Homeomorphism Groups of Manifolds*

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Abstract

In this paper we present the homeomorphism groups of manifolds, explaining why non–metrizable manifolds are better behaved, with regard to their homeomorphism groups, than metrizable manifolds. A proof that the natural topology on the homeomorphism group for a one dimensional metrizable manifold is the minimum group topology but the homeomorphism group does not admit a minimum group topology for a more than one dimensional metrizable manifold will be given. Likewise, examples demonstrating how badly behaved are the homeomorphism groups of continua, in comparison with homeomorphism groups of manifolds is also given.

1 Introduction

For a space X, write $H(X)$ for the group of all homeomorphisms of X. Endowed with the **compact–open–topology** , $H(X)$ is a topological group (under certain restrictions on X).

The compact–open–topology on $H(X)$ is the topology generated by the base consisting of all sets $\bigcap_{i=1}^{k} M(C_i, U_i)$, where C_i is a compact subset of X and U_i is an open subset of X for each $i = 1, 2, ..., k$ and where, for, $A, B \subset X$, $M(A, B) = \{f \in H(X) : f(A) \subset B\}$. Also we define **pointwise** topology on the group $H(X)$ by considering it as a subspace of the product X^X where X has the discrete topology, so a basic open neighbourhood of g

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in $H(X)$ has the form: $B(g, F) = \{h \in H(X) : g(x) = h(x) \text{ for all } x \in F\}$, where F is a finite subset of X.

In this paper we consider the homeomorphism groups of manifolds and of continua. We are concerned with the following two problems:

1. when does $H(X)$ determine X?
2. when does the algebraic structure of $H(X)$ determine the topological structure?

In 1963 Whittaker [7], considered the first problem on compact manifolds.

Theorem 1.1 *(Whittaker [7]) If M and N are compact manifolds, and $H(M)$ is algebraically isomorphic to $H(N)$, then there is a homeomorphism of M with N 'realising' the isomorphism of $H(M)$ and $H(N)$, i.e., there is a homeomorphism $h : M \to N$, such that $\phi : H(M) \to H(N)$ is an isomorphism, where $\phi(a) = h \circ a \circ h^{-1}$. Thus every group automorphism of $H(M)$ is an inner automorphism, and so must be continuous in any group topology on $H(M)$.*

In 1995 [5], Kawamura proved that Whittaker's result is true for Menger Manifolds.

Whittaker's result does not hold for non–compact metrizable manifolds.

Example 1.2 *[7]*

The homeomorphism groups $H((0, 1))$ and $H([0, 1])$ are topologically isomorphic.

Thus the algebraic structure of the homeomorphism group of a metrizable manifold need not uniquely determine the manifold. However, the algebraic structure does uniquely determine the natural group topology on the homeomorphism group.

Theorem 1.3 *(Kallman)[4]*

Let M be a metrizable manifold. Then $H(M)$ has a unique Polish group topology. Hence if $H(M)$ and $H(N)$ are algebraically isomorphic (where N is another separable metrizable manifold), then $H(M)$ and $H(N)$ are topologically isomorphic. It follows that every group automorphism of $H(M)$ is continuous.

In Section 2, we show that any non–metrizable manifold is uniquely determined by its homeomorphism group.

In section 3, we investigate minimum group topologies on $H(M)$. We prove that the compact open topology on $H(M)$ for a one dimensional metrizable manifold M is the minimum group topology but $H(M)$ does not admit a minimum group topology for a metrizable manifold M of dimension greater than one. We also remark that the usual (compact–open) topology on the homeomorphism group of the Cantor set is the minimum group topology, but that the usual topology on $H(I^\omega)$ is not minimal.

In section 4, we use Cook's curve [1] to construct further examples of continua which demonstrate how badly continua fail to satisfy the theorems of Whittaker and Kallman.

A space X is **Polish** if it is separable completely metrizable. A group $H(X)$ is a Polish, whenever X is a metrizable manifold. A $\boldsymbol{\pi}$**–base** for a space X is a collection $\mathcal{V}$ of non–empty open sets in X such that if R is any non–empty open set in X, then $V \subseteq R$ for some $V \in \mathcal{V}$.

2 Non–Metrizable Manifolds are 'Better'

Definition 2.1 *A π–base Π for a space X, is a* **Euclidean $\boldsymbol{\pi}$–base** *if each element of Π is homeomorphic to a Euclidean open set.*

Examination of Whittaker's proof of his theorem shows it holds not just for compact manifolds, but for any compact space with a Euclidean π–base. This enables us to show that a non–metrizable manifold is uniquely determined by its homeomorphism group.

Theorem 2.2 *If M and N are non–metrizable manifolds, and $H(M)$ and $H(N)$ are isomorphic as groups, then there is a homeomorphism $h : M \to N$ realising the isomorphism of $H(M)$ and $H(N)$.*

Hence every automorphism of $H(M)$ is inner.

Proof. Let αM and αN be the one point compactifications of M and N respectively. Any homeomorphism of M extends uniquely to an homeomorphism of αM, leaving the point at infinity fixed. Conversely, any homeomorphism of αM must leave the point at infinity fixed (it is the only point with uncountable character). Thus $H(M)$ is isomorphic to $H(\alpha M)$, and similarly $H(N)$ is isomorphic to $H(\alpha N)$.

Now αM and αN are compact and have a Euclidean π–base. Hence, if $H(M)$ and $H(N)$ are isomorphic, then $H(\alpha M)$ and $H(\alpha N)$ are also isomorphic, and there is a homeomorphism $h' : \alpha M \to \alpha N$ realising the isomomorphism of $H(\alpha M)$ and $H(\alpha N)$.

Restricting h' to M gives a homeomorphism of M with N (h' must take the point at infinity of αM to the point at infinity of αN), and this homeomorphism realises the isomorphism of $H(M)$ with $H(N)$. ■

Remark 2.3 *Theorem 2.2 also holds if the manifolds M and N are non compact and metrizable, have more than one* **end** *and have dimension at least 2. An end of a topological space X is a function $e : \mathcal{K} \to \mathcal{P}$ from the collection $\mathcal{K}$ of compact subsets of X to the power set of X such that:*

1. *for each $C \in \mathcal{K}$, $e(C)$ is a component of $X - C$;*
2. *for each $C, D \in \mathcal{K}$, if $C \subset D$ then $e(D) \subset e(C)$.*

3 Minimum Group Topologies

The standard group topology placed on the homeomorphism group of a manifold is the compact–open topology. Another compatible topology is the topology of pointwise convergence. It is not difficult to show that, for manifolds of dimension two or more, these two topologies are incomparable.

Lemma 3.1 *Let M be a metrizable manifold. Then the pointwise topology on $H(M)$ is not contained in the compact–open topology.*

Proof. Let d be a compatible metric for M, and pick any point x of M. The set B, where $B = \{h \in H(M) : h(x) = x\}$ is an open neighbourhood of the identity in the pointwise topology. For a manifold such as M, given any compact subspace K and $\epsilon > 0$, it is not difficult to find a homeomorphism h of M so that $d(x, h(x)) < \epsilon$ for all $x \in K$ but $h(x) \neq x$. This means that no basic neighbourhood of the identity, in the compact–open topology, is contained in B. ■

Lemma 3.2 *Let M be a metrizable manifold of dimension at least two. Then the compact–open topology on $H(M)$ is not contained in the pointwise topology.*

Proof. Let M be an n–manifold where $n \geq 2$, with metric d. We show that the compact–open topology is not contained in the pointwise topology. To this end, pick U an open subset of M homeomorphic to $(0,1)^n$ whose closure is homeomorphic to $[0,1]^n$. We may assume that d coincides with the standard metric on $\overline{U}$. Define $B = \{h \in H(X) : d(x, h(x)) < 1/4$, for all $x \in \overline{U}\}$. Then B is an open neighbourhood of the identity in the compact–open topology.

Take any finite subset F of M, and consider the basic neighbourhood $B(F)$ of the identity in the pointwise topology, where $B(F) = \{h \in H(M) : h|F = 1\}$. We show that no such $B(F)$ is contained in B, and hence B is not open in the pointwise topology.

To see this, pick any x and y in $U \setminus F$ such that $d(x,y) \geq 1/2$. Because $n \geq 2$, it is geometrically clear that we can find a h in $H(M)$ such that h restricted to F is the identity but $h(x) = y$. Then h is in $B(F)$ but not in B. ■

Corollary 3.3 *Let M be a metrizable manifold of dimension at least two. The compact–open and pointwise topologies on $H(M)$ are incomparable. Hence neither the compact–open topology nor the pointwise topology are minimal topologies on $H(M)$. Hence $H(M)$ does not admit a minimum group topology.*

Proof. Gartside (private communication) has shown that there is no group topology below both the compact–open and pointwise topologies, and hence $H(M)$ does not admit a minimum group topology. ■

In the one dimensional case, the situation is quite different. The theorem is stated and proved only for $H(I)$ (which is topologically isomorphic to $H(\mathbb{R})$ when both are given the compact–open topology), however it should be clear to the reader that the proof works with only minor modifications for any one dimensional manifold (with or without boundary).

Let X be any space, and let h be in $H(X)$.
Define $Move(h) = \{x : h(x) \neq x\}$.

Theorem 3.4 *The natural group topology on $H(I)$ is the minimum Hausdorff group topology.*

Proof. It suffices to find a family τ of subsets of $H(I)$ which must be open in any group topology on $H(I)$, so that if U is an open neighbourhood of the identity element 1 (in the compact open topology on $H(I)$), then there is a $T \in \tau$ such that $1 \in T \subseteq U$. In that case, all sets open in $H(I)$ in the usual topology must be open in any group topology on $H(I)$.

For $0 \leq a < b \leq 1$, pick $p, q \in H(I)$ (depending on a and b) so that $p, q \neq 1$, $Move(p) \subseteq (a,b)$, $Move(q) \subseteq (a,b)$ and $pq \neq qp$.

Define $T(a,b) = \{g \in H(I) : gpg^{-1}$ does not commute with $q\}$. Let τ be the collection of all finite intersection of sets of the form $T(a,b)$. We state and prove a series of claims which demonstrate that τ is as required.

(0) **Claim:** $1 \in T(a,b)$.

Because $1p1^{-1} = p$, and p does not commute with q.

(1) **Claim:** $T(a,b)$ is open in any (Hausdorff) group topology on $H(I)$. To see this, note that

$$T(a,b) = \Phi^{-1}[H(I) - \{1\}] \qquad \text{where} \qquad \Phi(g) = gpg^{-1}qgp^{-1}g^{-1}q^{-1},$$

and $H(I) - \{1\}$ is open in any Hausdorff topology, and Φ must be continuous in any group topology.

(2) **Claim:** If $g \in T(a,b)$, then there exists $x \in (a,b)$ such that $g(x) \in (a,b)$.

Suppose, for a contradiction, for all $x \in (a,b)$, $g(x) \notin (a,b)$. That means $g[(a,b)] \subseteq I - (a,b)$.

Then, Move $gpg^{-1} \subseteq g[(a,b)]$ (for if $y \notin g[(a,b)]$, then $g^{-1}(y) \notin (a,b)$, so $pg^{-1}(y) = g^{-1}(y)$, and $gpg^{-1}(y) = gg^{-1}(y) = y$).

Thus, Move $gpg^{-1}(\subseteq I - (a,b))$ and Move $q(\subseteq (a,b))$ are disjoint, so gpg^{-1} and q commute.

(3) **Claim:** Every open neighbourhood U of 1 in $H(I)$ (with the usual topology) contains a T in τ.

A basic neighbourhood of 1 has the form:

$$B_\epsilon = \{g \in H(I) : |g(x) - x| < \epsilon \quad \forall x \in I\}.$$

So, given $\epsilon > 0$, pick $n \in \mathbb{N}$ $(n > 1)$ so that $0 < 1/n < \varepsilon/3$. Let $T = T(0,1/n) \cap T(1/n, 2/n) \cap ... \cap T(1-1/n, 1)$.

Subclaim: If $h \in T$ then $h \in B_\varepsilon$.

Suppose not, say $h \in T - B_\varepsilon$. As $h \notin B_\varepsilon$, there exists $x \in I$ such that $|h(x) - x| \geq \varepsilon > 2/n$. Pick minimal i such that $x \in [i/n, (i+1)/n]$.

Using claim(2), pick x_0, x_1 such that $x_0 \in (i/n, (i+1)/n), x_1 \in ((i+1)/n, (i+2)/n)$ and $h(x_0) \in (i/n, (i+1)/n)$ while $h(x_1) \in ((i+1)/n, (i+2)/n)$.

By the intermediate value theorem, any homeomorphism of I is either increasing or decreasing.

Considering the four possible cases (illustrated in Figure 1 and Figure 2) we show that h is not monotone (as the diagrams show). (Note that of course $x_0 < x_1$.)

Case 1: $x_0 < x$ and $h(x_0) < h(x)$. Then $h(x) > h(x_1)$ and h is not monotone.

Case 2: $x_0 < x$ and $h(x_0) > h(x)$. Then $h(x) < h(x_1)$ and h is not monotone.

Case 3: $x_0 > x$ and $h(x_0) < h(x)$. Then $h(x_0) < h(x_1)$ and h is not monotone

Case 4: $x_0 > x$ and $h(x_0) > h(x)$.

Then $i \neq 0$. Pick $x_{-1} \in ((i-1)/n, i/n)$ such that $h(x_{-1}) \in ((i-1)/n, i/n)$. Now, $x_{-1} < x < x_0$, but $h(x_{-1}) > h(x)$, while $h(x) < h(x_0)$, so h is not monotone. ■

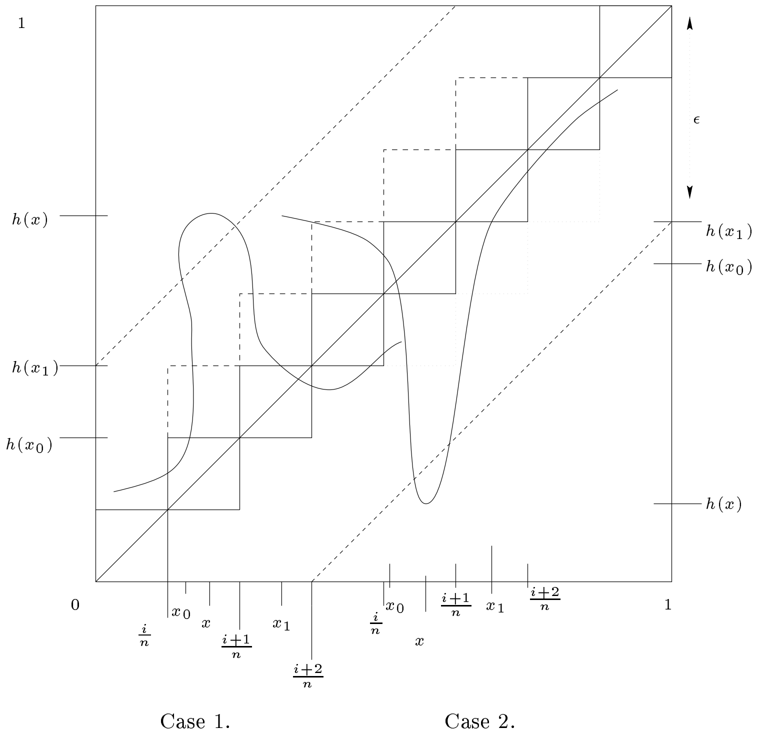

Figure 1: Cases 1 and 2 for h.

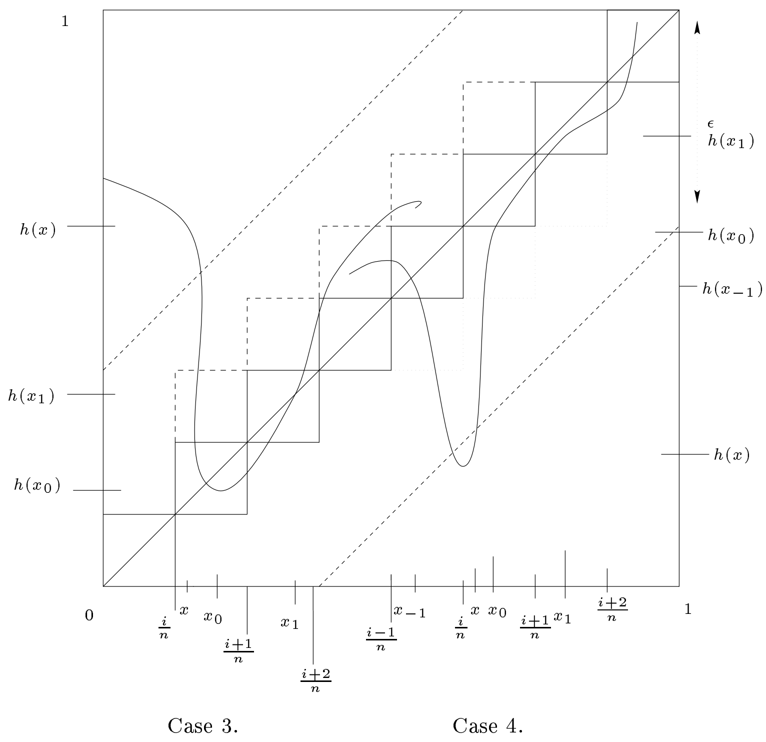

Figure 2: Cases 3 and 4 for h.

Spaces which are not manifolds

Gamarnik [2] has proved that the compact–open topology on $H(\mathcal{C})$ ($\mathcal{C}$ is Cantor set) is minimal. It follows that the compact–open topology is in fact the minimum topology on $H(\mathcal{C})$ (as for $H(I)$). On the other hand, Lemmas 3.1 and 3.2 hold for the Tychonoff cube, so neither the pointwise nor the compact–open topology are minimal group topologies on $H(I^\omega)$ (this answers a question of Stoyanov [6]).

4 Homeomorphisms of Continua

A **continuum** is a compact connected locally connected metric space. Continua are close analogues of compact manifolds. Indeed many theorems about the general topology of manifolds, also hold for spaces which locally are compact, connected and metrizable.

However the behaviour of homeomorphism groups of continua is quite different from that of manifolds. The existence of Cook's curve, a continuum

C which has the property (among many others) that every continuous injection of C into itself is the identity, makes this clear. For $H(C)$ is the trivial group, which is impossible for any manifold.

We use Cook's curve [1] to construct further examples of continua which demonstrate how badly continua can fail to satisfy the theorems of Whittaker and Kallman.

Theorem 4.1 *There are continua K_1 and K_2, and a compact, connected, locally connected non–metrizable space K_3, such that:*

(1) K_1 and K_2 are not homeomorphic, but $H(K_1)$ is topologically isomorphic to $H(K_2)$. (Indeed, there are $2^{\aleph_0}$ many pairwise non–homeomorphic continua each with the same homeomorphism group.)

(2) $H(K_1)$ admits a discontinuous group automorphism.

(3) Assuming $(2^{\aleph_0} = 2^{\aleph_1})$, $H(K_1)$ and $H(K_3)$ are algebraically isomorphic, but not homeomorphic.

Proof. As above denote Cook's curve by C. For distinct a and b in C, let $C(a,b)$ be the space obtained by taking $C \times \{0,1\}$ and identifying $(a,0)$ with $(a,1)$ and $(b,0)$ with $(b,1)$. (We will abuse notation and denote the equivalence class $\{(a,0),(a,1)\}$ by a, and similarly for b.) Note that $H(C(a,b)) = \mathbb{Z}_2$.

Fix a subset $\{x_n\}_{n\in\omega}$ of C so that $x_n \neq x_m$ if $n \neq m$. Let $K_1{}'$ be the space obtained by taking $\bigoplus_{n\in\omega} C(x_n, x_{n+1})$, and identifying x_{n+1} in $C(x_n, x_{n+1})$ with x_{n+1} in $C(x_{n+1}, x_{n+2})$. Now let K_1 be the one point compactification of $K_1{}'$. Then K_1 is a continuum.

Taking some different choice of $\{x_n\}_{n\in\omega}$, we obtain another continuum K_2.

Now fix a linear order of ω_1, $<$ say, so that every ordinal α has a unique successor, α_+, and a unique predecessor, α_-. Also, fix a subset $\{x_\alpha\}_{\alpha\in\omega_1}$ of C, so that $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. Let $K_3{}'$ be the space by taking $\bigoplus_{\alpha\in\omega_1} C(x_\alpha, x_{\alpha_+})$, and identifying x_{α_+} of $C(x_\alpha, x_{\alpha_+})$ with x_{α_+} of $C(x_{\alpha_+}, x_{\alpha_{++}})$. Define K_3 to be the one point compactification of $K_3{}'$. Then K_3 is compact, connected, locally connected but not metrizable.

Proof of (1). One can easily check that $H(K_1)$ and $H(K_2)$ are topologically isomorphic to $\mathbb{Z}_2^\omega$. To see this observe that any homeomorphism of K_1 (or K_2) takes each copy $C(x_n, x_{n+1})$ to itself, and each such copy can be 'flipped' or left alone independently of any other $C(x_m, x_{m+1})$.

The above identification of $H(K_1)$ holds *whatever the choice of* $\{x_n\}_{n\in\omega}$ in the definition of K_1. Moreover, *different choices of* $\{x_n\}_{n\in\omega}$ give rise to *topologically distinct continua* [1]. (In particular, K_1 and K_2 are not homeomorphic.)

Proof of (2). We need to show that $\mathbb{Z}_2^\omega$ has a discontinuous group automorphism. This and the proof of (3) depends on the the following:

Fact: Using Theorem 2.1 and Theorem 2.6 of [3, Chapter *IV*] we may prove that the group $\mathbb{Z}_2^\omega$ is the unique group A satisfying

(i) the cardinality of A is $2^{\aleph_0}$

(ii) every element of A has order two, and

(iii) there are no other relations between elements of A.

Note first that $\mathbb{Z}_2^\omega$ is the direct product of F and G, where $F = \{(i, 0, 0, \dots) : i \in \{0, 1\}\}$, and $G = \{(0, x_1, x_2, \dots) : x_r \in \{0, 1\}\}$.

Now let p be a non–trivial ultrafilter on ω, and set $F' = \{(0, 0, 0, \dots), (1, 1, 1, \dots)\}$ and $G' = \{(x_r)_{r\in\omega} : \exists S \in p \text{ such that } (x_r = 0 \text{ iff } r \in S)\}$. Then we can again check that $\mathbb{Z}_2^\omega$ is the direct product of F' and G'.

Since F and F' are isomorphic to $\mathbb{Z}_2$, we may pick an isomorphism $\alpha_0 : F' \to F$. Further, G and G' are isomorphic, since by the Fact above, they are both isomorphic to $\mathbb{Z}_2^\omega$; let $\alpha_1 : G' \to G$ be an isomorphism. Consequently, $\alpha : \mathbb{Z}_2^\omega \to \mathbb{Z}_2^\omega$, defined by $\alpha(f'g') = \alpha_0(f')\alpha_1(g')$ (where $f' \in F'$ and $g' \in G'$) is an automorphism. However, α is not continuous, because F is closed in $\mathbb{Z}_2^\omega$, but $\alpha^{-1}F = F'$ is not closed (F' is dense).

Proof of (3). Just as $H(K_1) = \mathbb{Z}_2^\omega$, so $H(K_3) = \mathbb{Z}_2^{\omega_1}$. Observe that $H(K_3)$ is not metrizable, and so definitely not homeomorphic to $H(K_1)$.

On the other hand, if we assume ($2^{\aleph_0} = 2^{\aleph_1}$), by the Fact above characterising $\mathbb{Z}_2^\omega$ algebraically, we see that $H(K_1) = \mathbb{Z}_2^\omega$ is algebraically isomorphic to $\mathbb{Z}_2^{\omega_1} = H(K_3)$. ■

References

[1] H. Cook, *Continua which admit only the identity mapping onto non–degenerate subcontinua*, Fund. Math. 60 (1967), 241–249.

[2] D. Gamarnik, *Minimality of the group* autohomeom(C), Serdica 17 (1991), 197–201.

[3] T. Hungerford, *Algebra*, New York, Holt, Rinehart and Winston, 1974.

[4] R. Kallman, *Uniqueness results for homeomorphism groups*, Trans. Amer. Math. Soc. 295 (1986), 389–396.

[5] K. Kawamura, *Isomorphisms of homeomorphism groups of menger manifolds*, Bull. Polish Acad. Sci. Math. 43 (1995), 329–333.

[6] L. Stoyanov, *Total minimality of the unitary groups*, Math. Z. 187 (1984), 273–283.

[7] J.V. Whittaker, *On isomorphic groups and homeomorphic spaces*, Ann. of Math. 78 (1963), 74–91.

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