic rectangular neighbourhood of $(l_{c,k}, a+1)$ $(a \in (-1, 0])$. We can see that these basic neighborhoods of $(l_{c,k}, a)$ are (apart from small deformations) of the form of the sets $V(l_{c,k}, a; n)$. Now consider the effect of $\psi.\phi$ on a rectangle about $(c, a + 1)$. We can see that these basic neighbourhoods of (c, a) $(a \in (-1, 0])$ are of the form of the sets $W(c, a; n)$. Hence, $W(c, a; n)$ is homeomorphic to $\mathbb{R}^2$. Thus M with the topology defined above, is a 2–manifold. It is easy to check $X \times (-1, 0]$ has the product topology (i.e. the identity map $\hat{B} \cup \hat{C} \to X \times (-1, 0]$ is a homeomorphism).

It remains to show that M is quasi–developable and has a G_δ diagonal. To do this we examine the stars of points of M in the open collections $\mathcal{G}_n$, $\mathcal{H}_n$ and $\mathcal{I}_n$.

Case: $x \in \mathbb{R}$ and $y > 0$.

$$\mathrm{st}((x, y), \mathcal{G}_{2n}) \subseteq U(x, y; n), \quad \mathrm{st}((x, y), \mathcal{H}_n) = \emptyset \quad \text{if } n > 1/y$$

and $\mathrm{st}((x,y),\mathcal{I}_n)=\emptyset$ if $n>1/y$.
Case: $x \in B$ and $a \in (-1,0]$.

$$\mathrm{st}((x,a),\mathcal{G}_n)=\emptyset, \mathrm{st}((x,a),\mathcal{H}_{2n}) \subseteq V(x,a;n)$$

and $\mathrm{st}((x,a),\mathcal{I}_n) \subseteq [x-1/n,x+1/n] \times ((0,1/n)\cup(a-1/n,a+1/n))$.
Case: $c \in C$ and $a \in (-1,0]$.

$$\mathrm{st}((c,a),\mathcal{G}_n)=\emptyset, \qquad \mathrm{st}((c,a),\mathcal{H}_n)=\emptyset, \qquad \mathrm{st}((c,a),\mathcal{I}_{2n}) \subseteq W(c,a;n).$$

It follows that $\{\mathcal{G}_n\}_{n\in\mathbb{N}} \cup \{\mathcal{H}_n\}_{n\in\mathbb{N}} \cup \{\mathcal{I}_n\}_{n\in\mathbb{N}}$, is a quasi–development for M.

Define $\mathcal{J}_n = \mathcal{G}_n \cup \mathcal{H}_n \cup \mathcal{I}_n$ ($n \in \mathbb{N}$). Computing stars of points in the open covers $\mathcal{J}_n$ (for the three cases as above), we have for n sufficiently large,

$$\mathrm{st}((x,y),\mathcal{J}_n) \subseteq U(x,y;n), \quad \mathrm{st}((c,a),\mathcal{J}_{2n}) \subseteq W(c,a;n),$$

$$\mathrm{st}((x,a),\mathcal{J}_{2n}) \subseteq V(x,a;n) \cup [x-1/n,x+1/n] \times ((0,1/n)\cup(a-1/n,a+1/n))$$

and hence $\bigcap_{n\in\mathbb{N}} \mathrm{st}((p,q),\mathcal{J}_n) = \{(p,q)\}$ for all $(p,q) \in M$. In other words, $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ is a G_δ–sequence for M. ■

Example 5.2 *A quasi–developable manifold M which has a G_δ–diagonal but is not perfect.*

Construction. Let B be a Bernstein subset of $\mathbb{R}$. We consider a special case of Example 5.1. Let $\{B_\alpha\}_{\alpha<2^{\aleph_0}}$ list all countable subsets of B with uncountable closure in $\mathbb{R}$. Inductively pick $x_\alpha \in \overline{B_\alpha} \setminus (B \cup \{x_\beta\}_{\beta<\alpha})$, and $l_{\alpha,n}$ and $r_{\alpha,n}$ sequences on B converging to x_α from the left and right (respectively). Set $C=\{x_\alpha\}_{\alpha<2^{\aleph_0}}$. Note that X is homeomorphic to Gruenhage's space [12]. Hence M is not perfect. ■

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