The p-local Rank of a Block

Jianbei An and Charles W. Eaton *

August 19, 1999

Abstract

We generalize the p-local rank of a finite group, introduced by G. Robinson [7], to a p-block of a finite group and show that this has properties analagous to most of those given in [7].

1 Introduction

Let G be a finite group and p a prime. Given a chain of p-subgroups

$$\sigma : Q_0 < Q_1 < \cdots < Q_n \tag{1.1}$$

of G, define the length $|\sigma| = n$, the final subgroup $V^\sigma = Q_n$, the initial subgroup $V_\sigma = Q_0$, the k-th initial subchain

$$\sigma_k : Q_0 < Q_1 < \cdots < Q_k,$$

and the normalizer

$$G_\sigma = N_G(\sigma) = N_G(Q_0) \cap N_G(Q_1) \cap \cdots \cap N_G(Q_n). \tag{1.2}$$

Write $\mathcal{C}(G|Q)$ for the set of those chains with initial subgroup Q and $\mathrm{Blk}(G)$ for the set of all p-blocks of G.

Let $(K, \mathcal{O}, k)$ be a p-modular system which splits for G, and consider a p-block B of G with defect group $D(B)$. Given a subgroup H of G, write $\mathrm{Blk}(H|B) = \{b \in \mathrm{Blk}(H) : b^G = B\}$ (in the sense of Brauer).

The Knörr-Robinson reformulation of Alperin's weight conjecture (see [5] and [1]) states that if B has positive defect then $\sum_{\sigma \in \mathcal{C}(G|1)} l(G_\sigma, B) = 0$, where $l(G_\sigma, B)$ denotes the number of simple kG_σ-modules belonging to p-blocks of G_σ whose Brauer correspondent is B. Clearly if $l(G_\sigma, B) \neq 0$ then there is some p-block b of G_σ with $b^G = B$. It thus makes sense to consider the set

$$\mathcal{C}(G, B) = \{\sigma \in \mathcal{C}(G) : \mathrm{Blk}(G_\sigma|B) \neq \emptyset\}.$$

*Research carried out whilst the second author was visiting the University of Auckland, supported by the Marsden Fund and a Royal Society Study Visit award.

1991 *Mathematics Subject Classification.* Primary 20C20

We say that σ is a *p-chain for B*, or a *B-chain* (as we will see we may associate with σ a chain of B-subgroups in the sense of Alperin [2]) if $\sigma \in \mathcal{C}(G, B)$. This is in part the motivation for this paper.

As demonstrated in [5], when considering this and similar conjectures we may restrict our attention to certain subsets of $\mathcal{C}(G)$, such as the following. A p-subgroup R of G is *radical* if $O_p(N_G(R)) = R$, where $O_p(N_G(R))$ is the maximal normal p-subgroup of the normalizer $N_G(R)$. A p-chain σ given by (1.1) is said to be *radical* if $Q_i = O_p(N_G(\sigma_i))$ for each i. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G and write

$$\mathcal{R}(G, B) = \{\sigma \in \mathcal{R}(G) : \mathrm{Blk}(N_G(\sigma)|B) \neq \emptyset\}.$$

By [7], the p-local rank $plr(G)$ of G is the number $plr(G) = \max\{|\sigma| : \sigma \in \mathcal{R}(G)\}$. We define the p-local rank $plr(B)$ of B to be the number

$$plr(B) = \max\{|\sigma| : \sigma \in \mathcal{R}(G, B)\}.$$

It is clear that $plr(B) \leq plr(G)$ for all $B \in \mathrm{Blk}(G)$. Also we can recover the p-local rank of G by considering the principal p-block. For suppose $plr(G) = n$ and $\sigma \in \mathcal{R}(G)$ with $|\sigma| = n$. If $b_0 = B_0(N_G(\sigma))$ is the principal block of $N_G(\sigma)$, then by [5, 3.2], b_0^G is well-defined and by Brauer's Third Main Theorem, $b_0^G = B_0 = B_0(G)$, so that $\sigma \in \mathcal{R}(G, B_0)$ and

$$plr(B_0) = plr(G).$$

The paper is organized as follows. In Section 2 we consider our definitions in terms of subpairs, and give the lemma which is the engineroom of this paper. In Sections 3 and 4 we prove that if B' is a p-block of a section of G corresponding to B under the Brauer correspondence and the block correspondence outlined in Section 4, then $plr(B') \leq plr(B)$, and give some conditions for equality/inequality. Finally, in Section 5 we give necessary and sufficient conditions for a p-block to have p-local rank one. We would like to draw the readers' attention to the fact that all proofs given here are direct, so the p-local structure is transparent(!) at each stage.

2 B-subgroups

Let R be a p-subgroup of G and $b_R \in \mathrm{Blk}(C_G(R)R)$. Then the pair (R, b_R) is called a *subpair* for G. If R is radical in G, then (R, b_R) is called a *radical subpair* for G. Following Alperin [2], for $B \in \mathrm{Blk}(G)$, the pair (R, b_R) is called a *B-subgroup* if $(1, B) \leq (R, b_R)$.

If (R, b_R) is a B-subgroup, then $b_R^G = B$ and $b_R^{N_G(R)}$ is a well-defined block of $N_G(R)$. In particular, $b_R^{N_G(R)} \in \mathrm{Blk}(N_G(R)|B)$ and $\mathrm{Blk}(N_G(R)|B) \neq \emptyset$. Conversely, if $\mathrm{Blk}(N_G(R)|B) \neq \emptyset$, then R is conjugate to a subgroup of $D(B)$ and there is a B-subgroup (R, b_R).

A B-subgroup (R, b_R) is called a *radical B-subgroup* if R is radical in G.

Suppose that $\sigma \in \mathcal{R}(G, B)$. Then there is a block $b \in \mathrm{Blk}(N_G(\sigma)|B)$ and we may suppose that $D(b) \leq D(B)$. Since $Q_0, \ldots, Q_n \trianglelefteq N_G(\sigma)$, it follows that $Q_i \leq D(b)$ and there is a B-subgroup $(Q_i, b_i) \leq (D(B), b_D)$ for each i, where $(D(B), b_D)$ is a fixed Sylow B-subgroup. Conversely if we can associate to $\sigma \in \mathcal{R}(G)$ such a chain of B-subgroups, then $\sigma \in \mathcal{R}(G, B)$. A radical p-chain σ is called a *radical B-chain* if $\sigma \in \mathcal{R}(G, B)$.

Lemma 2.1 *Let B be a p-block of G and (Q, b_Q) a B-subgroup. Then there is a radical B-subgroup (P, b_P) such that*

$$(Q, b_Q) \leq (P, b_P)$$

and $N_G(Q) \leq N_G(P)$. In particular, (Q, b_Q) and (P, b_P) are subpairs for $N_G(P)$. Moreover, suppose that $Q \leq H \leq G$ and Q is radical in H. Then $Q = P \cap N_H(Q) = P \cap H$ and $N_H(Q) = H \cap N_G(P)$.

Proof: Let $M_1 = N_G(Q)$ and $M_i = N_G(O_p(M_{i-1}))$ for all $i \geq 2$. Then

$$N_G(Q) = M_1 \leq M_2 \leq \ldots \leq M_m \tag{2.1}$$

for all $m \geq 2$ and since $Q \leq O_p(M_1)$ and $O_p(M_{i-1}) \leq O_p(M_i)$, it follows that

$$Q \trianglelefteq O_p(M_1) \trianglelefteq O_p(M_2) \trianglelefteq \ldots \trianglelefteq O_p(M_m). \tag{2.2}$$

Thus there is a minimal $m \geq 1$ such that $M_m = M_{m+1} = N_G(O_p(M_m))$, so $O_p(M_m) = O_p(M_{m+1})$ and $O_p(M_m)$ is radical in G. Let $L = M_m$ and $P = O_p(L)$, so that $N_G(Q) \leq L = N_G(P)$. Since $C_G(Q) \leq N_G(Q) \leq L$, it follows that $C_G(Q) = C_L(Q)$, (Q, b_Q) is a subpair for L and b_Q^L is a well-defined block of L. Thus (Q, b_Q) is contained in a Sylow b_Q^L-subgroup (D_1, b_{D_1}), where $D_1 =_L D(b_Q^L)$. Since $P \leq D_1$, it follows that there is a unique b_Q^L-subgroup (P, b_P) such that

$$(Q, b_Q) \leq (P, b_P) \trianglelefteq (D_1, b_{D_1}).$$

But $C_G(P) \leq C_G(Q) \leq L$, so (P, b_P) is also a subpair for G and $(1, B) \leq (Q, b_Q) \leq (P, b_P)$. Thus (P, b_P) is a B-subgroup.

Suppose that $Q \leq H \leq G$ and $Q = O_p(K)$, where $K = N_H(Q) \leq L$. Then $Q \leq K \cap P \trianglelefteq K$ and $Q = K \cap P$. Since $Q \trianglelefteq N_{H\cap P}(Q) \leq K$ and $N_{H\cap P}(Q) \leq K \cap P = Q$, it follows that

$$Q = P \cap K = P \cap H.$$

If $h \in L \cap H$, then h normalizes $O_p(L) \cap H = Q$, so that $h \in K$, $K \leq L \cap H \leq K$ and $K = L \cap H$.

$\square$

3 Subgoups and the p-local rank

Proposition 3.1 *Let H be a subgroup of G and $b \in \mathrm{Blk}(H|B)$. Suppose that*

$$\sigma : O_p(H) = Q_0 < Q_1 < \ldots < Q_n$$

is a radical b-chain with $n \geq 1$. Set $K_i = N_H(\sigma_i)$ for $i \geq 0$. Then there is a radical B-chain

$$\sigma^* : O_p(G) < P_1 < \ldots < P_n$$

of G such that for each $i \geq 1$, $K_i \leq N_{L_{i-1}}(Q_i) \leq L_i$, $K_i = L_i \cap H$, $C_G(Q_i) = C_{L_i}(Q_i)$ and $Q_i = P_i \cap K_i = P_i \cap H$, where $L_i = N_G(\sigma_i^*)$.

PROOF: We may suppose $D(b) \le D(B)$. Since σ is a radical b-chain, it follows that $\mathrm{Blk}(N_H(\sigma)|b) \neq \emptyset$ and we may suppose that

$$O_p(H) = Q_0 < Q_1 < \ldots < Q_n \le D(b) \le D(B). \tag{3.1}$$

Let $(D(B), b_D)$ be a Sylow B-subgroup. For each $i \ge 0$, there is a B-subgroup (Q_i, b_i) such that $(Q_i, b_i) \le (D(B), b_D)$.

By Lemma 2.1, there is a radical B-subgroup (P_1, B_1) such that $(Q_1, b_1) \le (P_1, B_1)$, $K_1 \le N_G(Q_1) \le L_1 = N_G(P_1)$, $C_G(Q_1) = C_{L_1}(Q_1)$, $Q_1 = P_1 \cap K_1 = P_1 \cap H$ and $K_1 = H \cap L_1$. Thus $O_p(G) \le P_1$ and by (3.1),

$$O_p(G) \cap H \le O_p(H) < Q_1 \le P_1 \cap H.$$

In particular, $P_1 \neq O_p(G)$,

$$\sigma_1^* : O_p(G) < P_1$$

is a radical B-chain and $L_1 = N_G(\sigma_1^*)$.

Suppose that $n \ge 2$ and there is a radical B-chain

$$\sigma_t^* : O_p(G) < P_1 < \ldots < P_t$$

with $1 \le t < n$ satisfying the conditions of the Proposition. Since σ is radical, it follows that $Q_{t+1} \le Q_n \le K_n \le K_t \le L_t$, $C_G(Q_{t+1}) \le C_G(Q_t) = C_{L_t}(Q_t)$ and (Q_{t+1}, b_{t+1}) is a subpair for L_t.

Apply Lemma 2.1 for $Q = Q_{t+1}$, $G = L_t$ and $H = K_t$. There is a radical subpair (P_{t+1}, B_{t+1}) for L_t such that $(Q_{t+1}, b_{t+1}) \le (P_{t+1}, B_{t+1})$, $K_t \le N_{L_t}(Q_{t+1}) \le L_{t+1} = N_{L_t}(P_{t+1})$, $Q_{t+1} = P_{t+1} \cap K_t = P_{t+1} \cap K_{t+1}$, $K_{t+1} = L_{t+1} \cap K_t$ and $C_{L_t}(Q_{t+1}) = C_{L_{t+1}}(Q_{t+1})$.

Since $K_t = L_t \cap H$, it follows that $K_{t+1} = L_{t+1} \cap L_t \cap H = L_{t+1} \cap H$ and $Q_{t+1} = P_{t+1} \cap K_t = P_{t+1} \cap H$. Similarly, since $C_G(P_{t+1}) \le C_G(Q_t) = C_{L_t}(Q_t) \le L_t$, it follows by Lemma 2.1 that

$$C_G(P_{t+1}) = C_{L_t}(P_{t+1}) = C_{L_{t+1}}(P_{t+1}),$$

so that (P_{t+1}, B_{t+1}) is a subpair for G. But $(1, B) \le (Q_{t+1}, b_{t+1}) \le (P_{t+1}, B_{t+1})$, so (P_{t+1}, B_{t+1}) is a B-subgroup. Since $Q_t = P_t \cap H < Q_{t+1} = P_{t+1} \cap H$, it follows that $P_t < P_{t+1}$,

$$\sigma_{t+1}^* : O_p(G) < P_1 < \ldots < P_t < P_{t+1}$$

is a radical B-chain of G and $L_{t+1} = N_G(\sigma_{t+1}^*)$. The result now follows by induction on n.

$\square$

Proposition 3.2 *Let H be a subgroup of G and $b \in \mathrm{Blk}(H|B)$. Then*

$$plr(b) \le plr(B).$$

PROOF: We may suppose $plr(b) \ge 1$. Suppose that σ is a radical b-chain given by (1.1) such that $|\sigma| = plr(b) = n$. Since Q_0 is radical in H, it follows that $O_p(H) \le Q_0$. If $O_p(H) < Q_0$, then

$$\sigma' : O_p(H) < Q_0 < Q_1 < \ldots < Q_n$$

is also a radical b-chain of H and so $plr(b) \geq |\sigma'| = plr(b) + 1$, which is impossible. Thus $O_p(H) = Q_0$. By Proposition 3.1, there is a radical B-chain σ^* of length n, so that $plr(b) \leq plr(B)$.

$\square$

We examine the situation where H contains $O^{p'}(G)$, the minimal normal subgroup with p-regular index in G. We first need the following in order to locate b-subgroups when moving from G to H:

Lemma 3.3 *Let $N \trianglelefteq G$ and $N \leq H \leq G$. Suppose that $B \in \mathrm{Blk}(G)$ and $b \in \mathrm{Blk}(H|B)$. Then*

(*i*) *there is $B' \in \mathrm{Blk}(N)$ such that B' is covered by both B and b,*

(*ii*) *B and b share a defect group D.*

PROOF: (i) By [3, 58.17] $b|B_{H\times H}$, and by [3, 61.4] B covers $B' \in \mathrm{Blk}(N)$ if and only if $B'|B_{N\times N}$ (and similarly for b). Suppose that B covers $B' \in \mathrm{Blk}(N)$ and b covers $B'' \in \mathrm{Blk}(N)$. Then $B''|b_{N\times N}$ and $b|B_{H\times H}$, so $B''|B_{N\times N}$, and B' is conjugate in G to B''. Hence we can take $B' = B''$.

(ii) Let $B' \in \mathrm{Blk}(N)$ be covered by both B and b as in part (i), and let $D(G)$ be a defect group for B, $D(H)$ a defect group for b such that $D(G) \cap N =_G D'$ for some defect group D' of B' and $D(H) \cap N =_H D''$ for some other defect group D'' of B' (see [3, 61.2]). We may assume that $D'' = D'$. But $D(G) \cap N = D(G)$ and $D(H) \cap N = D(H)$, so $D(G) =_G D' =_H D(H)$, and we may take $D(G) = D(H)$.

$\square$

Proposition 3.4 *Suppose that $O^{p'}(G) \leq H \leq G$ and $b \in \mathrm{Blk}(H|B)$. Then $\mathcal{R}(G,B) = \mathcal{R}(H,b)$, and in particular $plr(B) = plr(b)$.*

PROOF: Write $N = O^{p'}(G)$, and let D be a defect group for B and b as in Lemma 3.3.

Note that if X and Y are finite groups with $X \trianglelefteq Y$ and $[Y:X]_p = 1$, then $O_p(X) = O_p(Y)$, since $O_p(X)$ is the *unique* maximal normal p-subgroup of X. Also note that every p-subgroup of Y is contained in X.

Suppose that $\sigma \in \mathcal{R}(G,B)$. Then we may take $V^\sigma \leq D \leq H$. Now

$$O^{p'}(N_G(\sigma)) \leq N_H(\sigma) \leq N_G(\sigma),$$

so $O_p(N_H(\sigma)) = O_p(O^{p'}(N_G(\sigma))) = O_p(N_G(\sigma)) = V^\sigma$. Hence $\sigma \in \mathcal{R}(H)$. But $V^\sigma \leq D$, a defect group for b, so there is a b-subgroup (V^σ, b_{V^σ}). But $b_{V^\sigma}^{N_H(\sigma)}$ is a well-defined block, as is $(b_{V^\sigma}^{N_H(\sigma)})^H$, and $b_{V^\sigma}^H = b$, so $\sigma \in \mathcal{R}(H,b)$.

Ther argument that $\mathcal{R}(H,b) \subseteq \mathcal{R}(G,B)$ is similar.

$\square$

Proposition 3.5 *Let G_1 and G_2 be finite groups and $B = B_1 \times B_2$ a p-block of $G = G_1 \times G_2$, where each $B_i \in \mathrm{Blk}(G_i)$. Then*

$$plr(B) = plr(B_1) + plr(B_2).$$

PROOF: Let π_i be the natural projection of G onto G_i for $i = 1, 2$. If $\sigma \in \mathcal{R}(G)$ is given by (1.1), then let $\Omega_i = \{\pi_i(Q_j) : 0 \leq j \leq n\}$ and $|\Omega_i| = u_i$. Relabel the subgroups in Ω_i such that

$$\Omega_i = \{W_0 < W_1 < \ldots < W_{u_i}\}.$$

It follows by [7, 4.6] that each radical subgroup R of G is of the form $R = R_1 \times R_2$, where $R_1 = R \cap G_1$ and $R_2 = R \cap G_2$. Thus $N_G(R) = N_{G_1}(R_1) \times N_{G_2}(R_2)$, and it follows that $\pi_i(\sigma) : W_0 < W_1 < \ldots < W_{u_i}$ is a radical chain of G_i and

$$N_G(\sigma) = N_{G_1}(\pi_1(\sigma)) \times N_{G_2}(\pi_2(\sigma)).$$

In addition, if σ is a radical B-chain, then $\pi_i(\sigma)$ is a radical B_i-chain, so that $plr(B) \leq plr(B_1) + plr(B_2)$.

Conversely, suppose that $\sigma_1 : W_0 < W_1 < \ldots < W_{u_1}$ and $\sigma_2 : P_0 < P_1 < \ldots < P_{u_2}$ are radical B_1-chain and B_2-chain, respectively. Define

$$\sigma : Q_0 < Q_1 < \ldots < Q_{u_1+u_2},$$

such that $Q_i = W_i \times P_0$ for $0 \leq i \leq u_1$ and $Q_i = W_{u_1} \times P_{i-u_1}$ for $u_1 \leq i \leq u_1 + u_2$. Then σ is radical B-chain and $|\sigma| = |\sigma_1| + |\sigma_2|$. Thus $plr(B) \geq plr(B_1) + plr(B_2)$.

□

4 Quotient groups and the p-local rank

Let N be a normal subgroup of G and V an indecomposable OG-module which is indecomposable as an ON-module V_N, where O is $\mathcal{O}$ or k. Following [6], a block B of G *dominates* a block $\overline{B}$ of $\overline{G} = G/N$ through V if there is an $O\overline{G}$-module X such that $V \otimes \mathrm{Inf}(X)$ is a B-module, where $\mathrm{Inf}(X)$ is the inflation of X to G. This is equivalent, by [6], to the fact that B covers the block b of N containing V_N. In addition, by [6], $\overline{B}$ has a defect group QN/N such that

$$D \cap N \leq Q \leq D,$$

where $D = D(B)$.

Let $\overline{H} \leq \overline{G} = G/N$ and $\overline{b} \in \mathrm{Blk}(\overline{H}|\overline{B})$. We may suppose that $D(\overline{b}) \leq D(\overline{B}) = QN/N$. Since $QN/N \simeq Q/(Q \cap N)$, there is a subgroup W of Q such that $WN/N = D(\overline{b})$ and

$$D \cap N \leq Q \cap N \leq W \leq Q \leq D. \tag{4.1}$$

Moreover, suppose that W_1 and W_2 are subgroups of Q such that $Q \cap N \leq W_1 \leq W_2 \leq Q$ and $W_1N/N \trianglelefteq W_2N/N$. Then $W_1N/N \simeq \eta(W_1) \trianglelefteq \eta(W_2) \simeq W_2N/N$, where η is the natural group homomorphism from Q onto $Q/(Q \cap N)$. Since $Q \cap N = \ker(\eta) \leq W_1$, it follows that $W_1 \trianglelefteq W_2$.

Proposition 4.1 *Let N be a normal subgroup of G and $\overline{B} \in \mathrm{Blk}(G/N)$ which is dominated by a block B of G through V. Then*

$$plr(\overline{B}) \leq plr(B).$$

Moreovr, suppose that b is the block of N containing V_N and $plr(b) \geq 1$. Then $plr(\overline{B}) < plr(B)$.

Proof: Let $\overline{G} = G/N$ and $\sigma \in \mathcal{R}(\overline{G}, \overline{B})$ with $|\sigma| = n \geq 1$. Then there is a block $\bar{b} \in \mathrm{Blk}(N_{\overline{G}}(\sigma)|\overline{B})$, so we may suppose that $WN/N = D(\bar{b})$ for some subgroup $W \leq Q$ satisfying (4.1). Thus we may suppose that there are subgroups $Q_0, Q_1, \ldots, Q_n$ of Q such that

$$D \cap N \leq Q \cap N \leq Q_0 < Q_1 < \ldots < Q_n \leq Q \leq D \tag{4.2}$$

and $\sigma : Q_0N/N < Q_1N/N < \ldots < Q_nN/N$. Thus for each i with $0 \leq i \leq n$ there is a B-subgroup (Q_i, b_i) for G. Since σ is radical, it follows by the remark above that $Q_i \trianglelefteq Q_n$ for all $0 \leq i \leq n$.

By Lemma 2.1, there is a radical B-subgroup (P_1, b_{P_1}) containing (Q_1, b_1) and $N_G(Q_1) \leq L_1 = N_G(P_1)$, so that $O_p(G) \leq P_1$ and $Q_n \leq L_1$ as $Q_1 \trianglelefteq Q_n$. Since $\overline{O_p(G)} = O_p(G)N/N \trianglelefteq \overline{G}$ and $\overline{Q}_0$ is radical in $\overline{G}$, it follows that

$$\overline{O_p(G)} \leq \overline{Q}_0 < \overline{Q}_1 \leq \overline{P}_1,$$

and in particular, $O_p(G) < P_1$. Thus $\sigma_1^* : O_p(G) < P_1$ is a radical B-chain. In addition, since $N_G(Q_1) \leq L_1$, it follows that $\overline{N_G(Q_1)} = N_{\overline{G}}(\overline{Q}_1) \leq \overline{L}_1$, $\overline{Q}_1$ is a radical subgroup of $\overline{L}_1$, $\overline{P}_1 \leq \overline{Q}_1$ and $\overline{P}_1 = \overline{Q}_1$. Thus $N_{\overline{G}}(\overline{P}_1) = N_{\overline{G}}(\overline{Q}_1)$ and $\overline{L}_1 = N_{\overline{G}}(\overline{Q}_1)$.

Suppose that there is a radical B-chain

$$\sigma_t^* : O_p(G) < P_1 < \ldots < P_t$$

with $1 \leq t < n$ such that $Q_n \leq N_{L_{t-1}}(Q_t) \leq L_t$, $Q_t \leq P_t$, $\overline{Q}_t = \overline{P}_t$, $C_G(Q_t) = C_{L_t}(Q_t)$ and $N_{\overline{G}}(\sigma_t) = \overline{L}_t$, where $L_0 = G$. Thus $Q_{t+1} \leq L_t$, $C_G(Q_{t+1}) \leq C_G(Q_t) = C_{L_t}(Q_t) \leq L_t$ and $C_G(Q_{t+1}) = C_{L_t}(Q_{t+1})$. So (Q_{t+1}, b_{t+1}) is a subpair for L_t and by Lemma 2.1, $(Q_{t+1}, b_{t+1}) \leq (P_{t+1}, B_{t+1})$ for some radical subpair (P_{t+1}, B_{t+1}) for L_t and $N_{L_t}(Q_{t+1}) \leq L_{t+1} = N_{L_t}(P_{t+1})$. Since $C_G(P_{t+1}) \leq C_G(Q_{t+1}) \leq L_t$, it follows that $C_G(P_{t+1}) = C_{L_t}(P_{t+1}) \leq L_{t+1}$, so that $C_G(P_{t+1}) = C_{L_{t+1}}(P_{t+1})$ and (P_{t+1}, B_{t+1}) is a subpair for G. Since $(1, B) \leq (Q_{t+1}, b_{t+1}) \leq (P_{t+1}, B_{t+1})$, it follows that (P_{t+1}, B_{t+1}) is a B-subgroup. Since $Q_n \leq L_t$ and $Q_{t+1} \trianglelefteq Q_n$, it follows that $Q_n \leq L_{t+1}$.

Since P_{t+1} is radical in L_t and $N_{L_t}(Q_{t+1}) \leq L_{t+1}$, it follows that $P_t = O_p(L_t) \leq P_{t+1}$,

$$\overline{N_{L_t}(Q_{t+1})} = N_{\overline{L}_t}(\overline{Q}_{t+1}) = N_{\overline{G}}(\sigma_{t+1}) \leq \overline{L}_{t+1},$$

$\overline{Q}_{t+1}$ is radical in $\overline{L}_{t+1}$ and $\overline{P}_{t+1} \leq \overline{Q}_{t+1}$. Thus $\overline{P}_{t+1} = \overline{Q}_{t+1}$ and $N_{\overline{G}}(\sigma_{t+1}) = \overline{L}_{t+1}$. In particular, $\overline{Q}_t = \overline{P}_t < \overline{Q}_{t+1} = \overline{P}_{t+1}$ and $P_t \neq P_{t+1}$. Thus

$$\sigma_{t+1}^* : O_p(G) < P_1 < \ldots < P_t < P_{t+1}$$

is a radical B-chain and by induction on n, $plr(\overline{B}) \leq plr(B)$.

Suppose that $plr(b) > 0$, so that B covers b by [6]. We may choose above D so that $D(b) = D \cap N$. Thus

$$O_p(G) \cap N \leq O_p(N) < D \cap N \leq Q_0$$

as $plr(b) \geq 1$. By Lemma 2.1, there is a radical B-subgroup (W, b_W) containing the B-subgroup (Q_0, b_0) and $N_G(Q_0) \leq N_G(W)$. Thus $O_p(G) \cap N < D \cap N \leq Q_0 \leq W \cap N$ and $O_p(G) < W$. Repeat the proof above with G replaced by $N_G(W)$. Then there is a radical B-chain $O_p(Q) < W < P_1 < \ldots < P_n$ of length $n+1$, so that $plr(\overline{B}) < plr(B)$. $\square$

5 Blocks of p-local rank one

We say that a subgroup H of G is trivial intersection (TI) if $H^g \cap H = 1$ for every $g \in G - N_G(H)$. By [7, 7.1] if $plr(G) > 0$, then $plr(G) = 1$ if and only if $G/O_p(G)$ has TI Sylow p-subgroups. Hence if B is the principal p-block, then $plr(B) = 1$ if and only if B has defect groups intersecting pairwise in $O_p(G)$. We show that this may be generalized to any p-block.

We then highlight the consequence of this to conjectures such as those of Alperin [1], Dade [4] and of Robinson [7], although the consequence to Alperin's conjecture has been proved in [8].

Proposition 5.1 *Let $B \in \mathrm{Blk}(G)$ have defect group $D \neq O_p(G)$. Then $plr(B) = 1$ if and only if $D/O_p(G)$ is a TI set in $G/O_p(G)$. Further, in this case $O_p(G)$ is the only radical p-subgroup of G contained in D.*

PROOF: Suppose first that $D/O_p(G)$ is a TI subgroup of $G/O_p(G)$, and that $Q \neq O_p(G)$ is a radical p-subgroup of G such that $\mathrm{Blk}(N_G(Q)|B) \neq \emptyset$. Then Q is conjugate to a subgroup of D, and we may suppose that $Q \leq D$. Suppose that $Q \neq D$. Since $N_G(Q) \leq N_G(D)$, it follows that $N_D(Q) \trianglelefteq N_G(Q)$, and so $Q < N_D(Q) \leq O_p(N_G(Q))$, contradicting our assumption that Q is radical. Hence a radical B-chain σ of maximal length must satisfy $\sigma_1 : O_p(G) < D$, since D is a radical p-subgroup such that $\mathrm{Blk}(N_G(D)|B) \neq \emptyset$. But V^σ is G-conjugate to a subgroup of D, so $O_p(G) < D$ is a radical p-chain for B of maximal length, and $plr(B) = 1$.

Now suppose that $plr(B) = 1$, but that $D/O_p(G)$ is not TI. We show first that D cannot contain any radical p-subgroups of G other than $O_p(G)$ and itself.

Suppose that R is a radical p-subgroup of G with $O_p(G) < R < D$, and let (D, b_D) be a Sylow B-subgroup. Then there is a unique radical B-subgroup (R, b_R) such that $(1, B) < (R, b_R) < (D, b_D)$. Now $b_R^{N_G(R)}$ is a well-defined block of $N_G(R)$ and has defect group R, since otherwise $O_p(G) < R < D(b_R^{N_G(R)})$ is a radical B-chain and $plr(B) > 1$, contradicting our original assumption. But then $(b_R^{N_G(R)})^G$ must have defect group $R \neq D$ by Brauer's First Main Theorem, a contradiction since $(b_R^{N_G(R)})^G = B$ and $D(B) = D$.

Since $D/O_p(G)$ is not TI, we can choose a defect group $D_1 \neq D$ for B with $|D_1 \cap D|$ maximized. Write $Q = D_1 \cap D > O_p(G)$. As in the above argument, there is a

B-subgroup (Q, b_Q) such that $(Q, b_Q) < (D, b_D)$. By Lemma 2.1, there is radical B-subgroup (P, b_P) strictly containing (Q, b_Q) with $N_G(Q) \leq N_G(P)$ (since Q is not itself a radical B-subgroup).

Now we have seen that the only radical p-subgroups for B are either $O_p(G)$ or conjugate to D. By maximality, $Q = P \cap D = P \cap D_1$, hence we may take $P = D$, and $N_G(Q) \leq N_G(D)$.

Let (D_1, b_{D_1}) be a Sylow B-subgroup and $R = N_{D_1}(Q)$. Then $Q < R \leq D_1$, so there is a B-subgroup (R, b_R). Since

$$C_G(R) \leq C_G(Q) \leq N_G(Q) \leq N_G(D),$$

(R, b_R) is a subpair for $N_G(D)$, and $b_R^{N_G(D)}$ is defined.

Let $D(b_R^{N_G(D)})$ be a defect group for $b_R^{N_G(D)}$ containing R. Then $D \leq D(b_R^{N_G(D)})$ as $D \trianglelefteq N_G(D)$. Since $(b_R^{N_G(D)})^G = b_R^G = B$, it follows that $D(b_R^{N_G(D)})$ is G-conjugate to a subgroup of D, so $D = D(b_R^{N_G(D)})$. Hence $Q < R = N_{D_1}(Q) \leq D(b_R^{N_G(D)}) = D$, and $Q < |N_{D_1}(Q)| < |D_1 \cap D|$, contradicting our choice of Q.

□

Remark We note that there is already a notion of a *TI block* (see [1]), where B is TI if every nontrivial B-subgroup (Q, b_Q) is contained in a unique Sylow B-subgroup. However, it is not clear whether this is equivalent to B having TI defect groups. To distinguish between the two definitions, we say that a block with TI defect groups is a *TI defect block*.

Corollary 5.2 *Suppose that B is a p-block of G. Let $f : \mathcal{R}(G, B) \to A$ be a G-stable function dependent only on $N_G(\sigma)$ and V_σ, where A is an abelian group. If $D \neq O_p(G)$ is a defect group for B, $plr(G) > 0$ and any two defect groups for B intersect in $O_p(G)$, then*

$$\sum_{\sigma \in \mathcal{R}(G,B)/G} (-1)^{|\sigma|} f(N_G(\sigma), V_\sigma) = f(G, O_p(G)) + f(N_G(D), D) - f(N_G(D), O_p(G)).$$

Remark With analogy to the remarks in [5] we may replace $\mathcal{R}(G, B)$ by $\mathcal{D}_{max}$ in any alternating sum defined by such a function f, where $\mathcal{D}_{max}$ is the set of p-chains of G which are intersections of defect groups of B (here we do not make any assumptions on $plr(B)$. The proof is similar to that given in [5], so we do not include it here.

Acknowlegment

The authors would like to thank the Marsden Fund committee and The Royal Society for their support, and the University of Auckland for its generous hospitality.

References

[1] J.L. Alperin. Weights for finite groups. In *The Arcata Conference on Representations of Finite Groups, Proc. of Symposia in Pure Math.* **47** (1987), 369-379.

[2] J.L. Alperin. *Local representation theory* (Cambridge University Press, 1986).

[3] C.W. Curtis and I. Reiner. *Methods of representation theory II* (Wiley-Interscience, 1987).

[4] E.C. Dade. Counting Characters in Blocks, II.9. In *Representation theory of finite groups*, Ohio State Univ. Math. Res. Inst. Publ., 6 (de Gruyter, Berlin, 1997), 45-59.

[5] R. Knorr and G. Robinson. Some remarks on a conjecture of Alperin. *J. London Math. Soc.* (2) **39** (1989), 48-60.

[6] M. Murai. Normal subgroups and heights of character. *J. Math. Kyoto Univ.* **36** (1996), 31-43.

[7] G. Robinson. Local structure, vertices and Alperin's conjecture. *Proc. London Math. Soc.* (3) **72** (1996), 312-330.

[8] Zeng Jiwen. The problem of the Alperin conjecture for a block with a TI defect group. *Chinese Ann. Math. Ser. A* **19** (1998), no.4, 519-524.

Department of Mathematics
University of Auckland
Auckland, New Zealand

6 Turner Road
Coventry
West Midlands
CV5 8FT, United Kingdom