Domination Conditions for Tournaments

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Abstract

The definition for the domination graph of a tournament states that it has the same vertices as the tournament with an edge between two vertices if every other vertex is beaten by at least one of them. In this paper two new types of domination graphs are defined by using different relaxations of the adjacency definition. The first type is formed by reducing the number of vertices which must be dominated by a pair of vertices and the second by increasing the number of steps allowable for domination. Properties of these new types of domination graphs are presented with comparison between them where appropriate. In particular a full characterisation of each type is given for rotational tournaments.

1 Introduction

A *tournament* $T = (V, A)$ is a complete, directed graph, with a vertex set V and an arc set A. For $x, y \in V$ we denote an arc from x to y by $(x, y) \in A$ and say x *beats* y. For any vertex x let $out(x)$, the *out set* of x, be the set of vertices that x beats. Similarly let $in(x)$, the *in set* of x, be the set of vertices that beat x. A tournament T on n vertices is said to be *regular* if n is odd and every vertex has out degree $\frac{n-1}{2}$. Tournaments are well documented in Moon [9], Reid and Beineke [10], and Reid [11].

Given a tournament $T = (V, A)$, a pair of vertices x and y *dominate* T (in one step) if for all vertices $z \neq x, y$ either x beats z or y beats z. Two such vertices are called a *dominant pair*. The *domination graph* of a tournament T, denoted by $dom(T)$, is the graph with vertex set V and edges between those pairs of vertices which are dominant pairs. Numerically this condition states that xy is an edge in $dom(T)$ if and only if $|in(x) \cap in(y)| < 1$. Domination graphs of digraphs and tournaments have been characterised in a series of papers (see [1], [2], [3], [4], [5], [6]). For a tournament T, $dom(T)$ is either a cycle of odd length, with or without isolated and/or pendant vertices, or a forest of caterpillars.

A vertex in a tournament is called a *king* if it beats all other vertices in one or two steps; that is, vertex x is a king if for all $y \in V$, x beats y or x beats a vertex that beats y. Landau [7] proved that every vertex of maximum out degree in a tournament is a king; consequently, a tournament may have more than one king. A vertex which beats all other vertices in one step is called an *emperor*. A tournament has exactly one king if and only if that king is an emperor.

2 Relaxation of domination conditions

Other interesting types of domination graphs may be obtained by using different, more relaxed, conditions for a dominating pair of vertices. The first possibility is to reduce the number of vertices which must be dominated, whilst the second is to increase the number of allowable steps for domination. Pertinent definitions and results are presented in this section.

2.1 k-domination graphs

Let T be a tournament with n vertices. For $1 \leq k \leq n-2$, the *k-domination graph* of T is a graph with the same vertices as T and an edge between two distinct vertices if and only if that pair of vertices dominates in one step all of the other vertices of T with the possible exception of $k-1$ vertices. Thus two vertices x and y in T are adjacent in the k-domination graph provided they are not each dominated by some common set of k other vertices, that is $|in(x) \cap in(y)| < k$. We denote the k-domination graph of a tournament T by $dom_k(T)$. Clearly $dom_1(T) \subseteq \cdots dom_k(T) \subseteq dom_{k+1}(T) \cdots \subseteq dom_{n-2}(T)$ where $dom_1(T) = dom(T)$.

2.2 Royal graphs

For the domination graphs described in the previous section, domination by a pair of vertices was required to occur in just one step. Now we allow a domination condition more equivalent to that of a king, that is, domination by a pair of vertices in one or two steps. We define a *royal pair* to be a pair of vertices in a tournament T which together act as a king. That is, vertices x and y form a royal pair if, for all $z \in V \backslash \{x, y\}$, either x or y beats z in at most two steps. The *royal graph*, denoted by $roy(T)$, is the graph on the vertices of V with an edge between each pair of vertices which form a royal pair. It follows from their respective definitions that $dom_1(T)$ is a subgraph of $roy(T)$.

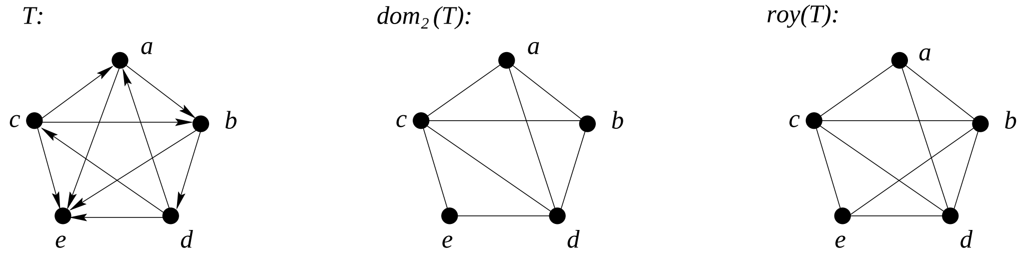

Figure 1: A tournament, its 2-domination graph and royal graph.

Lemma 1 *Let T be a tournament, then $roy(T)$ is connected.*

PROOF. Since every pair of vertices containing a king must be a royal pair, and every tournament must have at least one king, namely any vertex of highest out-degree, $roy(T)$ must be connected. ■

If all the n vertices in a tournament T are kings, then $roy(T)$ is the complete graph K_n. Rotational tournaments satisfy this condition. Maurer [8] showed that for large tournaments, it is likely that all vertices are kings, and so $roy(T)$ fails as a good representative of dominating pairs in this situation.

2.3 Comparing $dom_k(T)$ and $roy(T)$

The structures of $roy(T)$ and $dom_2(T)$ are strongly related. The next result shows that, depending on the presence of an emperor, one of these types of domination graphs is always a subgraph of the other.

Theorem 2 *Let T be a tournament.*

(i) If T has an emperor, then $roy(T) \subseteq dom_2(T)$.

(ii) If T has no emperor, then $dom_2(T) \subseteq roy(T)$.

PROOF.

(i) Suppose a vertex z is an emperor in T, then the only edges in $roy(T)$ are those in the star graph with centre at z. Since an emperor beats every other vertex in one step, every edge in $roy(T)$ is also in $dom_1(T)$.

(ii) Suppose T doesn't have an emperor and x and y are adjacent vertices in $dom_2(T)$. If xy is also an edge in $dom_1(T)$ then it is an edge in $roy(T)$. However, if xy is not an edge in $dom_1(T)$, then there is a vertex w which beats both x and y. Consequently, if there is an arc from $V \setminus \{w, x, y\}$ to w, then x and y must form a royal pair, otherwise w is an emperor. ■

Kings play a key role in determining when the graphs $dom_2(T)$ and $roy(T)$ are the same.

Theorem 3 *Let T be a tournament, if $dom_2(T) = roy(T)$ then every royal pair contains a king.*

PROOF. Assume there is a royal pair x and y where y beats x and neither x nor y is a king. Then there exists a vertex u which prevents x from being a king. This means u beats x and u beats every vertex in $out(x)$. Similarly, there is a vertex v which prevents y from being a king.

Case 1. Assume $u \notin out(y)$, then u beats y. Then xy is not an edge in $dom_2(T)$, since both u and v beat x and y. Thus $roy(T) \neq dom_2(T)$.

Case 2. Assume y beats u, and there is some vertex $t \neq v$ which beats both x and y. Then xy is not an edge in $dom_2(T)$, since both t and v beat x and y. Thus $roy(T) \neq dom_2(T)$.

Case 3. Assume y beats u, and v is the only vertex which beats both x and y. Then $V = \{v, y\} \cup out(x) \cup out(y)$.

Let A be the set of all vertices which beat x and all of the vertices in $out(x)$. Note that $u \in A$, hence A is not empty. Since $\{x, y\}$ is a royal pair there must exist some $w \in out(x)$ which beats v. Moreover, w beats y, or y would beat v in two steps (and therefore be a king).
<u>Case 3a.</u> Assume $|A| > 1$. Then the edge wx is not in $dom_2(T)$, since $|A| > 1$ and every vertex in A beats both w and x. But wx is an edge in $roy(T)$, since w beats y, w beats v, and w beats all vertices in $out(y)$ in at most two steps. Thus $roy(T) \neq dom_2(T)$.
<u>Case 3b.</u> Assume $|A| = 1$, and so $A = u$. Assume that there is some other vertex s which prevents $\{u, x\}$ from being a royal pair. That is, s beats u, s beats x, and s beats all the vertices in both $out(u)$ and $out(x)$. This means that $s \in A$, a contradiction. Hence $\{u, x\}$ is a royal pair. Moreover, since v and y dominate both of u and x, ux is not an edge in $dom_2(T)$.

In each possible case it has been shown that if T contains a royal pair where neither vertex is a king, then $roy(T) \neq dom_2(T)$. ■

The converse of Theorem 3 is not true. An unsolved problem is the characterisation of those tournaments T for which $dom_2(T)$ and $roy(T)$ are equal. The condition that every royal pair contains a king is not sufficient. This is illustrated in Figure 1 where the vertices b, c, and d are all kings but $roy(T)$ and $dom_2(T)$ are not equal.

We now generalise the definition of an emperor by defining a *sub*^{k}*-emperor* to be a vertex which beats all other vertices, except k of them, in one step. That is, a vertex $x \in V$ is a sub^k-emperor if and only if $|in(x)| \leq k$. The next lemma follows by generalising the proof of Theorem 2.

Lemma 4 *If T is a tournament with no sub^k-emperor, where $k \geq 1$, then $dom_{k+2}(T) \subseteq roy(T)$.*

It is not the case that $dom_i(T) \subseteq roy(T) \subseteq dom_{i+1}(T)$ for some i. There is a counterexample on eight vertices which has $dom_4(T) \not\subseteq roy(T)$ and $roy(T) \not\subseteq dom_4(T)$.

There are a number of interesting, but as yet unanswered, questions concerning the classification of tournaments which have a particular type of k-domination or royal graph. Rotational tournaments provide some partial answers and are a good source of examples. A sampling of these are presented in the next two sections.

3 Rotational tournaments

Let S be a subset of $\mathbf{Z}_n \backslash \{0\}$ with the property that $i \in S$ if and only if $n - i \in S$. Define a graph G to have a vertex set $v_1, v_2, \ldots, v_n$, with an edge joining vertices v_i and v_j if and only if $i - j \in S$. A graph G of this type is called a *circulant graph* and the set S is known as its *symbol*. A *rotational tournament* $T(S)$ is defined in a similar fashion with the symbol set S again being a subset of $\mathbf{Z}_n \backslash \{0\}$ but having slightly different restrictions, namely that it must contain $\frac{n-1}{2}$ elements and $s_i + s_j \neq 0$ for all $s_i, s_j \in S$. For such a rotational tournament (v_i, v_j) is an arc if and only if $j - i = s \bmod n$ for some $s \in S$. For example, if $\{1, 2, 3\}$ is the symbol for a tournament T on seven vertices, then vertex v_1 beats vertices v_2, v_3 and v_4, and is beaten by vertices v_5, v_6 and v_7.

Rotational tournaments are a subclass of regular tournaments. They are vertex transitive, that is, for every pair of vertices v_i and v_j, there is an automorphism which maps v_i to v_j. Let S be a symbol set for a rotational tournament T with n vertices. It is routine to show that if r is relatively prime to n, then rS is also a valid symbol set for some tournament. Proposition 5 shows that this tournament will be isomorphic to T.

Proposition 5 *A rotational tournament T on n vertices with symbol set S is isomorphic to any tournament T_r with symbol set rS where r is relatively prime to n.*

PROOF. Suppose $S = \{s_1, \ldots, s_t\}$, and the vertices of T and T_r are labelled $v_1, \ldots, v_n$. The symbol for T_r is $\{rs_1, \ldots, rs_t\}$. Define a mapping ϕ from the vertices of T to the vertices of T_r by $\phi(v_\ell) = v_{r\ell}$. Since r and n are relatively prime, ϕ reorders the vertices. It also gives an isomorphism between T and T_r. Suppose vertex v_i beats vertex v_j for some $1 \leq i, j \leq n$ in T, then $j - i \in S$. Consequently, $r(j - i) = rj - ri \in rS$. Hence $v_{ri} = \phi(v_i)$ beats $v_{rj} = \phi(v_j)$ in T_r. Hence ϕ is an isomorphism. ■

Proposition 5 does not determine all possible symbol sets for a rotational tournament. For example, the symbols $\{1, 3, 4, 7\}$ and $\{1, 4, 6, 7\}$ form isomorphic tournaments on nine vertices, but one is not a multiple of the other.

Let T be a tournament on n vertices. The *arc reversal* of T is defined to be the tournament with the same vertex set as T but with the direction of each arc reversed. By Proposition 5 it is clear that a tournament is isomorphic to its arc reversal.

Let U_n, $n \geq 3$, be the rotational tournament with symbol set $\{1, 3, \ldots, n-2\}$, namely all odd numbers from 1 to $n-2$. Fisher et al. [3] showed that if T is a rotational tournament on n vertices then either $dom_1(T)$ is the cycle graph C_n and T is U_n or $dom_1(T)$ is the empty graph. A related result is now proved, namely that each domination graph $dom_k(T)$, for $k = 1, 2, \ldots, \frac{n-3}{2}$, is either the empty graph or a circulant graph. (Note that for a rotational tournament the out degree of every vertex is $\frac{n-1}{2}$, hence any pair of vertices must beat at least $\frac{n-1}{2}$ other vertices, consequently in the definition for $dom_k(T)$ it is only necessary to consider $k = 1, 2, \ldots, \frac{n-3}{2}$.)

Theorem 6 *Let T be a rotational tournament with symbol set S and vertices labelled $v_1, v_2, \ldots, v_n$. Let D be the sequence of all possible differences modulo n between pairs of elements in S. For each element d in D let $N(d)$ be the number of times the value d occurs in D. Then, for $1 \leq k \leq \frac{n-3}{2}$, $dom_k(T)$ is either the empty graph or a circulant graph whose symbol set consists of those values d in D for which which $N(d) < k$.*

Before presenting the proof of Theorem 6 a short example is given. Let T be the rotational tournament on nine vertices with symbol set $S = \{1, 5, 6, 7\}$. Form the difference table for this symbol set by calculating the difference between all pairs of elements in S and counting how many times each difference occurs. The difference table for this example is

d	1	2	3	4
$N(d)$	2	1	1	2

.

(Note: if $s_1, s_2 \in S$ then $d_1 = s_1 - s_2 \in D$ if and only if $d_2 = s_2 - s_1 = n - d_1 \in D$, hence it is only necessary to count differences in the range 1 to $\frac{n-1}{2}$). Since there are no values of d such that $N(d) = 0$ it follows that $dom_1(T)$ is the empty graph. There are four values of d such that $N(d) \leq 1$, hence $dom_2(T)$ is the circulant graph with symbol set $\{2, 3, 6, 7\}$. Similarly it follows that $dom_3(T)$ is the circulant graph with symbol set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, that is, $dom_3(T)$ is the complete graph K_9.

Proof. Since T is vertex transitive it suffices to consider any pair of vertices v_i and v_j in T. Let $N(i-j)$ be the number of times the value $i-j$ occurs in the difference table for the symbol set S. By counting the number of vertices $z = v_\ell$ which beat both v_i and v_j the equality $N(i-j) = |in(v_i) \cap in(v_j)|$ is obtained. For if v_ℓ beats v_i and v_j then $i - \ell = s_i \bmod n$ and $j - \ell = s_j \bmod n$ for some $s_i, s_j \in S$, and consequently $i - j = s_i - s_j$. Thus the number of vertices which beat both v_i and v_j equals $N(i-j)$.

Thus the symbol set S_k for $dom_k(T)$ consists of the values of those differences d, from pairs of elements in $\{1, 2, \ldots, n\}$, for which $N(d) < k$. As observed in the example $\ell \in S_k$ if and only if $n - \ell \in S_k$; consequently this symbol set fulfils the appropriate condition for a circulant graph. ■

Let $\overline{U}_n$, $n \geq 7$, represent the rotational tournament with symbol set $\{1, 2, 4, \ldots, n-3\}$, namely one plus all even numbers except $n-1$. It follows from Theorem 6 that $dom_1(\overline{U}_n)$ is the empty graph and $dom_{\frac{n-3}{2}}(\overline{U}_n)$ is the complete graph K_n. Fisher et al. [4] showed that for a tournament with n vertices the maximum possible number of edges in $dom_1(T)$ is n. It is not surprising that more than n edges may occur in $dom_k(T)$, and in the fact the maximum possible number of edges, namely $\frac{n(n-1)}{2}$, is obtained in $dom_{\frac{n-3}{2}}(\overline{U}_n)$.

Since every vertex in $\overline{U}_n$ is a king it is clear that $roy(\overline{U}_n)$ is complete. However $dom_1(\overline{U}_n)$, $n \geq 7$ is empty. This gives a situation where the graphs of $dom_1(T)$ and $roy(T)$ are as different as possible. The next section expands the construction of the symbol set used in $\overline{U}_n$ to give a family of rotational tournaments where, for each member, the k-domination graph is empty and the $(k+1)$-domination graph is not.

4 Construction of empty dom_k

In this section an infinite family of rotational tournaments on n vertices is described, in which for sufficiently large n, the k-domination graph is empty and the $(k+1)$-domination graph is not empty.

Let $E_{n,k}$, $k \geq 1$, be a rotational tournament on n vertices with symbol

$$S_{n,k} = \{1, 2, \ldots, 2k-1, 2k, 2k+2, 2k+4, \ldots, n-2k-3, n-2k-1\}.$$

For example, the symbol for $E_{17,2}$ is $S_{17,2} = \{1, 2, 3, 4, 6, 8, 10, 12\}$, and the symbol for $E_{17,3}$ is $S_{17,3} = \{1, 2, 3, 4, 5, 6, 8, 10\}$. Notice that the symbol sets for $E_{n,1}$ and $\overline{U}_n$ are equal.

Theorem 7 *The k-domination graph $dom_k(E_{n,k})$ is empty when $n > 8k$.*

Proof. Let D be the sequence of all possible differences modulo n between pairs of elements in $S_{n,k}$. Let $N(d)$ be the number of times a particular difference d occurs in D. By Theorem 6 it suffices to show that $N(d) \geq k$ for all $d \in D$. Odd and even values of d are considered separately. Recall that it is only necessary to consider $1 \leq d \leq \frac{n-1}{2}$.

If d is even then each of the k differences in the sequence

$$[(d+2)-2], [(d+4)-4], \ldots, [(d+2k)-2k]$$

has value d and occurs in D provided $d + 2k < n - 2k + 1$. The condition $n > 8k$ ensures this inequality holds.

The case for d odd is similar. The required sequence of k differences, each with value d, is

$$[(d+1)-1], [(d+3)-3], \ldots, [(d+2k-1)-(2k-1)]$$

and it is clear that $d + 2k - 1 < n - 2k + 1$.

Hence $N(d) \geq k$ irrespective of whether d is even or odd, so $dom_k(E_{n,k})$ is the empty graph. ■

Note that this construction works for $E_{17,2}$ where $n > 8k$ and fails for $E_{17,3}$ where $n < 6k$. Observations seem to suggest this construction will in fact work when $n > 6k$, and fail when $n < 6k$. A computer check shows that the condition $n > 6k$ is satisfactory for all constructions of $E_{n,k}$ with $k < 35$.

Theorem 8 *The k-domination graph $dom_{k+1}(E_{n,k})$ is not empty.*

Proof. Let D, d and $N(d)$ be as in Theorem 7. It suffices to show that there is a value of d such that $N(d) < k + 1$, for then there will be edges in

$dom_{k+1}(E_{n,k})$. Let $d = 2k+1$, since d is odd the only pairs of elements in $S_{n,k}$ with difference equal to $2k+1$ are given by the k differences

$$[(1+2k+1)-1], [(3+2k+1)-3], \ldots, [(2k-1+2k+1)-(2k-1)].$$

(None of the even symbol set elements between 1 and $2k$ can be used to form a difference with value $2k+1$ in this fashion, since adding $2k+1$ to any of them gives a larger odd value which isn't in the symbol set.) Thus $N(d) = k < k+1$ giving the required result. ■

Theorems 7 and 8 give an example of a family of rotational tournaments, where for any member T, $dom_k(T)$ is empty and the next graph in the chain of domination graphs, namely $dom_{k+1}(T)$ is not empty. Moreover $roy(T)$ is complete.

It was observed that there are some rotational tournaments where, for a particular value of k, the k-domination graph is empty while the $(k+1)$-domination graph is complete. By Theorem 6, a necessary condition for this is that $N(d) = k$ for all d in the tournament difference set. A counting argument on the difference table gives a second necessary condition, namely $n = 4k+3$. Investigation showed that for $k = 1, 2, 4, 5, 7$, that is n prime, there was a tournament for which this 'empty-to-complete' domination condition is satisfied, whereas for $k = 3$ and 6, no rotational tournament with this property existed.

An obvious conjecture is that this 'empty-to-complete' domination condition holds some rotational tournament on n vertices when n is a prime of the form $4k+3$.

Acknowledgements

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