The Characterization of Finite Simple Groups with no Elements of Order Six by Their Element Orders

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ABSTRACT. We show that all but A_6 non-abelian finite simple groups with no elements of order 6 are characterized by their element orders.

INTRODUCTION

Let G be a finite group and $\pi_e(G)$ the set of orders of all elements in G. Then $\pi_e(G)$ is a subset of positive integers $\mathbb{Z}^+$. Let Ω be a subset of $\mathbb{Z}^+$ and $h(\Omega)$ the number of non-isomorphic finite groups K such that $\pi_e(K) = \Omega$. Then $h(\pi_e(G)) \geq 1$ for any finite group G. Following [16] we say that a finite group G is *non-distinguishable* if $h(\pi_e(G)) = \infty$, and *distinguishable* if $h(\pi_e(G)) < \infty$. Moreover, G is *k-distinguishable* if $h(\pi_e(G)) = k$; and *characterizible* if $h(\pi_e(G)) = 1$. From [4], [14, Theorem 1.1] and [5], we have the following propositions.

Proposition 1. *Suppose a finite group G has an elementary abelian normal subgroup. Then G is non-distinguishable. In particular, a solvable finite group is non-distinguishable.*

Proposition 2. *The following non-abelian finite simple groups are characterizible.*

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(a) *Alternating groups* A_n *for* $n \in \{5, 7, 8, 9, 11, 12, 13\}$.

(b) *All sporadic simple groups except* J_2.

(c) *Simple groups* $L_2(q)$ *with* $q \neq 9$, *Suzuki-Ree groups*, $L_3(4)$, $L_3(7)$, $L_3(8)$, $L_4(3)$, $S_4(7)$, $U_3(4)$, $U_4(3)$, $U_6(2)$, $G_2(3)$, $O_8^-(2)$ *and* $O_{10}^-(2)$.

In this paper, we proved that, except for A_6, all non-abelian finite simple groups with no elements of order 6 are characterizible, that is we proved the following Theorem.

Theorem. *Let* G *be a finite group and* H *a finite non-abelian simple group with no elements of order* 6 *such that* $H \not\simeq A_6$. *Then* $G \simeq H$ *if and only if* $\pi_e(G) = \pi_e(H)$.

The alternating group A_6 is not characterizible, in fact, infinitely many groups share the same element orders with A_6 (see [2, Lemma 2]).

In the following, we suppose a group is always a finite group and a simple group is always non-abelian. We denote by $\pi(G)$ the set of prime divisors of the order $|G|$ of G. The other notation are standard following from [3] and [13].

1. PRELIMINARY RESULTS AND LEMMAS

The results of the paper depends on the classification of finite simple groups and the following lemmas.

Lemma 1.1. *Let* G *be a simple group with no elements of order* 6. *Then* G *is isomorphic to one of the following groups:*

(a) $L_2(q)$ *with* $q \not\equiv \pm 1 \pmod{12}$;

(b) $Sz(2^{2n+1})$ *with* $n \geq 1$;

(c) $L_3(2^n)$ *with* $n \not\equiv 0 \pmod{6}$;

(d) $U_3(2^n)$ *with* $n \geq 2$ *and* $n \not\equiv 3 \pmod{6}$.

Proof. See [8] and [9].

Remark. From paper [9], a unitary group $U_3(2^n)$ has no elements of order 6 whenever $n \geq 2$ and $n \not\equiv 3, 5 \pmod{6}$. But, in fact, $U_3(2^n)$ with $n \equiv 5 \pmod{6}$ also has no elements of order 6. Indeed, suppose $n = 6k + y$ for some integers $k \geq 0$ and $0 \leq y \leq 5$. Then $2^n + 1 \equiv 2^y + 1 \equiv 0 (\bmod 9)$ if and only if $y = 3$, that is $n = 6k + 3$.

From paper [6], $SU_3(2^n)$ has a single conjugacy class of involutions each with a centralizer of order $2^n(2^n + 1)$. If $U_3(2^n)$ with $n \equiv 5 \pmod{6}$ has elements of order 6, it follows that 3 divides 2^n+1, so then 9 must divides 2^n+1 (note that $Z(SU_3(2^n))$ is cyclic of order $gcd(3, 2^n+1)$), which is impossible.

The *prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is the set $\pi(G)$ and two vertices $p, r \in \pi(G)$ are adjacent if and only if G contains an element of order pr. Denote by $t(G)$ the number of connected components of $\Gamma(G)$, and by $\pi_i = \pi_i(G)$ for $i = 1, 2, \ldots, t(G)$ the connected components of $\Gamma(G)$. If $|G|$ is even, then suppose 2 is a vertex of π_1. We shall also use the following unpublished result of K.W. Gruenberg and O.H. Kegel, (see [19, Theorem A]).

Lemma 1.2. *If G is a finite group whose prime graph has more than one component, then G has one of the following structures:* (a) *Frobenius or 2-Frobenius;* (b) *simple;* (c) *an extension of a π_1-group by a simple group;* (d) *simple by π_1; or* (e) *π_1 by simple by π_1.*

A group G is called 2-Frobenius if there exists a normal series $1 \leq H \leq K \leq G$ of G such that H is the Frobenius kernel of K and K/H is the Frobenius kernel of G/H.

Suppose $\epsilon = +$ or $-$ and let $L_3^\epsilon(q) = L_3(q)$ or $U_3(q)$ according as $\epsilon = +$ or $-$. For simplicity, we always identify $q - \epsilon$ with $q - \epsilon 1$.

Lemma 1.3. *Suppose $q = 2^n$ and $\eta = \eta_\epsilon = \frac{1}{\gcd(3, q-\epsilon)}$. Then*

$$\pi_e(L_3^\epsilon(q)) = \{1, 2, 4, s : s|(q-\epsilon), s|2\eta(q-\epsilon), s|\eta(q^2-1) \text{ or } s|\eta(q^2+\epsilon q+1)\}.$$

In addition, if $(q, \epsilon) \in \{(2,+), (4,+)\}$, then $\pi_1(L_3(q)) = \{2\}$, $\pi_2(L_3(q)) = \{3\}$ for $q \in \{2, 4\}$, $\pi_3(L_3(q)) = \{7\}$ or $\{5\}$ according as $q = 2$ or 4 and $\pi_4(L_3(4)) = \{7\}$; if $(q, \epsilon) \notin \{(2,+), (4,+)\}$, then $\pi_1(L_3^\epsilon(q)) = \{r \text{ prime} : r|2\eta(q-\epsilon) \text{ or } r|\eta(q+\epsilon)\}$ and $\pi_2(L_3^\epsilon(q)) = \{r \text{ prime} : r|\eta(q^2+\epsilon q+1)\}$. In particular, $t(L_3^\epsilon(q)) \geq 2$.

Proof. The set $\pi_e(L_3^\epsilon(q))$ is determined by calculations from conjugacy $\mathrm{SL}_3^\epsilon(2^n)$-classes given, say by [6, Tables VII-1 and IX-1], and $\pi_i(L_3^\epsilon(q))$ for all i are given by [12].

2. THE PROOF OF THE THEOREM

From Lemma 1.1, if H is a simple group with no elements of order 6, then H is isomorphic to $L_2(q)$, $Sz(2^{2n+1})$ or $L_3^\epsilon(2^n)$ for suitable q or n. By [1] and [17], $L_2(q)$ ($q \neq 9$ and $L_2(9) \simeq A_6$) and $Sz(2^{2n+1})$ ($n \geq 1$) are characterizible. It suffices to show that $H = L_3^\epsilon(2^n)$ is characterizible.

Theorem 2.1. *Let G be a finite group and $H = L_3^{\epsilon}(2^n)$ with $n \not\equiv 0$ or $3 \,(\mathrm{mod}\, 6)$ according as $\epsilon = +$ or $-$. Then $G \simeq H$ if and only if $\pi_e(G) = \pi_e(H)$.*

Proof. It is clear that if $G \simeq H$, then $\pi_e(G) = \pi_e(H)$.

Let $q = 2^n$ with $n \not\equiv 0$ or $3 \,(\mathrm{mod}\, 6)$ according as $\epsilon = +$ or $-$, and suppose $\pi_e(G) = \pi_e(H)$. By Lemma 1.3, $\pi_e(L_3^{\epsilon}(q)) = \{1, 2, 4, s : s|(q-\epsilon), s|2\eta(q-\epsilon), s|\eta(q^2-1)$ or $s|\eta(q^2 + \epsilon q + 1)\}$, where $\eta = \frac{1}{\gcd(3, q-\epsilon)}$. Thus G has no elements of order 6 and $t(G) \geq 2$.

(1). G is non-solvable.

Suppose G is solvable and K is its $\{2, p, r\}$-Hall subgroup, where $p, r \in \pi(G)$ such that $p|q+\epsilon$ and $r|\eta(q^2+\epsilon q+1)$. Since G has no elements of order $2p$, $2r$ and pr, it follows that K is a solvable group all of whose elements are of prime power orders. By [11, Theorem 1], $|\pi(K)| \leq 2$, which is impossible.

(2). G is neither Frobenius nor 2-Frobenius.

Suppose G is a Frobenius group with complement C. By the structure of non-solvable Frobenius complements (see [15, Theorem 18.6]), we have that C has a normal subgroup C^* of index ≤ 2 such that $C^* \simeq \mathrm{SL}_2(5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \emptyset$. Since $\pi_e(\mathrm{SL}_2(5)) = \{1, 2, \ldots, 6, 10\}$, it contradicts with $6 \notin \pi_e(G)$. imilarly, G is not 2-Frobenius.

By Lemma 1.2, G has a normal series $1 \leq N < G_1 \leq G$ such that (a) N and G/G_1 are π_1-groups; (b) $\bar{G} = G_1/N$ is simple with $6 \notin \pi_e(\bar{G})$. We claim that

(3). $\bar{G} \simeq H$.

By Lemma 1.1, $\bar{G}$ is isomorphic to one of the groups, $L_2(q')$ with $q' \not\equiv \pm 1 (\mathrm{mod}\, 12)$, $Sz(2^{2m+1})$ with $m \geq 1$, $L_3^{\beta}(2^m)$ with $m \not\equiv 0$ or $3 (\mathrm{mod}\, 6)$ according as $\beta = +$ or $-$.

Case (i). Suppose $\bar{G} \simeq L_2(q')$ and $q' = 2^m$ with $m \geq 2$. Since the vertexes set of π_i is a subset of the set $\pi(G)$ and different vertexes sets are disjoint, each vertex of π_i is coprime to any vertex of π_j for $i \neq j$ and both G/G_1 and N are π_1-groups, it follows that $\eta(q^2 + \epsilon q + 1) \in \pi_e(\bar{G})$. Note that $\eta(q^2 + \epsilon q + 1)$ is the largest or second largest number in the set

$$\{1, 2, 4, (q-\epsilon), 2\eta(q-\epsilon), \eta(q^2-1), \eta(q^2+\epsilon q+1)\}$$

according as $\epsilon = +$ or $-$, and in addition, G and hence $\bar{G}$ have no element x such that $\eta(q^2 + \epsilon q + 1)$ is a proper divisor of $|x|$. It follows that $\eta(q^2 + q + 1) = q' + 1$ and $\eta(q^2 - q + 1) \in \{q' + 1, q' - 1\}$, which is impossible as both q and q' are powers of 2.

Suppose $q' \equiv \gamma (\mathrm{mod}\, 4)$ and $q' = p'^m$, where p' is an odd prime and $\gamma = +$ or $-$. A similar proof to above shows that $\eta(q^2+\epsilon q+1) \in \{p', \frac{1}{2}(q'+\gamma)\}$, since $\eta(q^2+\epsilon q+1)$ is odd and $\frac{1}{2}(q'-\gamma)$ is even. If $\eta(q^2+\epsilon q+1) = p'$, then $m = 1$, since otherwise $\frac{1}{2}(q'+1) > \frac{1}{2}(q'-1) > p'$. Thus $\frac{1}{2}(q'+1) = \frac{1}{2}(p'+1) = \frac{1}{2}(\eta(q^2+\epsilon q+1)+1) \in \pi_e(G)$.

If $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1) = 2$, then $\eta(q^2+\epsilon q+1) = 3$, which is impossible as 3 is not a divisor of $\eta(q^2+\epsilon q+1)$. If $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1) = 4$, then $\eta(q^2+\epsilon q+1) = 7 = p'$, so that $\gamma = -$, and $(q, \epsilon) = (2, +)$ or $(4, +)$. In the former case, $L_2(p') = L_2(7)$, which is isomorphic to $H = L_3(2)$. In the later case, $H = L_3(4)$, $\bar{G} = L_2(7)$ (note that $\pi_e(L_2(7)) = \{1, 2, 3, 4, 7\} \subseteq \pi_e(L_3(4)) = \{1, 2, 3, 4, 5, 7\}$), and by Proposition 2, $L_3(4)$ is characterizible, so that $\pi_e(G) \neq \pi_e(H)$, which is impossible.

If $r \geq 5$ is a common prime factor of $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1)$ and $q-\epsilon$, then $q \equiv \epsilon (\mathrm{mod}\, r)$ and $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1) \equiv \frac{3\eta+1}{2} (\mathrm{mod}\, r)$. But $\frac{3\eta+1}{2} \not\equiv 0 (\mathrm{mod}\, r)$, so r is not a common factor of $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1)$ and $q-\epsilon$ (respectively, and $2\eta(q-\epsilon)$). Similarly, a prime $r \geq 5$ is not a common factor of $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1)$ and $q+\epsilon$. Thus $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1)$ is a power of 3. Moreover, if $q > 2$ and $\eta = 1$, then $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1) = \frac{1}{2}(q^2+\epsilon q+2) \equiv \frac{1}{2}\epsilon(-1)^n \not\equiv 0 (\mathrm{mod}\, 3)$, which is impossible. If $\eta = \frac{1}{3}$, then $\frac{1}{2}(\eta(q^2+\epsilon q+1)+1) = \frac{1}{6}(q^2+\epsilon q+4) = 3^t$ has no solution for n and t, which is a contradiction. If $q = 2$, then $\epsilon = +$ and $\eta = 1$, so $\eta(q^2+\epsilon q+1) = 7$, which is discussed above.

Suppose $\eta(q^2+\epsilon q+1) = \frac{1}{2}(q'+\gamma)$. Then $\frac{1}{2}(q'-\gamma) = \eta(q^2+\epsilon q+1) - \gamma \in \pi_e(G)$, which is impossible.

It follows that $\bar{G} \not\simeq L_2(q')$ except when $q' = 7$ and $H = L_3(2)$, in which case $\bar{G} = L_2(7) \simeq H$.

Case (ii). Suppose $\bar{G} \simeq Sz(2^{2m+1})$ with $m \geq 1$. A similar proof to that of Case (i) shows that $\eta(q^2+q+1) = 2^{2m+1}+2^{m+1}+1$ and $\eta(q^2+q+1) \in \{2^{2m+1}-1, 2^{2m+1}+2^{m+1}+1\}$, which are impossible. Thus $\bar{G} \not\simeq Sz(2^{2m+1})$.

Case (iii). Suppose $\bar{G} \simeq L_3^{-\epsilon}(2^m)$, where $m \geq 2$ when $\epsilon = +$. A similar proof to that of Case (i) shows that $\eta(q^2+q+1) = \mu(2^{2m}-1)$ and

$$\eta(q^2-q+1) \in \{\mu(2^{2m}+2^m+1), \mu(2^{2m}-1)\},$$

where $\mu = 1/\gcd(3, 2^m+\epsilon)$. If $\eta(q^2+\epsilon q+1) = \mu(2^{2m}-1)$ and $\mu(2^m+\epsilon) \neq 1$, then an odd prime divisor r of $\mu(2^m+\epsilon)$ divides both $\eta(q^2+\epsilon q+1)$ and $2\mu(2^m+\epsilon) \in \pi_e(\bar{G})$. In particular, $2r \in \pi_e(\bar{G})$ and so $2r \in \pi_e(H)$. A contradiction, since by Lemma 1.3, $2r \notin \pi_e(H)$ for $r | \eta(q^2+\epsilon q+1)$. If $\mu(2^m+\epsilon) = 1$, then $\eta(q^2+\epsilon q+1) = \mu(2^{2m}-1) = 2^m-\epsilon$

has no solution. Similarly, $\eta(q^2-q+1)=\mu(2^{2m}+2^m+1)$ can also induce a contradiction. It follows that $\bar{G}\not\simeq L_3^{-\epsilon}(2^m)$.

Case (iv). Suppose $\bar{G}\simeq L_3^{\epsilon}(2^m)$. A similar proof to that of Case (i) shows that $\eta(q^2+q+1)=\mu(2^{2m}+2^m+1)$ and

$$\eta(q^2-q+1)\in\{\mu(2^{2m}-1),\mu(2^{2m}-2^m+1)\},$$

where $\mu=1/\gcd(3,2^m-\epsilon)$. A similar proof to that of Case (iii) shows that $\eta(q^2-q+1)\neq\mu(2^{2m}-1)$, so $\eta(q^2+\epsilon q+1)=\mu(2^{2m}+\epsilon 2^m+1)$ and $m=n$. Thus $\bar{G}\simeq L_3^{\epsilon}(2^n)\simeq H$.

(4). $N=1$.

Suppose $N\neq 1$. Since $\pi(N)\subseteq\pi_1(G)$, N is a normal π_1-subgroup of G. By $t(G)\geq 2$, a subgroup R/N of order $d\in\pi_2(G/N)$ acts fixed-point freely on N, so by a result of Thompson, [10, p.337], N is nilpotent. Replacing N by a factor group, we may suppose N is an elementary abelian r-subgroup, where r is a prime such that $r=2$, $r|q+\epsilon$ or $r|q-\epsilon$.

Case (i). Suppose $r|q+\epsilon$. Since $2r\notin\pi_e(G)$, it follows that a Sylow 2-subgroup P_2 of $\bar{G}$ acts fixed-point freely on N, so that P_2 is cyclic or generalized quaternion, a contradiction since G_1 is non-solvable.

Case (ii). Suppose $r=2$. Let Q be a cyclic subgroup of order $\eta(q^2+\epsilon q+1)$ in G_1/N, and $N(Q)=N_{G_1/N}(Q)$. By [7, Theorem 3.1], $N(Q)=Q{:}\mathbb{Z}_3$ and suppose $[N](Q{:}\mathbb{Z}_3)$ is its preimage in G_1. It follows by [10, Chapter 3, Theorem 4.4] that G_1 contains an element of order 6, which is impossible.

Case (iii). Suppose $r|q-\epsilon$. If $r=3$, then repeat the proof of Case (ii) above, so that by [1, Lemma 7], G_1 has an element of order 9 and $9\in\pi_e(H)$. Since $3|q-\epsilon$, it follows that $9|q-\epsilon$, $n\equiv 0$ or $3(\bmod 6)$ according as $\epsilon=+$ or $-$, and by Lemma 1.3, $6\in\pi_e(H)$, which is also impossible. If $r\neq 3$, we denote $S\in Syl_2(G_1)$. Let $\bar{G}=G_1/N\simeq L_3(q)$ (The case of $U_3(q)$ is similar since the centralizer of every involution has a normal Sylow 2-subgroup in the two classes simple groups [18]). We have the following observations about $\bar{G}$: The case of $q=4$ is discussed, see Proposition 2.(c). $Z(S)$ is an elementary abelian group of order q, and for any $1\neq x\in Z(S)$, $C_{\bar{G}}(\bar{x})\simeq SK$, where K is of order $(q-1)/\gcd(3,q-1)$ and acts on S as the set of diagonal matrices. And all elements of $Z(S)$ are conjugate in G_1.

By co-prime action ([10, Chapter 5, Theorem 3.16]), $N=\langle C_N(x)|1\neq x\in Z(S)\rangle$ and

$C_N(x) \neq 1$. Without lose any generality, we may assume that

$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

put $U = C_N(x)$, $\bar{C} = C_{\bar{G}}(\bar{x})$, so $\bar{C}$ acts on U. Assume $C_N(f) = 1$ for each element f of order 4 in S. Then $\bar{f}$ inverts each element of U, so $[\bar{f}, \bar{C}] \subseteq C_{\bar{C}}(U)$. Put $D = \langle f \in S | f^2 = x\rangle$. As $q > 4$, $S/Z(S)$ is the direct sum of two non-isomorphic K-submodules of order q. As D is K-invariant, $D = S$. Thus $[\bar{S}, \bar{C}] \subseteq C_{\bar{C}}(U)$. But $\bar{S} = [\bar{S}, \bar{C}]$, so $[S, U] = 1$ which is a contradiction.

Therefore $N = 1$ and $G_1 \simeq L_3^\epsilon(q)$. Now G_1 is normal in G and by $t(G) \geq 2$, $C_G(G_1) = 1$, so

$$G_1 \leq G \leq \mathrm{Aut}(G_1),$$

since $N_G(G_1)/C_G(G_1) \leq \mathrm{Aut}(G_1)$. Finally, we show that

(5). $G = G_1$.

Suppose $G = G_1{:}T$ and $q = 2^n$, where $T \leq \mathrm{Out}(G_1)$. In the notation of Kleidman and Liebeck, [13, Propositions 2.2.3 and 2.3.5] and the fact that the automorphism group of $L_3^\epsilon(q)$ splits over $L_3^\epsilon(q)$,

$$\ddot{A} = \mathrm{Out}(G_1) = \begin{cases} \langle \ddot{\delta}, \ddot{\phi}, \ddot{\iota}\rangle \simeq \mathbb{Z}_{(3,q-1)}{:}\mathbb{Z}_n{:}\mathbb{Z}_2 & \text{if } \epsilon = +, \\ \langle \ddot{\delta}, \ddot{\phi}\rangle \simeq \mathbb{Z}_{(3,q+1)}{:}\mathbb{Z}_{2n} & \text{if } \epsilon = -, \end{cases}$$

where $\delta = \mathrm{diag}\,\{\lambda, 1, 1\} \in \mathrm{GL}_3^\epsilon(q)$, $\phi((a_{ij})) = (a_{ij}^2)$ for each matrix $(a_{ij}) \in \mathrm{GL}_3^\epsilon(q)$ and ι is the inverse-transpose of $\mathrm{GL}_3(q)$. Let

$$L = \mathrm{GL}_2^\epsilon(q) = \left\{ \begin{pmatrix} \det(X)^{-1} & 0 \\ 0 & X \end{pmatrix} : X \in \mathrm{GL}_2^\epsilon(q) \right\} \leq K = \mathrm{SL}_3^\epsilon(q).$$

Then $L \simeq \mathbb{Z}_{q-\epsilon} \times L_2(q)$ and $L \leq C_K(\delta)$, where $C_K(\delta)$ is the fixed-point set of δ in K. So

$$C_L(\ddot{A}) = C_{\mathbb{Z}_{q-\epsilon}}(\ddot{A}) \times C_{L_2(q)}(\ddot{A})$$

and $C_{L_2(q)}(\ddot{\phi}) = L_2(2) = \langle \sigma, \tau\rangle \simeq S_3 \leq G_1$, where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $|\sigma| = 3$, $|\tau| = 2$ and $\tau^{-1}\sigma\tau = \sigma^\tau = \sigma^{-1}$. If $\epsilon = -$, then $T \leq \langle \ddot{\delta}, \ddot{\phi}\rangle$.

Suppose $\epsilon = +$ and $w = \ddot{\delta}^k\ddot{\phi}^\ell\ddot{\iota} \in T$ is a 2-element for some integers k and ℓ. Then $\sigma^{w\tau} = \sigma$ and 6 is a divisor of $|\sigma w\tau|$, which is impossible. Thus $T \leq \langle \ddot{\delta}, \ddot{\phi}\rangle$, since $w \in$

$\langle w^2\rangle \subseteq \langle\ddot{\delta},\ddot{\phi}\rangle$ when w is odd. So $L_2(2) \le C_{G_1}(\langle\ddot{\delta},\ddot{\phi}\rangle) \le C_{G_1}(T)$ and neither 2 nor 3 is a factor of $|T|$.

Since $|\ddot{\delta}| = 1$ or 3, it follows that $T \cap \langle\ddot{\delta}\rangle = 1$ and $T\langle\ddot{\delta}\rangle/\langle\ddot{\delta}\rangle \simeq T$. But $\langle\ddot{\delta},\ddot{\phi}\rangle/\langle\ddot{\delta}\rangle \simeq \langle\ddot{\phi}\rangle$ is cyclic, so is T. If $T = \langle w\rangle$, then $w \in \{\ddot{\phi}^k, \ddot{\delta}\ddot{\phi}^k, \ddot{\delta}^{-1}\ddot{\phi}^k\}$ for some k.

If $w = \ddot{\delta}\ddot{\phi}^k$ with k odd, then by [13, Propositions 2.2.3 and 2.3.5], $w^2 = \ddot{\delta}\ddot{\phi}^k\ddot{\delta}\ddot{\phi}^{-k}\ddot{\phi}^{2k} = \ddot{\phi}^{2k}$. If $w = \ddot{\delta}\ddot{\phi}^k$ with k even, then $w^3 = \ddot{\phi}^{3k}$. Since $T = \langle w\rangle = 3D\langle w^2\rangle = \langle w^3\rangle$, we may suppose $T = \langle w\rangle$ with $w = \ddot{\phi}^\ell$ for some ℓ. Similarly, if $w = \ddot{\delta}^{-1}\ddot{\phi}^k$, then we may also suppose $T = \langle w\rangle$ with $w = \ddot{\phi}^\ell$ for some ℓ.

If $G_1 = L_3(q)$, then $C_{G_1}(\ddot{\phi}) = L_3(2) \le C_{G_1}(\ddot{\phi}^\ell) = C_{G_1}(T)$. If $G_1 = U_3(q)$, then $T = \langle w^2\rangle = \langle\ddot{\phi}^{2\ell}\rangle$ and $C_{G_1}(\ddot{\phi}^2) = U_3(2) \le C_{G_1}(\ddot{\phi}^{2\ell}) = C_{G_1}(T)$. Since $L_3^\epsilon(2)$ has an element of order 4, it follows that $4|w| \in \pi_e(G) = \pi_e(H)$. But $4t \notin \pi_e(H)$ for any $t \ge 1$, so $T = 1$ and $G = G_1$. This proves Theorem 2.1.

Since up to now, only finitely many non-abelian finite simple groups are found to be non-distinguishable, we raise the following conjecture, which is opposite to Proposition 1.

Conjecture. *Almost all non-abelian finite simple groups are characterizible.*

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