FEW-BODY KREIN'S FORMULA

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Selfadjoint extensions of symmetric operators with infinite deficiency indices are discussed. In particular the operators describing the system of several quantum particles are investigated in detail and a few-body analog of Krein's formula for generalized resolvents is proven. The conditions for the semiboundedness of the simplest M-body quantum Hamiltonian with point interactionsin in the three-dimensional space are derived

1 Introduction

Possibly G. R. Kirchhoff [18] was the first mathematcal physicist who noticed that for electrostatic problems in composite domains consisting of two standard parts joined by several small openings the *ansatz* for the solution of the corresponding partial differential equation may be obtained in the form of linear combination of solutions of nonperturbed problems in standard domains and singular solutions- in fact Green functions - attached to the openings. At that moment this important observation was considered just as a technical device for constructing of approximate solutions ("Kirchhoff method") and the general role of it has not been noticed yet. In 1933 Fermi [14] made another important step. He suggested a solvable model for *point interaction* between neutrons and nuclei. But only few decades later the idea of Fermi was non - formally decoded by mathematicians [10] and the operator extension theory was recognized as a general and powerful tool for constructing solvable models with *zero-range potentials* for quantum systems [11]. Now we have an impressive number of solvable models of one-body quantum and acoustic systems constructed from very simple standard elements, such as matrices and differential equations with constant coefficients in simple domains [5,6,33,35,36].

In general extension theory construction the role of the Kirchhoff's Green functions in the corresponding ansatz is played by the elements of the proper the deficiency subspaces.

Operator extension approach provides a large class of solvable models of wave processes which can be described roughly by the following condition: "the wave lengths exceed the size of characteristic details of the perturbation considered ", for instance , the width of the opening in acoustic problems. In quantum mechanics this geometric condition is replaced by the *low-energy* or *slow dephasing conditions.* In particular this approach gives a rigorous substitution for so-called "energy-dependent potentials" for low-energy nuclear physics [22], and for scattering problems appearing in design of nano-electronic devices and networks similar to ones considered in [7].

The mathematical background for all mentioned *one-* and *two-body* models is given by Krein's formula for generalized resolvents of symmetric operators with finite deficiency indices. The corresponding few-body problems , see [22,25,34] require considering operators with *infinite deficiency indices and special algebraic structure* involving existence of several tensor decompositions - *splittings* of the *underlying* Hilbert space and underlying *nonperturbed* Hamiltonian and a special construction of the system of corresponding integral equations in terms of *Faddeev components.* These problem are studied in detail in the recent book [6], where the most general approach to few-body problems with singular finite rank interactions is developed.

In actual paper we show that the abstract form of this construction is equivalent to some analog of Krein's formula for operators with infinite deficiency indices and several disjoint deficiency subspaces. It appeared that the structure of the family of deficiency subspaces corresponds to the physical structure of *channels* which is encoded in corresponding *splittings* of the configuration space (see e.g. [8]). The deficiency elements are distributed along the channel's *interaction planes or cylinders* with densities playing the role of Faddeev components.

The mentioned characteristics of geometric and algebraic structures of the family of deficiency elements appeared in [6,33] as a fiber structure [1] of the deficiency subspace attached to the interaction cylinder. Corresponding Krein's formula was derived in [33] as *fiber Krein's formula.* In fact even for simplest n-body problems $n > 2$ the base of the corresponding fiber space - the joining of all relevant interaction cylinders - *is not a smooth variety*, as in [6,33], but has some conic singularities at the intersections of them, that is at the corresponding *equidomoid* (see [22]). In the simplest case of three particles with point interactions in R^3 the singular manifold contains only one point - " the triple collision point",- and the apriory estimates of operators appeared in corresponding Faddeev equations require usage of Hardy inequality [34]. In general case the singular variety may be much more sophisticated, it may contain edge points or be unbounded, which actually creates serious obstacles in M-body scattering problems even for smooth potentials with $M \geq 4$ see [30]. In the third section of the current paper we consider few-body Krein's formula for an "elementary brick" of the whole construction which is an abstract analog of the interchange-reaction between two clusters. The semiboundedness of the corresponding few-body hamiltonians are discussed. In particular,if the corresponding deficiency subspaces satisfy zero-intersection condition $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, then the semiboundedness of *every* selfadjoint extension follows from invertibility of the corresponding Krein Q-functions for large negative values of the spectral parameter. But if the intersection of the deficiency subspaces is infinitedimensional, then the invertibility of the corresponding Krein Q-functions for large negative values of the spectral parameter implies the semiboundedness of the *Friedrichs extension* of the Hermitian operator with deficiency subspace $\mathcal{N}_1 \cap \mathcal{N}_2$ defined on smooth

[1] Representing the deficiensy elements as direct products , see (9) we reveal this structure in standard sense.

functions vanishung near the intersection of supp$\mathcal{N}_1 \cap \mathcal{N}_2$.

In the last section of the paper we consider as an example of our approach the Hamiltonian of the system of M distinguishable quantum particles with the generalized pairwise point interactions in the Hilbert space $L_2(R^{3M-3})$. New conditions for the semiboundedness of the Hamiltonian constructed via operator extension technique are proven.

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2 Two-body Krein's formula in rigged Space.

In our paper we use the approach to Operator Extensions based on analysis of the corresponding hermitian symplectic forms which is very well known for differential and difference equations , see for instance [31] and was developed first for common extensions of abstract and differential operators in [32], see also [6,33] and the bibliography there. Professor S.Novikov informed the second author recently that I.M. Gelfand allways underlined the importance of analysis of symplectic hermitian boundary forms in operator theory and in mathematical physics.

In what follows we use the following statement, see [6,33] which contains in particular important Krein's formula, see [19, 20]. Let a be a selfadjoint operator [2] with the domain Dom(a) and acting in the Hilbert space h. Consider any finite dimensional subspace $n = n_i$ such that $\mathrm{Dom}(a) \cap n = \{0\}$. Then the restriction a_0 of the operator a to the domain $\mathrm{Dom}(a_0) = \frac{1}{a-iI} h \ominus n$ is a symmetric densely defined operator. For the classical description of all generalized resolvents of the operator a_0 with the finite dimensional deficiency subspaces $n_i \equiv n,\ n_{-i} = \frac{a+iI}{a-iI} n,\ D_0 \cap n = \emptyset$ see [1].

Theorem 2.1 *The domain* $\mathrm{Dom}(a_0^*)$ *of the adjoint operator* a_0^* *may be described using the boundary values* $\xi_\pm^u \subset n$ *as follows*

$$\mathrm{Dom}(a_0^*) = \left\{ u = u_0 + \frac{a}{a-iI}\xi_+^u + \frac{1}{a-iI}\xi_-^u \right\}, \tag{1}$$

where $u_0 \in \frac{1}{a-iI} h \ominus n \equiv \mathrm{Dom}(a_0)$. *The action of the adjoint operator is given by the following formula*

$$a^* \left(u_0 + \frac{a}{a-iI}\xi_+^u + \frac{1}{a-iI}\xi_-^u \right) = a u_0 - \frac{1}{a-iI}\xi_+^u + \frac{a}{a-iI}\xi_-^u. \tag{2}$$

The symplectic hermitian boundary form of the adjoint operator on elements $u, v \in \mathrm{Dom}(a_0^*)$ *is given by*

$$\mathcal{J}(u,v) \equiv < a_0^* u, v > - < u, a_0^* v > = < \xi_-^u, \xi_+^v > - < \xi_+^u, \xi_-^v > .$$

[2] We use the nonstandard notation - a small character for operator - just to underline that it is splitted from A.

The selfadjoint extensions of the symmetric operator a_0 are determined uniquely by the Lagrangian planes of the simplectic boundary form. Some of these planes can be determined by the boundary conditions

$$\xi_- = \gamma \xi_+,$$

with any fixed selfadjoint boundary operator $\gamma : n \to n$ [3]. *The selfadjoint extensions of a_0 determined by the latter boundary conditions will be denoted by a_γ in what follows. All selfadjoint extensions of a_0 are defined by these Lagrangian planes and all their J-unitary transformations.*

The resolvent of any selfadjoint extension a_γ is described by Krein's formula :

$$\frac{1}{a_\gamma - \lambda I} = \frac{1}{a - \lambda I} + \frac{a + iI}{a - \lambda I} P_n \frac{I}{\gamma - P_n \frac{1+\lambda a}{a - \lambda I} P_n} P_n \frac{a - iI}{a - \lambda I} \tag{3}$$

Proof of this "one - body version" can be found, for instance, in [6,27]. The case of non-densely defined symmetric operators is considered in [20,21].

Notice, that the elements $\xi^u_\pm$ play the role of boundary values for the elements from the domain of the adjoint operator, see [15]. In general case when $n \cap \mathrm{Dom}(a_0) \neq \{0\}$ Krein's formula (3) can determine resolvents of operator relations, see for instance [12,16,17]. In the case where the operator a can be represented as an orthogonal sum of two operators $a = a^0 \oplus a^1$ acting in a certain orthogonal decomposition of the Hilbert space $h = h^0 \oplus h^1$ and the deficiency subspace possesses a similar decomposition $n = n^1 \oplus n^2$, $n^s \in h^s$ this formula describes the interaction between the two *channels* $[h^0, a^0]$ and $[h^1, a^1]$ via the *nonadditive* perturbation γ. If $n_0 \cap \mathrm{Dom}(a^0) = \{0\},$, $n_1 \cap \mathrm{Dom}(a^1) \neq \{0\}$, Krein's formula may give a resolvent of some operator relation, see [12,34], but still the restriction of it onto the minimal reducing subspace of the total space $h = h^0 \oplus h^1$ containing $\mathrm{Dom}(a^0_0)$ *is an operator* if the spectrum of a^0 is absolutely continuous, see [20,21].

Krein's formula may be generalized in different ways. The most obvious but still useful *few-channel* generalization of it can be obtained assuming that the operator a, the corresponding Hilbert space h and the deficiency subspaces are given in the form of orthogonal decompositions :

$$h = \sum_s \oplus h_s, \quad a = \sum_s \oplus a_s, \quad n = \sum_s \oplus n_s,$$

and the boundary operator γ is defined by some operator matrix γ_{st} corresponding to the decomposition of the deficiency subspace. The calculation of components of the generalized resolvent requires elimination of some variables and solution of the matrix equations similar to ones considered in [9].

More subtle generalization of Krein's formula has been suggested in [33] as an intermediate step for the construction of a solvable model of the three-body scattering. We derive here a modified version of this *fiber Krein's formula*, see Theorem 2.3, in terms of

[3] Of course one can consider also infinite dimensional deficiency subspaces , see below, and even sometimes unbounded operators γ. We restrict ourselves at the moment to the case of finite dimensional deficiency subspaces.

rigged Hilbert spaces. Using the ideas of rigged spaces the construction of the extension a_γ can be reduced to some abstract version of the potential theory.

Consider a dense linear variety $\mathrm{Dom}(a) \equiv h_a \in h$ - the domain of a positive selfadjoint operator a - and the space h^a of all bounded linear functionals defined on it:

$$h_a \subset h \subset h^a.$$

The spaces $h_a,\ h,\ h^a$ form *Gelfand triplet* or *rigged Hilbert space*, associated with the operator a. In our case the spaces h_a and h^a are Hilbert ones with the following dot products

$$\begin{aligned} [u,v]_a &= <(a+iI)u,(a+iI)v>, \\ [\rho,\tau]^a &= <(a+iI)^{-1}\rho,(a+iI)^{-1}\tau> \end{aligned}$$

for h_a and h^a respectively. In what follows we use notation $<,>$ to denote both the dot-product in the original Hilbert spaces and the action of functionals. In particular we consider below linear functionals ρ which are bounded on $\mathrm{Dom}(a) \equiv h_a$ and presented as $< \dfrac{1}{a-iI}f,\rho >$. For instance for a given $u = u_0 + \frac{a}{a-iI}\xi_+^u + \frac{1}{a-iI}\xi_-^u \in D_{a_0}$, we define the generalized boundary density $\rho^u \in h^a$ by the formula

$$\rho^u = (a+iI)\xi_+^u \ \in (a+iI)n_i \equiv n^i \in h^a,$$

and the *regularized boundary value* of u in h_a as follows

$$u_- \equiv \frac{1}{a-iI}\xi_-^u \ \in \frac{I}{a-iI}n_i \in h_a = \mathrm{Dom}(a),$$

then the boundary form $\mathcal{J}(u,v)$ may be rewritten as

$$\mathcal{J}(u,v) =< u_-,\rho^v > - < \rho^u, v_- >, \tag{4}$$

where the brackets $<\cdot,\cdot>$ denote the action of the functionals on the functions. Note that any element $u \in \mathrm{Dom}(a)$ possesses the following representation

$$u \equiv \frac{1}{a-iI}f = \frac{1}{a-iI}(I-P_n)f + \frac{I}{a-iI}P_n f \equiv u_0 + u_-,$$

where $u_0 \in \mathrm{Dom}(a_0)$ and $u_- \in \frac{I}{a-iI}n_i$ is the regularized boundary value of u. The generalized density for any element $u \in D_0$ is trivial, $\rho^u = 0$. We call $\rho^u,\ u_-$ the *rigged boundary values* of the element $u \in \mathrm{Dom}(a_0^*)$. It is important that the densities of deficiency elements $\nu_{\bar\lambda}$ for different values of the spectral parameter λ

$$\nu_{\bar\lambda} = \frac{a+iI}{a-\lambda I}\nu,\ \ \nu \in n \equiv n_i,$$

may be chosen from the standard subspace $n^i \in h^a$ which does not depend on the spectral parameter:

$$\nu_{\bar\lambda} = \frac{1}{a-\lambda I}\rho,\ \ \rho \in (a+iI)n_i = n^i \in h^a,$$

similarly to the densities of potentials which appear in potential theory. But the regularized values of these deficiency elements do depend on λ so that all spectral information on the constructed extension may be derived from this dependence via Krein's formula, see for instance [6,33,34].

The next statement is just a reformulation of the preceding Theorem 2.1 in terms of generalized densities and boundary values:

Theorem 2.2 *The boundary conditions defining some Lagrangian plane ofthe symplectic boundary form in terms of rigged variables ρ^u, u_- are parametrized by the corresponding boundary operator $\Gamma : h^a \to h_a$; acting in the relevant Gelfand triplet $h_a \subset h \subset h^a$ as follows*

$$u_- = \Gamma \rho^u.$$

This operator is connected to the operator $\gamma : n \to n$ parametrizing the boundary conditions in the previous Theorem 2.1 by the formula:

$$\Gamma = \frac{1}{a - iI} \gamma \frac{1}{a + iI}.$$

In terms of the rigged variables ρ^u, u_- the calculation of the resolvent of the extension $a^\Gamma \equiv a_\gamma$ is reduced to the solution of the equation for the density ρ^u of the element $u = \frac{1}{a^\Gamma - \lambda} f$:

$$[\Gamma - \frac{1}{a - iI} P_n \frac{1 + \lambda a}{a - \lambda I} P_n \frac{1}{a + iI}]\rho = \frac{1}{a - iI} P_n \frac{a - iI}{a - \lambda I} f,$$

which gives the Krein's formula in the rigged spaces

$$\frac{1}{a^\Gamma - \lambda} f = \frac{1}{a - \lambda} f + \frac{I}{a - \lambda} \rho.$$

Proof. Changing notations in formulas (1) and (2) we get

$$u = u_0 + u_- + \frac{a}{a^2 + I} \rho^u \tag{5}$$

$$u_0 + u_- = \frac{1}{a - \lambda I} f - \left[\frac{1}{a - \lambda} - \frac{a}{a^2 + 1} \right] \rho^u. \tag{6}$$

Then using the boundary operator Γ acting in the rigged spaces together with $P_n(a - iI)u_0 = 0$ and (2) implies

$$\left[\Gamma - \frac{1}{a - iI} P_n \frac{1 + \lambda a}{a - \lambda I} P_n \frac{1}{a + iI} \right] \rho = \frac{1}{a + iI} P_n \frac{a + iI}{a - \lambda I} f, \tag{7}$$

and then (4) gives the result announced. □

Note, that the reduction of the calculation of the resolvent of the operator $a_\gamma \equiv a^\Gamma$ to the solution of (6) results in effective lowering of the dimension of the space of the solutions due to the presence of the projection P_n onto the deficiency subspaces if $\dim n < \infty$. If for the operator a_0 we have $\dim n = \infty$, then the solution of (6) may be reduced to Riesz - Schauder problem with a compact operator, if some additional conditions are satisfied. In what follows we show, that equation (6) in the "few- body" situation plays the role of the system of Faddeev equations and the densities play the role of Faddeev components.

Let us consider the decomposition - *splitting* - of the Hilbert space $\mathcal{H}$ which is *compatible* with the corresponding *splitting* of a certain selfadjoint operator $\mathcal{A}$:

$$\mathcal{H} = H \times h$$
$$\mathcal{A} = A \times I_h + I_H \times a.$$

We assume that A and a are positive selfadjoint operators acting in the Hilbert spaces H and h respectively. Then the operator $\mathcal{A}$ given by (8) can be defined on the algebraic tensor product $\mathrm{Dom}(A) \times \mathrm{Dom}(a)$ of the domains of the operators A and a. The operator defined defined this way is in fact essentially selfadjoint. In what follows we are not going to distinguish the symmetric operator $\mathcal{A}$ defined on the algebraic tensor product and the closure of it.

We chose a certain finite dimensional subspace $n \in h, n \cap \mathrm{Dom}(a) = \{0\}$ and consider the restriction $a \to a_0$ defined on $\mathrm{Dom}(a_0) \equiv \frac{1}{a-iI} h \ominus n$. Obviously this restriction is a densely defined symmetric operator with the deficiency subspace $n_i = n$. The dual deficiency subspace n_{-i} of it coincides with $\frac{a+iI}{a-iI} n$. The adjoint operator a^* is densely defined on $\mathrm{Dom}(a_0^*) = \mathrm{Dom}(a_0) + n_i + n_{-i}$ and the subspaces n_i, n_{-i} are the eigenspaces of a^* with the eigenvalues $\mp i$ respectively. The subspace

$$\frac{a+iI}{a-\bar{\mu}I} n \equiv n_\mu$$

is the deficiency subspace which corresponds to the value μ of the spectral parameter.

Consider the restriction $\mathcal{A}_0$ of $\mathcal{A}$ defined by the formula

$$\mathcal{A}_0 = A \times I_h + I_H \times a_0.$$

Obviously it is a symmetric operator with infinite deficiency indices if the Hilbert space H is infinite dimensional.

Lemma 2.1 *The operator $\mathcal{A}_0$ has the deficiency subspace $\mathcal{N}_i \equiv \mathcal{N}$:*

$$\mathcal{N} = \overline{\left\{ \frac{1}{\mathcal{A}+iI}(a+iI_h)\ \nu \times e \right\}}, \tag{8}$$

where $\nu \in n, e \in H$

Proof may be obtained by straightforward verification of the formulas

$$< \mathcal{N}, (\mathcal{A}-iI)\mathrm{Dom}(\mathcal{A}_0) >= 0, \ \ \mathcal{H} \ominus (\mathcal{A}-iI)\mathrm{Dom}(\mathcal{A}_0) = \mathcal{N}.$$

Remark. The deficiency subspace N is equal to the closure of the set of all deficiency elements described by the densities e from the Hilbert space H. Such deficiency elements will be called regular. Using only regular elements one can define a symmetric extension of the operator $\mathcal{A}_0$, but not a selfadjoint operator. Nevertheless this extension is essentially selfadjoint. In what follows we are going to use only regular deficiency elements having in mind that all operators defined in this way are essentially selfadjoint only. The corresponding unique selfadjoint operators can be obtained via the closure procedure.

One may check in a similar way that for any complex value λ of the spectral parameter we have also

$$\mathcal{N}_{\bar{\lambda}} = \overline{\frac{a+iI}{\mathcal{A}-\lambda I} n \times H}. \tag{9}$$

In what follows we describe the selfadjoint extensions of the splitted operator

$$\mathcal{A}_0 = A \times I_h + I_H \times a_0,$$

provided the deficiency subspaces n_i of the operator a_0 are finite dimensional. In fact we shall solve slightly more general problem concerning extensions of the operator

$$A \times I_{h\oplus E} + I_H \times [a_0 \oplus b_0] = [A + a_0] \oplus [A + b_0] \, .$$

The program of construction of selfadjoint extensions of splitted and reduced operators via some boundary conditions

$$\mathcal{A} = A + a \longrightarrow \mathcal{A}_0 = A + a_0, \text{ in } \mathcal{H} = H \times h,$$

$$\mathcal{B} = A + b \longrightarrow \mathcal{B}_0 = A + b_0, \text{ in } H \times E \tag{10}$$

was realized first in [10] for unique extension component $\mathcal{A}_0$ only and then in [34] for zero-range potentials with inner structure. Both approaches used the following straightforward regularization of the boundary values of outer component of deficiency elements [10,28]

$$\frac{1}{\mathcal{A}-\lambda I}\rho \to \frac{1}{\mathcal{A}-\lambda I}\rho - \frac{\mathcal{A}}{\mathcal{A}^2+1}\rho, \;\; \rho \in (\mathcal{A}+iI)\mathcal{N}.$$

In [34] the semiboundedness of the total Hamiltonian constructed using operator extension technique based on this regularization has been proven. Later investigating the solvable model for three-body scattering with "inner structure" K.Makarov noticed [26], that this straightforward regularization of the boundary values, being *translation invariant*, still is not compatible with the separation of variables of the third particle on the interaction cylinder for the given pair. He suggested another *pairwise* regularization of the boundary values for elements from the domain of adjoint operator.

In the current paper we suggest using similar regularization for general M-body problem both for inner and outer components of the boundary values and characterize it as *compatible with the splitting* or *pairwise* regularization. For instance, if just one inner channel is present and the corresponding inner Hamiltonian is defined by the operator $\mathcal{B} = B + b$ then the regularization is defined as

$$\frac{I}{\mathcal{A}\oplus\mathcal{B}-\lambda I}\rho \longrightarrow \frac{I}{\mathcal{A}\oplus\mathcal{B}-\lambda I}\rho - \frac{a\oplus b}{[a\oplus b]^2+I}\rho, \; \rho \in (a\oplus b+iI)\mathcal{N} \subset h^{a\oplus b} \times H. \tag{11}$$

In the next section we derive the Krein's formula for the selfadjoint extension of the orthogonal sum of three "channels". This construction may serve as an elementary brick for more sophisticated multychannel solvable models. The analysis of this elementary brick permits to discover new conditions for semiboundedness of Hamiltonians with zero - range potentials. On this base we consider in the last section the classical example of M-particle quantum Hamiltonian with generalized zero-range potential constructed by operator extension procedure and derive new konditions for the semiboundedness of it.

In the remaining part of this section we derive simplest generalizations of Krein formula : the *fiber* Krein formula and two-body Krein formula in erns of rigged boundary values. Both of them serve as steps to more advanced Krein formula discussed in the following section. Note first of all that the deficiency elements of the operator $\mathcal{A}_0 \oplus \mathcal{B}_0$ in the tensor decomposition of the Hilbert Space $\mathcal{H} \oplus [E \times H] = [h \oplus E] \times H$ are represented in terms of vector-valued densities associated with the Gelfand triplet $h_{a\oplus b} \subset h \subset h^{a\oplus b}$,

$$\nu_{\lambda}^{\mathcal{A}} = \frac{a+iI}{\mathcal{A}-\lambda I}(\nu \times e) =$$
$$\frac{1}{\mathcal{A}-\lambda I}(\rho_\nu \times e), \; e \in H.$$

We denote the generalized vector-valued densities $\rho \equiv (\rho_\nu \times e) \in h^a \times H$ by the same symbol and call them *two-body densities.* The dual regularized boundary values will be denoted by u_-. We denote by $P \equiv P_n$ the orthogonal projection onto $n \times H$ in $\mathcal{H}$. The operator P is connected to the orthogonal projection onto n in h by the obvious formula:

$$P \equiv P_n = P_n^h \times I_H.$$

Similarly for any selfadjoint operator $a : \mathrm{Dom}(a) \to h$ we denote the operator $a \times I_H$ in the product space $h \times H$ just by a, having in mind that the precise meaning of the notation is clear from the context.

The following statement gives the "two-body" [4] characterization of densities and regularized boundary values for some deficiency elements. Just for the sake of brevity we formulate this statement for deficiency elements $\nu_{\lambda}^{\mathcal{A}}$ of $\mathcal{A}$. Obviously similar statement is true also for deficiency elements for the orthogonal sum $\mathcal{A}+a\oplus b = [a+A] \oplus [b+A]$ defined by proper vector-valued densities and regularized boundary values of them.

Lemma 2.2 *The following decomposition holds for any deficiency element $n_{\lambda}^{\mathcal{A}} = \frac{1}{\mathcal{A}-\lambda I}\rho_\nu$ of the operator $\mathcal{A}_0$:*

$$\frac{1}{\mathcal{A}-\lambda I}\rho_\nu =$$
$$\frac{a}{a^2+I}\rho_\nu + \frac{1}{a-iI}P_n\frac{I+a(\lambda-A)}{\mathcal{A}-\lambda}\frac{1}{a+iI}\rho_\nu + u_0,$$

where $u_0 \in \mathrm{Dom}(a_0) \times H$

[4] In fact we discuss here the separation of the whole system into two subsystems, one of them is a cluster and another one is a subsystem of free particles (or clusters). But each cluster is in fact a "particle" with internal degrees of freedom. Hence the example "cluster + free particles" may be considered as a representative example for few-body complex. This is our motivation for using the term " two-body" for general situation.

Proof. Notice, that for any deficiency element of $\mathcal{A}$ the following decomposition holds in proper Gelfand triplet:

$$\frac{1}{\mathcal{A}-\lambda I}\rho_\nu = \frac{a}{a^2+1}\rho_\nu + \frac{I}{a-iI}\frac{I+a(\lambda-A)}{\mathcal{A}-\lambda I}\frac{I}{a+iI}\rho_\nu.$$

inserting the projection operator P we transform the latter expression as follows

$$\begin{aligned}\frac{1}{\mathcal{A}-\lambda I}\rho_\nu &= \frac{a}{a^2+1}\rho_\nu + \frac{I}{a-iI}P\frac{I+a(\lambda-A)}{\mathcal{A}-\lambda I}P\frac{I}{a+iI}\rho_\nu \\ &+\frac{I}{a-iI}[I-P]\frac{I+a(\lambda-A)}{\mathcal{A}-\lambda I}P\frac{I}{a+iI}\rho_\nu.\end{aligned}$$

Note, that the second projection in front of $\frac{I}{a+iI}\rho_\nu$ does not play any role since $\frac{I}{a+iI}\rho_\nu \in \mathcal{N}$. On the other hand $\frac{I}{a-iI}[I-P]\frac{I+a(\lambda-A)}{\mathcal{A}-\lambda I}P\frac{I}{a+iI}\rho_\nu \in \mathrm{Dom}(a)$. Hence the density and the regularized value of the deficiency element are defined by

$$\begin{aligned}\rho_\nu &= \rho_\nu, \\ u_- &= \frac{I}{a-iI}P\frac{I+a(\lambda-A)}{\mathcal{A}-\lambda I}P\frac{I}{a+iI}\rho_\nu\end{aligned}$$

□

To derive the *fiber* Krein formula in two- channel situation we consider the total two - channel deficiency subspace $\mathcal{N}\oplus\mathcal{M}$ of the operator $\mathcal{A}_0\oplus\mathcal{B}_0$ and the corresponding space $\mathcal{D}$ of vector - valued densities $\mathcal{D}=(a+iI)\mathcal{N}\oplus(b+iI)\mathcal{M}\subset h^a\oplus E^b$,

$$\rho=\begin{pmatrix}\rho_a\\ \rho_b\end{pmatrix}\in\mathcal{D}=(a+iI)\mathcal{N}\oplus(b+iI)\mathcal{M}\subset h_-\oplus E_-.$$

We need also the dual space $\mathcal{C}$ of regularized boundary values

$$u_-=\begin{pmatrix}u_a\\ u_b\end{pmatrix}\in\mathcal{C}=(a+iI)^{-1}\mathcal{N}\oplus(b+iI)^{-1}\mathcal{M}\subset h_a\oplus E_b$$

The latter Lemma and the "channel- version of it" with $a\oplus b$ instead of a is essentially different from the similar " one-body" regularization of the boundary value in the Theorem 2.2 via subtracting the term $\frac{a}{a^2+1}\rho$, which involves the total operator a. Using for the total Hamiltonian $\mathcal{A}$ the splitted operator $\mathcal{A}=A\times I_h+I_H\times a\equiv A+a$ we achieve the regularization by subtracting the term which involves the component a only. Thus the local values of the corresponding densities and regularized values *are compatible with the separation of variables related to the component* A. We say that the regularization $\frac{1}{\mathcal{A}-\lambda I}\rho_\nu\to\frac{1}{\mathcal{A}-\lambda I}\rho_\nu-\frac{a}{a^2+I}\rho_\nu$ *is compatible with the splitting* $\mathcal{A}\to(A+a)$ or call it *two-body regularization*, still having in mind, that each of the objects a, A may have a sophisticated inner structure, see the footnote on the previous page.

Consider a bounded operator Γ acting from $\mathcal{D}$ to C and associated with the 2×2 operator matrix within the decompositions above:

$$\Gamma=\begin{pmatrix}G_{hh} & G_{hE}\\ G_{Eh} & G_{EE}\end{pmatrix}.$$

We assume , that Γ is defined via some finite Hermitian block matrix γ acting from $n \oplus m$ into $n \oplus m$:

$$\gamma = \begin{pmatrix} \gamma_{hh} & \gamma_{hE} \\ \gamma_{Eh} & \gamma_{EE} \end{pmatrix}, \Gamma = \begin{pmatrix} \frac{I}{a-iI} & 0 \\ 0 & \frac{I}{b-iI} \end{pmatrix} \gamma \begin{pmatrix} \frac{I}{a+iI} & 0 \\ 0 & \frac{I}{b+iI} \end{pmatrix}.$$

Theorem 2.3 *The boundary condition*

$$u_- = \Gamma \rho$$

defines a Lagrangian plane of the boundary form

$$J(u,v) = < u_-, \rho_v > - < \rho_u, v_- > .$$

Let us denote by $\mathcal{A}_\Gamma$ the corresponding selfadjoint extension of the operator

$$\mathcal{A}_0 = A \times I_{h \oplus E} + I_H \times [a_0 \oplus b_0].$$

The resolvent of $\mathcal{A}_\Gamma$ is given by the formula

$$\frac{1}{\mathcal{A}_\Gamma - \lambda I} f = \frac{1}{\mathcal{A} - \lambda I} f + \frac{1}{\mathcal{A} - \lambda I} \rho,$$

where the vector-valued density ρ satisfies the equation

$$\begin{pmatrix} \Gamma_{hh} - \frac{I}{a-iI} P_n \frac{I+(\lambda I - A)a}{A+a-\lambda I} P_n \frac{I}{a+iI} & \Gamma_{hE} \\ \Gamma_{Eh} & \Gamma_{EE} - \frac{I}{b-iI} P_m \frac{I+(\lambda I-A)b}{A+b-\lambda I} P_m \frac{I}{b+iI} \end{pmatrix} \begin{pmatrix} \rho_h \\ \rho_E \end{pmatrix} =$$
$$\begin{pmatrix} \frac{I}{a-iI} P_n \frac{a-iI}{a-\lambda I} f_h \\ \frac{I}{b-iI} P_m \frac{b-iI}{a-\lambda I} f_E, \end{pmatrix}$$

and $(f_h,\ f_E)$ are the components of $f \in \mathcal{H}$ with respect to the orthogonal decomposition $\mathcal{H} = (H \times h) \oplus (H \times E)$.

Proof. Let us consider the equation involving the adjoint operator

$$\begin{pmatrix} \mathcal{A}_0 & 0 \\ 0 & \mathcal{B}_0 \end{pmatrix}^* u - \lambda u = f \in (H \times h) \oplus (H \times E).$$

Using the two-body regularization we represent the element u as

$$\begin{aligned} u &= \frac{a \oplus b}{[a \oplus b]^2 + I} \rho + \frac{I}{a \oplus b - iI} \left[\frac{I + (\lambda I - A) a \oplus b}{A + a \oplus b - \lambda I} \right] \rho + \frac{I}{\mathcal{A} - \lambda I} f \\ &= \frac{a \oplus b}{[a \oplus b]^2 + I} \rho + \\ & \quad \frac{I}{a \oplus b - iI} P \left[\frac{I + (\lambda I - A) a \oplus b}{A + a \oplus b - \lambda I} \right] \rho + \frac{I}{a \oplus b - iI} P \frac{a \oplus b - iI}{\mathcal{A} - \lambda I} f \\ & \quad + \frac{I}{a \oplus b - iI} [I - P] \left[\frac{I + (\lambda I - A) a \oplus b}{A + a \oplus b - \lambda I} \right] \rho + \frac{I}{a \oplus b - iI} [I + P] \frac{a \oplus b - iI}{\mathcal{A} - \lambda I} f \\ &\equiv \frac{a \oplus b}{[a \oplus b]^2 + I} \rho + u_- + u_0, \end{aligned}$$

where $u_0 \in D_A \times D_{a_0 \oplus b_0}$ and u_- is the regularized boundary value of the element u. Then using the boundary conditions we get ρ as a solution of the inhomogeneous equation

$$\left\{\gamma - P\left[\frac{I + (\lambda I - A)\, a \oplus b}{A + a \oplus b - \lambda I}\right] P\right\} \frac{I}{a \oplus b + iI}\, \rho = P \frac{a \oplus b - iI}{\mathcal{A} - \lambda I} f. \tag{12}$$

Inserting the result into the expression for u we get *fiber Krein's formula* derived first in [27]

$$\begin{aligned} u &= \frac{1}{A + a \oplus b - \lambda I} f \\ &+ \frac{a \oplus b + iI}{A + a \oplus b - \lambda I} P \left[\frac{I}{\gamma - P \frac{I + (\lambda - A)(a \oplus b)}{A + a \oplus b - \lambda I} P}\right] P \frac{a \oplus b - iI}{A + a \oplus b - \lambda I} f. \end{aligned}$$

Using rigged variables $\rho = (\rho_h \oplus \rho_E) = (a + iI)\xi_h \oplus (b + iI)\xi_E$ one can reduce the latter formula to the announced form. □

Remark. In applications one often needs to calculate the restriction of the resolvent calculated to only one of the orthogonal components of the Hilbert space $\mathcal{H} = (H \times h) \oplus (H \times E)$. Let vector f have only one nonzero component $f = (f_h \oplus 0)$. Then the variable ρ_E may be excluded using the standard techniques of the operator matrices, see [36] :

$$\rho_E = \frac{I}{\Gamma_{EE} - \frac{I}{b - iI} P_m \frac{I + (\lambda - A)b}{\mathcal{A} + b - \lambda I} P_m \frac{I}{b + iI}} \Gamma_{Eh} \rho_h .$$

Then the problem is reduced to the calculation of the density ρ_h from the equation

$$\begin{aligned} &\left[\Gamma_{hh} - \frac{I}{a - iI} P_n \frac{I + (\lambda - A)a}{\mathcal{A} + a - \lambda I} P_n - \Gamma_{hE} \frac{I}{\Gamma_{EE} - \frac{I}{b - iI} P_m \frac{I + (\lambda - A)b}{\mathcal{A} + \lfloor - \lambda \mathcal{I}} P_m \frac{I}{b + iI}} \Gamma_{Eh}\right] \rho_h \\ = &\ \frac{I}{a - iI} P_n \frac{a - iI}{A + a - \lambda I} f_h . \end{aligned}$$

The interaction constructed above, though formed by operator extensions tools, is quite strong since the deficiency indices of operators are infinite. Still the discrete spectrum of the perturbed operator may be characterized in terms of solutions of the homogeneous equation

$$[\gamma_{hh} - P_n \frac{I + (\lambda - A)a}{A + a - \lambda I} P_n - \gamma_{hE} \frac{I}{\gamma_{EE} - P_m \frac{I + (\lambda - A)b}{A + b - \lambda I} P_m} \gamma_{Eh}] \rho_h = 0 \tag{13}$$

The operator $\mathcal{A}_\Gamma$ just defined is semibounded from below just because it admitts separation of variables. Really, it posesses the tensor decomposition

$$\mathcal{A}_\Gamma = A \times I_{h \oplus E} + I_H \times (a \oplus b)_\gamma ,$$

where $(a \oplus b)_\gamma$ is a certain selfadjoint extension of the symmetric operator $a_0 \oplus b_0$. The operator $a_0 \oplus b_0$ has finite deficiency and the operator A is positive, therefore the operator $\mathcal{A}_\Gamma$ is semibounded.

3 Few-body Krein's formula.

In the previous section we discussed "two - body" Krein's formula which gives the description of the simplest two-body Hamiltonians corresponding to two channels only [8]. We may interpret the channel $A+a$, $H\times h$ as a channel of free particles and the channel, $(A+b,\ H\times E)$ as a channel where one cluster described by the Hamiltonian b coexists with a group of free particles described by the Hamiltonian A.

But even description of the simplest three-body quantum processes requires investigation of several channels, for instance (for distinguishable particles)

$$[(1),(2),(3)],\ [(3,1)(2)],\ [(1,2)(3)],\ [(2,3)(1)],\ [(1,2,3)].$$

The corresponding *equidomoid* - the intersection of the interaction planes of these channels $L_{31}\equiv(x:x_3=x_1)$, L_{12}, L_{23}, contains only one common point 0, so called *triple collision point.* Construction of the interaction via operator-extensions techniques in this situation has been described in [2,3,4,10,19,23,28,31]. The description of the similar procedure for M-body Hamiltonians requires considering channels with non smooth and unbounded equidomoids. For instance, to switch the interaction in the 4- body quantum problem we should split the 4- body free Hamiltonian into several channels $[(1)(2)(3)(4)],$, $[(1,2)(3)(4)]$, $[(1,3)(2)(4)]$, $[(1,4)(2)(3)]$, $[(2,3)(4)(1)]$, $[(2,3)(4)(1)]$, $[(3,4)(1)(2)]$ $[(1,2)(3,4)]$, $[(1,3)(2,4)]$, $[(1,4)(2,3)]$, $[(1,2,3)(4)]$, $[(2,3,4)(1)]$, $[(3,4,1)(2)]$, $[(4,1,2)(3)]$, $[(1,2,3,4)]$. The intersections of the interaction planes of some channels, for instance $[(1,2,3)(4)]$, and $[(2,3,4)(1)]$, are unbounded.

In this section we consider the general model of interaction between three channels, one of them is free channel and two others are channels corresponding to a certain cluster and a group of free particles. A representative example of clusters with a bounded equidomoid is given by the triple of channels

$$[(1)(2)(3)(4)],\ [(1,2)(3)(4)],\ [(3,4)(1)(2)],$$

an exampel of the triple of channels with an unbounded equidomoid is given by the triple of channels

$$[(1)(2)(3)(4)],\ [(1,2)(3)(4)],\ [(1,4)(3)(2)].$$

Consider a Hilbert space $\mathcal{H}$ and a selfadjoint operator $\mathcal{A}$ in $\mathcal{H}$ admitting two decompositions ("splittings")

$$\mathcal{H}=H_1\times h_1,\ \mathcal{A}=A_1\times I_{h_1}+I_{H_1}\times a_1,$$
$$\mathcal{H}=H_2\times h_2,\ \mathcal{A}=A_2\times I_{h_2}+I_{H_2}\times a_2,$$

where $A_s,\ a_t,\ s,t\in(1,2)$ are selfadjoint positive operators acting in the Hilbert spaces $H_s,\ h_t$ respectively.

Let $b_s,\ s\in(1,2)$ be selfadjoint operators acting in the Hilbert spaces E_s. We consider the spaces

$$\mathcal{H}_1=H_1\times E_1,\ \text{and}\ \mathcal{H}_2=H_2\times E_2,$$

and the selfadjoint operators

$$\mathcal{B}_1 = A_1 \times I_{E_1} + I_{H_1} \times b_1,$$
$$\mathcal{B}_2 = A_2 \times I_{E_2} + I_{H_2} \times b_2,$$

acting in these spaces. Restricting the operators $a_s,\ s = 1,\,2$ to $a_{s0},\ \mathrm{Dom}(a_{s0}) = \frac{I}{a_s - iI} h_s \ominus \{n_s\},\ s = 1, 2$ and the operators $b_s,\ s = 1,\,2$ to $b_{s0},\ \mathrm{Dom}(b_{s0}) = \frac{I}{b_s - iI} E_s \ominus \{m_s\},\ s = 1, 2$ we get the symmetric operator$\mathcal{A}_0$ in $\mathcal{H}$ with the total deficiency subspace $\overline{\mathcal{N}_1 + \mathcal{N}_2} \equiv N$ generated by the densities ρ_s

$$\mathcal{N}_s = \left\{ \frac{I}{\mathcal{A} + iI} \rho_s \right\},\ \rho_s \in n^{a_s} \times H_s\ s = 1, 2.$$

and two symmetric operators $\mathcal{B}_{s0}$ with the deficiency subspaces $\mathcal{M}_s$ generated by the corresponding densities ν_s:

$$\mathcal{M}_s = \left\{ \frac{I}{\mathcal{B} + iI} \nu_s \right\}\ \nu_s \in m^{b_s} \times H_s\ s = 1, 2.$$

In what follows we denote by P_{ns} the orthogonal projections in $\mathcal{H}$ onto the deficiency subspaces of operators $a_s \times I_{H_s}$, and by $P_{m1},\ P_{m2}$ the orthogonal projections onto the deficiency subspaces $b_{0s} \times I_{H_s}$.

In this section we discuss an analog of Krein's formula describing selfadjoint extensions of the orthogonal sum generated by the fixed splitting of the operator $\mathcal{A}$

$$\mathbf{A} = \mathcal{A}_0 \oplus \mathcal{B}_{10} \oplus \mathcal{B}_{20}$$

in the Hilbert space

$$\mathbf{H} = \mathcal{H} \oplus [H_1 \times E_1] \oplus [H_2 \times E_2].$$

Our aim is to derive the conditions which guarantee the semiboundedness of the constructed extension- the " non- collaps condition". This condition is relevant to the condition of *physical stability* of this model quantum system when interacting with electromagnetic field. We plan to discuss this interesting question in forthcoming publications.

We consider first the case when the deficiency subspaces $\mathcal{N}_1, \mathcal{N}_2$ have trivial or finitedimensional intersection, $\mathcal{N}_1 \cap \mathcal{N}_2 = \{0\}$, but generally we admitt even infinitedimensional intersections. This condition is not restrictive. For example in the first case when the intersection is finite-dimensional the semiboundedness of *all* selfadjoint extensions may be easily derived. For the infinitedimensional intersection the semiboundedness of some special extension constructed in two steps is proven.

For deficiency subspaces $\mathcal{N}_{\bar\lambda,1}, \mathcal{N}_{\bar\lambda,2}$ corresponding to the spectral parameter $\bar\lambda$ we derived in the section 1 the representation via densities

$$\nu^s_{\bar\lambda} = \frac{I}{\mathcal{A} - \lambda \mathcal{I}} \rho_s,\ \rho_s \in n^{as} \times H_s,\ s \in (1, 2)$$

from the corresponding channel subspaces $n^{as} \times H_s$. For M-body quantum Hamiltonians constructed with underlying Laplacian in $L_2(R^{3M-3})$ the free-particles component H of the channel-space connected to the k particles cluster $(b,\,E)$ is formed as a space of square - integrable functions on some plane L or cylinder $\Omega \times L$, $\dim L = 3k$.

$$H = L_2(L).$$

Different channels will be marked by different indices : $H_s,\,A_s,\,L^s,\,a_s,\,E_s,\,b_s$. For two given channels $H_s,\,A_s,\,E_s, b_s,\ s=1,2$ consider the corresponding equidomoid $L^1 \cap L^2$. We may form a dense linear set in the total deficiency subspace of elements which densities $\rho_s^\delta, \delta > 0$ vanish in a δ- neighborhood of the equidomoid $L^1 \cap L^2$. The corresponding general construction involves consideration of the exhausting sequences of deficiency subspaces $\mathcal{N}_s^\delta \longrightarrow \mathcal{N}_s,\ s = 1,2$ for $\delta \to 0$, satisfying the condition

$$\sin\left(\mathcal{N}_1^\delta, \mathcal{N}_2^\delta\right) = \delta \geq 0.$$

Then

$$\mathcal{N}_1^\delta \oplus \mathcal{N}_2^\delta \equiv \mathcal{N}^\delta \longrightarrow \mathcal{N},\ \ \delta \to 0,$$

and the whole construction below can be developed for $\mathcal{N}^\delta$ with subsequent passing to the limit, provided necessary apriory estimates are satisfied. Together with the deficiency subspaces $\mathcal{N}^\delta$ we consider the corresponding operators $\mathcal{A}_0^\delta$, defined on $\mathrm{Dom}(\mathcal{A}_0^\delta) = \frac{1}{\mathcal{A}-iI}\left[\mathcal{H} \ominus \mathcal{N}^\delta\right]$ and their adjoints $\left(\mathcal{A}_0^\delta\right)^* \equiv \mathcal{A}_0^{\delta *}$. The boundary form of this operator can be given in terms of the proper densities ρ^δ which can be decomposed as a sum of channel-densities $\rho_s^\delta \in h^{as} \times H_s$:

$$\rho^\delta = \rho_1^\delta \oplus \rho_1^\delta$$

and the corresponding regularized boundary values.

Lemma 3.1 *Consider the elements from $\mathcal{N}_{\bar\lambda}$ generated by the densities distributed along the interaction planes outside some δ-neighbourhood of the equidomoid:*

$$u^\delta = \frac{1}{\mathcal{A}-\lambda I}\left[\rho_1^{u,\delta} + \rho_2^{u,\delta}\right] \equiv u_1 + u_2,$$
$$v^\delta = \frac{1}{\mathcal{A}-\lambda I}\left[\rho_1^{v,\delta} + \rho_2^{v,\delta}\right] \equiv v_1 + v_2.$$

Then in the corresponding boundary form can be represented as

$$< \mathcal{A}_0^* u^\delta,\, v^\delta >_{\mathcal{H}} - < u^\delta,\, \mathcal{A}_0^* v^\delta >_{\mathcal{H}} =$$
$$\sum_{s,t=1,2} < u_{s,t},\, \rho_s^v > - < \rho_s^u,\, v_{s,t} >$$

where the regularized boundary values $u_{s,t},\ v_{s,t}$ are defined as follows

$$u_{s,t} = \frac{I}{a_s - iI} P_s \frac{a_s - iI}{\mathcal{A}-\lambda I}\rho_t^\delta - \delta_{st}\frac{a_t}{a_t^2 + I}\rho_t^\delta +$$

$$\delta_{st}\frac{1}{a_t - iI} P_t \frac{I + (\lambda I - A_t)a_t}{\mathcal{A}-\lambda I}\frac{1}{a_t + iI}\rho_t.$$

Proof. The proof may be obtained the same way as the proof of the corresponding statement in [28]. Note that the boundary value of the function $u_2 = \frac{I}{\mathcal{A}-\lambda I}\rho_2$ generated by the density ρ_2 being considered on the support of the density ρ_1 does not need any regularization since the bilinear form

$$< \frac{I}{\mathcal{A}-\lambda I}\rho_2^\delta,\ \rho_1^\delta >$$

is bounded untill we stay on some distance (more than δ) from the equidomoid. Hence the corresponding operator

$$P_{1n}\frac{a_1 - iI}{\mathcal{A}-\lambda I} : n^2 \times H_2 \rightarrow n_1 \times H_1$$

is bounded on the described class of densities ρ^δ. This permits to represent the function u_2 in the form

$$\begin{aligned}
\frac{I}{\mathcal{A}-\lambda I}\rho_2^\delta &= \frac{I}{a_1 - iI}P_{1n}\frac{a_1 - iI}{\mathcal{A}-\lambda I}\rho_2^\delta + \frac{I}{a_1 - iI}\left[I - P_{1n}\right]\frac{a_1 - iI}{\mathcal{A}-\lambda I}\rho_2 \\
&\equiv u_{12} + u_{12}^0,
\end{aligned}$$

where $u_{12} \in h_a$ is a regularized boundary value and u_{12}^0 is an element from the domain of the restricted operator a_{10}.

Vice versa, for the boundary value of the element $u_1 \in \frac{1}{a_1 - iI}\mathcal{N}_1$, generated by the density ρ_1 we have to use the regularization,since generally the operator

$$P_{1n}\frac{a_1 - iI}{\mathcal{A}-\lambda I} : h^1 \times H_2 \rightarrow n_1 \times H_1$$

is unbounded, as simple examples show. Hence ve have , for instance:

$$\begin{aligned}
u_1 &= \frac{I}{\mathcal{A}-\lambda I}\rho_1 \\
&= \frac{I}{\mathcal{A}-\lambda I}\rho_1 - \frac{a_1}{a_1^2+1}\rho_1 + \frac{a_1}{a_1^2+1}\rho_1 \\
&= \frac{a_1}{a_1^2+1}\rho_1 + \frac{I}{a_1 - iI}P_{1n}\frac{I + a_1\left(\lambda I - A_1\right)}{\mathcal{A}-\lambda I}P_{1n}\frac{I}{a_1 + iI}\rho_1 \\
&\quad + \frac{I}{a_1 - iI}\left[I - P_{1n}\right]\frac{I + a_1\left(\lambda I - A_1\right)}{\mathcal{A}-\lambda I}P_{1n}\frac{I}{a_1 + iI}\rho_1 \\
&\equiv \frac{a_1}{a_1^2+1}\rho_1 + u_{11} + u_{10},
\end{aligned}$$

where u_{11} is the regularized boundary value of u_1 in $\frac{I}{a_1 - iI}\mathcal{N}_1$ and $u_{10} \in \mathrm{Dom}(a_{10})$. Now the same arguments as in the previous section permitt to calculate the boundary form.

$\square$

We may restrict the operators $\mathcal{B}_s,\ s = 1,2$ as it has been described -n the previous section and consider the boundary forms of the adjoint operators $\mathcal{B}_{s0}^*,\ s = 1,2$ representing them in terms of the corresponding rigged variables $\nu^{u_s} \in E^s \times H_s$, $U_s \in E_s \times H_s$ as follows

$$\mathcal{J}(\mathcal{B}_{0s}^*) =< \mathcal{B}_{0s}^* u, v > - < u, \mathcal{B}_{0s}^*, v >=< U_s, \nu^{v_s} > - < \nu^{u_s}, V_s >,\ s = 1,2.$$

Our nearest aim is to construct the Hamiltonian describing the exchange between two clusters via extension of the restricted operators

$$\mathcal{A}_0 \oplus \mathcal{B}_{10} \oplus \mathcal{B}_{20}$$

to some selfadjoint operator in the total Hilbert space

$$\mathbf{H} = \mathcal{H} \oplus [H_1 \times E_1] \oplus [H_2 \times E_2].$$

Practically we need to construct the resolvent of this extension. We reduce the construction of the resolvent to the solution of some intermediate equation. In the classical case when the deficiency subspace is finitedimensional this intermediate equation is just a matrix equation involving Krein's Q- functions, see the previous section. In our case this equation is obviously infinitedimensional and it coincides with the Faddeev equation in three-body scattering problem. We retain this term even in general case.

We derive first the equation for corresponding rigged variables and then investigate the solvability of this equation both for general situation and for some basic examples.

We construct the selfadjoint extension of $\mathcal{A}_0 \oplus \mathcal{B}_{10} \oplus \mathcal{B}_{20}$ via *pairwise* boundary conditions attached to the interaction cylinders L^s and connecting the rigged variables ρ_s, u_s of the adjoint operator $\mathcal{A}_0$ with rigged variables ν_s, U_s for the operators $\mathcal{B}_s$:

$$\begin{pmatrix} u_s \\ U_s \end{pmatrix} = \Gamma_s \begin{pmatrix} \rho_s \\ \nu_s \end{pmatrix}, \tag{14}$$

here Γ_s is a *constant* hermitian matrix . The dimension of it is equal to $(\dim\, n_s + \dim\, m_s) \times (\dim\, n_s + \dim\, m_s)$, where n_s, m_s are the deficiency indices of the operators a_{s0}, b_{s0} :

$$\Gamma_s = \begin{pmatrix} \Gamma_{h_s,h_s} & \Gamma_{h_s,E_s} \\ \Gamma_{E_s,h_s} & \Gamma_{E_s,E_s} \end{pmatrix}.$$

The described boundary conditions incorporate the data of a_s, b_s which correspond to the particles interacting at given cylinder L^s. The other data, which correspond to free particles, described by A^s may be separated as in [23].

One may check directly, that this boundary conditions (14) define a symmetric operator on the dense domain in the total Hilbert space corresponding to the densities ρ^δ with a compact support outside the equidomoid. If we prove, that this operator is semibounded, then we can accomplish the construction of the corresponding selfadjoint operator by Friedrich's procedure. It means, that we need to derive the operator equation corresponding to the equation for densities similar to (11) and then prove, that it has a unique solution on a dense domain in the total Hilbert space for large negative values of the spectral parameter.

We fulfill the described program deriving in this section the relevant system of equations for densities in abstract model. The resulting statement summarizing our analysis will be formulated at the end of this section. Basing on this statement we investigate in the following section the conditions of semiboundedness of M body quantum hamiltonian with a generalized point interaction (with inner structure).

Consider the solutions u, U_1, U_2 of the adjoint equations:

$$(\mathcal{A}_0^* - \lambda I)\, u = f,$$

$$(\mathcal{B}_{10}^* - \lambda I)\, U_1 = f_1 \in E_1 \times H_1,$$

$$(\mathcal{B}_{20}^* - \lambda I)\, U_2 = f_2 \in E_2 \times H_2. \tag{15}$$

Note that on a dense domain of $f \in \frac{I}{a_1 - iI}\mathcal{H}$ the element $\frac{I}{\mathcal{A}-\lambda I} f$ may be represented as

$$\frac{I}{a_1 - iI} P_{1n} \frac{a_1 - iI}{\mathcal{A} - \lambda I} f + \frac{I}{a_1 - iI} \left[I - P_{1n}\right] \frac{a_1 - iI}{\mathcal{A} - \lambda I} f \equiv F_{11} + F_{10},$$

where F_{11} is a regularized boundary value and F_{10} belongs to the domain of a_{10}.

Solutions of the latter equations can be represented by the following ansatz

$$\begin{aligned}
u &= \frac{I}{\mathcal{A} - \lambda I}\rho_1 + \frac{I}{\mathcal{A} - \lambda I}\rho_2 + \frac{I}{\mathcal{A} - \lambda I} f \\
&= \frac{a_1}{a_1^2 + I}\rho_1 + \frac{1}{a_1 - iI}\frac{I + (\lambda - A_1)\, a_1}{\mathcal{A} - \lambda I}\frac{1}{a_1 + iI}\rho_1 \\
&\quad + \frac{I}{\mathcal{A} - \lambda I}\rho_2 + \frac{1}{\mathcal{A} - \lambda I} f \\
&= \frac{a_2}{a_2^2 + I}\rho_2 + \frac{1}{a_2 - iI}\frac{I + (\lambda - A_2)\, a_2}{\mathcal{A} - \lambda I}\frac{1}{a_2 + iI}\rho_2 \\
&\quad + \frac{I}{\mathcal{A} - \lambda I}\rho_1 + \frac{1}{\mathcal{A} - \lambda I} f, \\
U_1 &= \frac{b_1}{b_1^2 + I}\nu_1 + \frac{1}{b_1 - iI}\frac{I + (\lambda - A_1)\, b_1}{\mathcal{B}_1 - \lambda I}\frac{1}{b_1 + iI}\nu_1, \\
U_2 &= \frac{b_1}{b_2^2 + I}\nu_2 + \frac{1}{b_2 - iI}\frac{I + (\lambda - A_1)\, b_2}{\mathcal{B}_2 - \lambda I}\frac{1}{b_2 + iI}\nu_2.
\end{aligned}$$

To calculate the regularized boundary values of the solution u (on the proper interaction cylinders L^1, L^2) we multiply u by $(a_1 - iI)$ and $(a_2 - iI)$ respectively and apply the projections $P_{1n},\ P_{2n}$ onto the deficiency subspaces $\mathcal{N}_1,\ \mathcal{N}_2$. If taking into account, that the domain of $\mathcal{A}_0$ is

$$\frac{I}{a_1 - iI}\,(h_1 \ominus n_1) \times \mathrm{Dom}(A_1) \cap \frac{I}{a_2 - iI}\,(h_2 \ominus n_2) \times \mathrm{Dom}(A_2),$$

we get the following expressions for the regularized values u_1, u_2 of the solution u

$$\begin{aligned}
u_1 &= \frac{I}{a_1 - iI} P_{1n} \frac{I + a_1\,(\lambda - A_1)}{\mathcal{A} - \lambda I}\frac{I}{a_1 - iI}\rho_1 \\
&\quad + \frac{I}{a_1 - iI} P_{1n}\frac{a_1 - iI}{\mathcal{A} - \lambda I}\rho_2 - \frac{I}{a_1 - iI} P_{1n}\frac{a_1 - iI}{\mathcal{A} - \lambda I} f, \\
u_2 &= \frac{I}{a_2 - iI} P_{2n} \frac{I + a_2\,(\lambda I - A_2)}{\mathcal{A} - \lambda I}\frac{I}{a_2 - iI}\rho_2 \\
&\quad + \frac{I}{a_2 - iI} P_{2n}\frac{a_2 - iI}{\mathcal{A} - \lambda I}\rho_1 - \frac{I}{a_2 - iI} P_{2n}\frac{a_2 - iI}{\mathcal{A} - \lambda I} f.
\end{aligned}$$

The regularized values of the solutions of the adjoint equations exist if $\rho_s = \rho_s^\delta,\ s=1,2$ and belong to the domain of A_s.

The regularized values of the solutions U_1, U_2 may be obtained similarly to the regularized values of the solutions appeared in Theorem 3:

$$\begin{aligned} U_1 &= \frac{1}{b_1 - iI} P_{1m} \frac{I + b_1\,(\lambda I - A_1)}{A_1 + b_1 - \lambda I} \nu_1 \\ &+ \frac{I}{b_1 - iI} P_{1m} \frac{b_1 - iI}{\mathcal{B}_1 - \lambda I} f_1, \\ U_2 &= \frac{I}{b_2 - iI} P_{2m} \frac{I + b_2\,(\lambda I - A_2)}{A_2 + b_2 - \lambda I} \nu_2 \\ &+ \frac{I}{b_2 - iI} P_{2m} \frac{b_2 - iI}{\mathcal{B}_2 - \lambda I} f_2, \end{aligned}$$

Now we apply the boundary conditions (14) and receive the following system of equations for rigged variables $\rho_s,\ \nu_s,\ s=1,2$

$$\left[\Gamma^1 - Q(a_1,b_1)\right] \begin{pmatrix} \rho_1 \\ \nu_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{a_1 - iI} P^1_n \frac{a_1 - iI}{\mathcal{A} - \lambda I} f \\ \frac{1}{b_1 - iI} P^1_m \frac{b_1 - iI}{\mathcal{B} - \lambda I} f_1 \end{pmatrix}$$

$$\left[\Gamma^2 - Q(a_2,b_2)\right] \begin{pmatrix} \rho_2 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{a_2 - iI} P^2_n \frac{a_2 - iI}{\mathcal{A} - \lambda I} f \\ \frac{1}{b_2 - iI} P^2_m \frac{b_2 - iI}{\mathcal{B} - \lambda I} f_2 \end{pmatrix} \tag{16}$$

where

$$Q(a_s,b_s) = \begin{pmatrix} \frac{1}{a_s - iI} P_{sn} \frac{1 + a_s(\lambda I - A_s)}{A_s + a_s - \lambda I} P_{sn} \frac{1}{a_s + iI} & 0 \\ 0 & \frac{1}{b_s - iI} P_{sn} \frac{1 + b_s(\lambda I - A_s)}{A_s + b_s - \lambda I} P_{sn} \frac{1}{a_s + iI} \end{pmatrix}.$$

Together with the latter system it is convenient to use the equivalent system with respect to non- rigged variables

$$\begin{pmatrix} \frac{1}{a_s - iI} \rho_s \\ \frac{1}{b_s - iI} \nu_s \end{pmatrix} \equiv \begin{pmatrix} e_{h_s} \\ e_{E_s} \end{pmatrix},$$

$$[\gamma^s - q^s] \begin{pmatrix} e_{h_s} \\ e_{E_s} \end{pmatrix} = \begin{pmatrix} P^s_n \frac{a_s - iI}{\mathcal{A} - \lambda I} f \\ P^s_m \frac{b_s - iI}{\mathcal{B} - \lambda I} f_s \end{pmatrix} = \mathcal{F}_s,\ s=1,2. \tag{17}$$

Here the rigged Q-functions $Q(a_s,b_s)$ are connected to the non-rigged Q-functions q_s via multiplication by fectors $\frac{I}{a_s - iI}, \frac{I}{b_s - iI}$ from the left and from the right side.

The last system can be simplified by eliminating variables of the inner parts of both channels ν_s from each second equation.

Practically we need to solve the equations [15,16] and then to restore the Krein formula basing on [14]. At the same time these equations play the role of Faddeev equations for the components represented by the channel's densitied ρ_1, ρ_2.

We analyse the solvability of these equations assuming $\dim\, m_s = \dim\, n_s = 1$ and considering separately three cases :

$$1)\ \ n_s \cap \mathrm{Dom}(\sqrt{a_s}) = \{0\},\ \ \gamma^s_{EE} + P_{ms} b_s P_{ms} \neq 0,\ \ s=1,2,$$

$$2)\ \ \gamma^s_{EE} + P_{ms} b_s P_{ms} = 0,\ s = 1, 2,$$

and

$$3)\ \ n_1 \cap \mathrm{Dom}(\sqrt{a}_1) = \{0\},\ \ \gamma^1_{EE} + P_{m1} b_1 P_{m1} \neq 0,\ \gamma^2_{EE} + P_{m2} b_2 P_{m2} = 0\,.$$

Denoting by $\tilde{O}(\frac{I}{\lambda})$ the vectors and operators which admit the estimate by $|\frac{1}{\lambda}|$ for large negative λ and denoting by $\mathcal{F}_s$ the expression which appears in the right hand side after the elimination the inner variables e_{Es} we get the following systems in these cases.

In the first case we have:

$$\Bigg[\gamma^s_{hh} + \int_0^\infty \frac{\alpha\,(A_s - \lambda I)}{A_s + \alpha - \lambda I} P_n d\mathcal{E}^s_\alpha P_n -$$

$$\gamma^s_{hE}[\gamma^s_{EE} + P_{sm} b_{sm} P_{sm} + \tilde{O}(\frac{I}{\lambda})]^{-1} \gamma^s_{Eh}\Bigg]\, e_h + P_{ns} \frac{a_s - iI}{\mathcal{A} - \lambda I}(a_t + iI) e_t =$$

$$P_{ns} \frac{a_s - iI}{\mathcal{A} - \lambda I} f - \gamma^s_{hE}[\gamma^s_{EE} + P_{ms} b_s P_{ms} + \tilde{O}(\frac{I}{\lambda})]^{-1}$$

$$P_{ms} \frac{b_s - iI}{\mathcal{B}_s - \lambda I} f_{E_s} \equiv \mathcal{F}_s. \tag{18}$$

In the second case we have:

$$P_{sn} \frac{a_s - iI}{\mathcal{A} - \lambda I}(a_t + iI) e_{ht} + \Bigg[\Bigg[\int_0^\infty \frac{\alpha\,(A_s - \lambda I)}{A + \alpha - \lambda I} P_{sn} d\mathcal{E}^s_\alpha P_{sn} - \gamma^s_{hh} -$$

$$\gamma^s_{hE}\left([\lambda I - A_s][P_{sm}[I + b_s^2] P_{sm}]^{-1} - \frac{P_{sm}[I + b_s^2] b_s P_{sm}}{P_{sm}[I + b_s^2] P_{sm}} + \tilde{O}(\frac{1}{\lambda})\right) \gamma^s_{Eh}\Bigg]\, e_{sh} =$$

$$P_{sn} \frac{a_s - iI}{\mathcal{A} - \lambda I} f_h - \left([\lambda I - A][P_{sm}[I + b^2] P_{sm}]^{-1} - \frac{P_{sm}[I + b_s^2] b P_{sm}}{P_{sm}[I + b_s^2] P_{sm}} + \tilde{O}(\frac{1}{\lambda})\right)$$

$$P_{sm} \frac{b_s - iI}{\mathcal{B}_s - \lambda I} f_{Es} \equiv \mathcal{F}_s, \tag{19}$$

and for the last one we have

$$P_{2n} \frac{a_2 - iI}{\mathcal{A} - \lambda I}(a_1 + iI) e_{h1} + \Bigg[\int_0^\infty \frac{\alpha(A_2 - \lambda I)}{A_2 + \alpha - \lambda I} P_{2n} d\mathcal{E}^2_\alpha P_{2n} - \gamma^2_{hh} - \gamma^2_{hE}$$

$$\Bigg([\lambda I - A_2][P_{2m}[I + b_2^2] P_{2m}]^{-1} - \frac{P_{2m}[I + b_2^2] b_2 P_{2m}}{P_{2m}[I + b_2^2] P_{2m}}$$

$$+\tilde{O}(\frac{1}{\lambda})\Bigg) \gamma^2_{Eh}\Bigg]\, e_{2h} = P_{2n} \frac{a_2 - iI}{\mathcal{A} - \lambda I} f_h -$$

$$\gamma_h E\left([\lambda I - A_2](P_{2m}[I + b_2^2] P_{2m})^{-1} - \frac{P_{2m}[I + b_2^2] b_2 P_{2m}}{P_{2m}[I + b_2^2] P_{2m}} + \tilde{O}(\frac{1}{\lambda})\right)$$

$$P_{2m}\frac{b_2 - iI}{\mathcal{B}_2 - \lambda I} f_{E2} \equiv \mathcal{F}_2, \tag{20}$$

We have used here the notation $\mathcal{E}^s_\alpha$ for the family of spectral projections of the operator a_s.

The only essential complication of each of the above systems is the presence of *crossing terms* in them, for instance :

$$P_{1n}\frac{a_1 - iI}{\mathcal{A} - \lambda I}(a_2 - \lambda I)e_{h2},\ P_{2n}\frac{a_2 - iI}{\mathcal{A} - \lambda I}(a_1 - \lambda I)e_{h1}.$$

If the whole construction is reduced to the domain of densities with compact support $\rho^\delta_s,\ s = 1, 2,\ \delta > 0$ then these crossing terms are not essential, since the angles between the deficiency subspaces $\mathcal{N}_1$, $\mathcal{N}_2$ is positive and the crossing terms are bounded. Then the constructed extensions are semibounded from below.

We consider now the class of densities $\hat{\rho}_s$ with compact support, but will not specify $\delta,\ \delta > 0$. We assume the smoothness of densities with respect to the corresponding channel operators A_s.This class of densities is a dense linear set which will be used to derive the apriory estimates and develop the Friedrich's procedure. For this class of densities the bilinear forms of crossing terms obviously exist. Our aim is to find the conditions which guarantees the uniform (in δ) domination the forms of crossing terms by the quadratic forms of the diagonal terms of the matrix $\Gamma - Q$.

Retaining only leading terms and crossing terms in equations corresponding to the cases considered we get the following results:

$$1)\ \left[\int_0^\infty \frac{\alpha(A_s - \lambda I)}{A_s + \alpha - \lambda I} P_{sn} d\mathcal{E}^s_\alpha P_{sn} + \tilde{O}(1)\right] e_s + P_{ns}\frac{a_s - iI}{\mathcal{A} - \lambda I}(a_t + iI)e_t = \mathcal{F}_s,\ s,t = 1, 2. \tag{21}$$

$$\begin{aligned} 2)\ \int_0^\infty \frac{\alpha\,(A_s - \lambda I)}{A_s + \alpha - \lambda I} P_{sn} d\mathcal{E}^s_\alpha P_{sn} e_{sh} + \gamma^s_{hE}\frac{(A_s - \lambda I)}{P_{sm}[I + b_s^2]P_{sm}}\gamma^s_{Eh} e_{sh} + \\ \tilde{O}(1)e_{sh} + P_{sn}\frac{a_s - iI}{\mathcal{A} - \lambda I}(a_t + iI)e_{th} = \mathcal{F}_s,\ \ s,t = 1, 2. \end{aligned} \tag{22}$$

$$3)\ \int_0^\infty \frac{\alpha(A_1 - \lambda I)}{A_1 + \alpha - \lambda I} P_{1n} d\mathcal{E}^1_\alpha P_{1n} e_{1h} + \tilde{O}(1)e_{1h} + P_{n1}\frac{a_1 - iI}{\mathcal{A} - \lambda I}(a_2 + iI)e_{2h} = \mathcal{F}_1.$$

$$\begin{aligned} P_{2n}\frac{a_2 - iI}{\mathcal{A} - \lambda I}(a_1 + iI)e_{1h} + \int_0^\infty \frac{\alpha(A_2 - \lambda I)}{A_2 + \alpha - \lambda I} P_{2n} d\mathcal{E}^2_\alpha P_{2n} e_{2h} - \\ -\gamma^2_{hE}\frac{\lambda I - A_2}{P_{2m}[I + b_2^2]P_{2m}}\gamma^2_{Eh} e_{2h} + \tilde{O}(1)e_{2h} = \mathcal{F}_2, \end{aligned} \tag{23}$$

Here only the leading diagonal terms are presented in explicite form. The dominated diagonal terms are denoted by $\tilde{O}(1)$.

Theorem 3.1 *Provided some of equations (21) , (22), (23) have unique solutions in proper rigged spaces for large negative* λ*, the corresponding operators defined by the corresponding boundary conditions are semibounded and we can accomplish the extension procedure for these operators by Friedrichs method. Then the corresponding resulting few - body Krein's formulae for resolvents of these extensions may be obtained from the solution of the system of adjoint equations (15) submitted to the boundary conditions (14).*

We continue the didcussion of solvability of the equations (21) , (22), (23) in the following section.

It is clearly seen now that the relevant densities in rigged-space setting of the problem play the role of Faddeev components, see [13] and each of systems (21) , (22), (23) is a system of Faddeev equations for the resolvent of the corresponding few-body Hamiltonian generated by switching on the interaction via the boundary conditions between two groups of *quantum particles* arranged in two different clusters and the channel of free particles. This means that during the collision both clusters are virtually desroyed and then recreated. Of course we may consider more subtle setting of the problem when only partial destruction takes place. This requires more delicate treating of intersections of deficiency subspaces.

The discussed problem for three channels (two scattering channels and a free channel) may be considered, to some extent, as an elementary brick of the real problem involving several channels. The relevant densities also play the role of Faddeev components, see [13].

4 Semiboundedness of the few-body Schrödinger Operator with generalized point interaction.

In this section we begin with discussion of the following *basic example*. Consider M distinguishable particles in Euclidean Space R_3 with Hamiltonian represented by Laplace operator with respect to cartesian coordinates in $R^{(3M-3)}$ associated with the center of masses, and the corresponding Hilbert space $\mathcal{H} = L_2\left(R^{(3M-3)}\right)$. Selecting a subspace $l \in R^{(3M-3)}$, $\dim l = 3M-3-3k$ we assign the Hilbert space $h = L_2(l)$ to it. This space may play the role of inner Hilbert space of some cluster and a complementary space $H = L_2(L)$, supported by the complementary plane L , $\dim L = 3k$, (such that $l \times L = R_{(3M-3)}$) will play the role of the Hilbert space of the complementary group of free particles with respect to the common center of masses [5]. The direct product of these spaces gives the total space: $\mathcal{H} = h \times H$. The space L plays the role of the interaction plane, so that the elements from the deficiency subspace $\mathcal{N}$ of the total Hamiltonian $\mathcal{A}_0 = -\triangle_0 = [-\triangle_l]_0 - \triangle_L \equiv a_0 + A$ reduced by the condition of vanishing near L

$$a = -\triangle_\perp \longrightarrow -\triangle_{\perp 0} \equiv a_0,\ \mathcal{A} \longrightarrow a_0 + A$$

are represented via densities supported by this plane:

$$e_{\bar\lambda} = \frac{I}{-\triangle - \lambda I}\rho$$

[5]The latter condition means that further splitting of these particles is not observed during the process considered below . But we do not exclude the possibility, that some of them can be splitted in other processes.

In terms of Fourier transform $\mathcal{F}$ the condition supp$\rho \in L$ means that $\mathcal{F}\rho \equiv \tilde{\rho}$ depends only on the longitudinal component of the momentum in the orthogonal decomposition $p = p_\perp + p_{||}$ with respect to L , $p_\perp \perp L$ or $p_\perp \in l$, dim$L = 3(M-k)$, dim$l = 3(k-1)$:

$$\tilde{\rho}(p) \equiv \tilde{\rho}(p_{||}),$$

hence

$$\tilde{e_\lambda} = \frac{\tilde{\rho}(p_{||})}{p_\perp^2 + p_{||}^2 - \lambda}.$$

Generally the splitting of Laplace operator with respect to the decomposition $\mathcal{H} = L_2(l) \times L_2(L)$ is given in the Fourier representation by the formula

$$\mathcal{A} = -\triangle \xrightarrow{\mathcal{F}} p_\perp^2 + p_{||}^2 \equiv a + A.$$

In the case of finite deficiency index of a_0 the dot products in representation of the boundary forms with respect to the rigged variables are reduced to the integration over the plane L, for instance

$$< u,\, \rho_L^v > - < \rho_L^u,\, v > = \int_L [< u,\, \rho_L^v >_h - < \rho_L^u,\, v >_h]\, dL.$$

We concider two splittings of the configuration space $R^(3M-3) = l^s \times L^s,\ s = \pm$ and two corresponding pairs of complementary planes $l^s, L^s,\ s = \pm$. We assign the proper index to all relevant quantities, for instance we consider two pairs of mutually complementary subspaces of the Hilbert space : $h_s, H_s; \mathcal{H} = h_s \times H_s$. We associate these pairs with two clusters, $h_s = L_2(l^2), H_s = L_2(L^s),$. Both L^1, L^2 play roles of interaction planes in the extension described procedure below. In what follows, if not mentioned specially, we consider the restriction of the Laplacian by the conditions of vanishing near *both* planes $L^1 \cup L^2$

$$\mathcal{A} \longrightarrow \mathcal{A}_0.$$

The domain of the restricted operator is defined as intersection of domains $\mathcal{A}_s = a_{0s} + A_s,\ s = 1, 2$. The total deficiency subspace is a sum of deficiency subspaces $\mathcal{N}_1$, $\mathcal{N}_2$

$$\overline{\mathcal{N}_1 + \mathcal{N}_2}$$

of the splitted operators $\mathcal{A}_1$, $\mathcal{A}_2$. It is obvious that the restricted operator may have infinite deficiency.

The following obvious statement permitts to consider in our model only the problem of recombination of several two - body clusters.

Lemma 4.1 *Consider the symmetric operator defined as a Laplacian $-\triangle$ on all compactely supported smooth functions vanishing near the plane L_k, dim$L_k = 3(M-k)$. If $k > 2$ then this operator is essentially selfadjoint.*

Proof. Really, the deficiency elements of this operator are Fourier-represented as

$$\frac{\tilde{\rho}(p_{||})}{p_\perp^2 + p_{||}^2 - \lambda}$$

with proper generalised densities $\tilde{\rho}(p_{||})$. These elements are square integrable if

$$\int \frac{|\tilde{\rho}(p_{||})|^2}{|p_\perp^2 + p_{||}^2 - \lambda|^2} dp_{||} dp_\perp < \infty.$$

The last condition is fulfilled for proper densities if

$$\dim p_\perp = 3(M-1) - 3(M-k) = 3(k-1) < 4$$

since it implies the convergence of the integral

$$\int \frac{1}{|p_\perp^2 + p_{||}^2 - \lambda|^2} dp_\perp \approx |p_{||}^2 - \lambda|^{\frac{3k-3}{2}-2}.$$

Hence only $k = 1, 2$ are possible and the densities for the deficiency elements corresponding to two- body clusters should satisfy the condition

$$\int \frac{|\tilde{\rho}(p_{||})|^2}{\sqrt{p_{||}^2 + 1}} dp_{||}.$$

□

Note that the last statement may be used generally to estimate the dimension of intersections of the deficiensy subspaces generated by densities supported by some interaction planes. In particular one can show that the dimension of the intersection of the deficiency subspaces $\dim \mathcal{N}_1 \cap \mathcal{N}_2$ is finite in this case.

Remaining in the Hilbert space of square-integrable functions we can construct solvable models with point interactions only for processes of scattering and recombination of several two - body clusters for which the densities ρ satisfy the condition

$$\int_L \frac{|\tilde{\rho}^2(p_{||})}{\sqrt{p_{||}^2 + 1}} dp_{||}^{3M-6} < \infty.$$

For larger clusters, $k > 2$, the interaction planes are too slim so that the restriction of the Laplacian onto smooth functions vanishing near them is essentially selfadjoint.

Consider the linear space of smooth densities ρ_s with compact support [6] on interaction planes L^s. Since the complementary planes l^s are three-dimensional, the deficiency subspaces n_s of operators $a_{s0} = -\triangle$ in $L_2(l^s)$ and the corresponding projections P_{sn} onto them are one-dimensional. The relevant projections onto $n_s \times H_s = n_s \times L_2(L^s)$ are represented as $P_{sn} \times I_{H_s}$. We use for them the former notations $P_{sn} \times I_{H_s} \equiv P_{sn}$.

We investigate now the solvability of analogs of equations (21), (22), (23) in our model.

Having in mind that the operator-extension construction can be developed for two-body clusters only we see that all interaction planes are $3M-6$-dimensional. Let us consider the intersection L^{st} of interaction planes $L^s,\ L^t$ in the momentum space. We denote by p_{st} the component of the Fourier variable (momentum) p along L^{st} and by $q_s,\ q_t$ the complementary

[6]Vanishing near the equidomoid and at infinity

components of the momentum in L^s, L^t respectively which are orthogonal to L^{st} Since the angle between q_s, q_t is positive, we have generally the direct decomposition of the total momentum space R_{3M-3}, $\dim L_{st} = 3M-6$, $\dim\mathcal{L}_s = \dim\mathcal{L}_t = 3$:

$$p = p_{st} + q_s + q_t; R^{3M-3} \equiv L_{st} \oplus [\mathcal{L}_s + \mathcal{L}_t], \; q_s \in \mathcal{L}_s, \; s \neq t, \; s,t = 1,2.$$

Note, that $p^s_{||} = p_{st} + q_s$, $p^t_{||} = p_{st} + q_t$. In the next statement we use the fact that the spectral representation of operators A_s is given by Fourier transform.

Lemma 4.2 *The following estimates for the bilinear forms of crossing terms via diagonal leading terms are valid for large negative λ :*

$$|\int \frac{\rho^s(p^s_{||})\overline{\rho^t(p^t_{||})}}{p^2-\lambda} dp| \leq$$

$$\frac{C_1}{\sqrt{-\lambda}} \left[\int (|p^s_{||}|^2 - \lambda)||\rho^s(p^s_{||})|^2 dp^s_{||} + \int (|p^t_{||}|^2 - \lambda)|\rho^t(p^t_{||})|^2 dp^t_{||}\right] \tag{24}$$

$$|\int \frac{\rho^s(p^s_{||})\overline{\rho^t(p^t_{||})}}{p^2-\lambda} dp| \leq$$

$$\frac{C_2}{|-\lambda|^{1/4}} \left[\int (|p^s_{||}|^2 - \lambda)|\rho^s(p^s_{||})|^2 dp^s_{||} + \int (|p^t_{||}|^2 - \lambda)|^{1/2}|\rho^t(p^t_{||})|^2 dp^t_{||}\right], \tag{25}$$

with certain positive constants C_1, C_2

Proof. Introducing new functions f_s, f_t as follows $f_s(q_s + p_{st}) \equiv \sqrt{p^2_{st} + q^2_s - \lambda} \;\; \rho^s(p^s_{||})$ and denoting by J Jacobians corresponding to the nonorthogonal pais of coorditates q_s, q_t we get the first estimate :

$$\begin{aligned}
&\int \frac{|\rho^s(p^s_{||})\overline{\rho^t(p^t_{||})}|}{|p^2-\lambda|} dp \\
\leq\; & \frac{J}{2}\int \frac{|f_s(q_s+p_{st})|^2 dp_{st}dq_s dq_t}{|p^2_{st}+q^2_t-\lambda| \; |p^2_{st}+(q_s+q_t)^2-\lambda|} \\
&+\frac{J}{2}\int \frac{|f_t(q_t+p_{st})|^2 dp_{st}dq_s dq_t}{|p^2_{st}+q^2_s-\lambda| \; |p^2_{st}+(q_s+q_t)^2-\lambda|} \\
\leq\; & \frac{J}{2}\int |f_s(q_s+p_{st}|^2) dp_{st}dq_s \; \sup_{p_{st},q_s} \int \frac{dq_t}{|p^2_{st}+q^2_t-\lambda| \; |p^2_{st}+(q_s+q_t)^2-\lambda|} \\
&+\frac{J}{2}\int |f_t(q_t+p_{st})|^2 dp_{st}dq_t \; \sup_{p_{st},q_t} \int \frac{dq_s}{|p^2_{st}+q^2_s-\lambda| \; |p^2_{st}+(q_s+q_t)^2-\lambda|} \\
\leq\; & \frac{C_1}{\sqrt{|\lambda|}} \left[\int |f_s(q_s+p_{st})|^2 dp_{st}dq_s + \int |f_t(q_t+p_{st})|^2 dp_{st}dq_t\right].
\end{aligned}$$

Returning to the original functions ρ we get the first estimate.

To prove the second estimate we need proper distribution of powers in the brackets of the denominator. Now we are using the following notations

$$f_s(q_s+p_{st}) \equiv \sqrt{q^2_s+p^2_{st}-\lambda}\; \rho^s, \;\; f_t(q_t+p_{st}) \equiv (q^2_t+p^2_{st}-\lambda)^{1/4}\; \rho^t.$$

Then we may estimate the second integral as

$$\int \frac{|\rho^s(p_{||}^s)\overline{\rho^t(p_{||}^t)}|}{p^2-\lambda} dp$$

$$= \int \frac{|f(q_s+p_{st})\overline{f(q_2+p_{st})}|dq_s dq_t dp_{st}}{|p_{st}^2+q_s^2-\lambda|^{\frac{1}{2}}\ |p_{st}^2+(q_s+q_t)^2-\lambda||p_{st}^2+q_t^2-\lambda|^{frac14}}$$

$$\leq \frac{J}{2}\int \frac{|f(q_s+p_{st})|^2 dq_s dq_t dp_{st}}{|p_{st}^2+q_s^2-\lambda|^{\frac{1}{2}}\ |p_{st}^2+(q_s+q_t)^2-\lambda|^{5/4}}$$

$$+\frac{J}{2}\int \frac{|f(q_t+p_{st})|^2 dq_s dq_t dp_{st}}{|p_{st}^2+q_s^2-\lambda||p_{st}^2+(q_s+q_t)^2-\lambda|^{3/4}}$$

$$\leq \frac{C_2}{|\lambda|^{1/4}}\left[\int |f_s(q_s+p_{st})|^2 dq_s dp_{st} + \int |f_t(q_t+p_{st})|^2 dq_t dp_{st}\right]$$

□

Remark. The estimates (24), (25) permit to investigate the solubility of equations (22), (23) for large negative λ, but still not the equation (31). Really, the attempt to estimate the crossing terms for the equation (21) we get the following integral for "renormalized densities" $f_s(q_s+p_{st}) \equiv (q_s^2+p_{st}^2-\lambda)^{1/4}\rho^s(q_s+p_{st})$, $f_t(q_t+p_{st}) \equiv (q_t^2+p_{st}^2-\lambda)^{1/4}\rho^t(q_t+p_{st})$:

$$\int \frac{|f_s(q_s+p_{st})f_t(q_t+p_{st})|dq_s\, dq_t\, dP_{st}}{|q_t^2+p_{st}^2-\lambda|^{1/4}|p_{st}+(q_s+q_t)^2-\lambda||q_s^2+p_{st}^2-\lambda|^{1/4}}. \tag{26}$$

The straightforward estimate as above gives just

$$\frac{J}{2}\int |q_s^2+p_{st}^2-\lambda|^{\epsilon}|f_s(q_s+p_{st})|^2 dq_s\ dp_{st}$$

$$\sup_{q_s,p_{st}}\int \frac{dq_t}{|p_{st}+(q_s+q_t)^2-\lambda|\ |q_t^2+p_{st}^2-\lambda|^{1/2+\epsilon}}+$$

$$\frac{J}{2}\int |q_t^2+p_{st}^2-\lambda|^{\epsilon}|f_t(q_t+p_{st})|^2 dq_t\ dp_{st}$$

$$\sup_{q_t,p_{st}}\int \frac{dq_s}{|p_{st}+(q_s+q_t)^2-\lambda|\ |q_s^2+p_{st}^{-}\lambda|^{1/2+\epsilon}} \leq$$

$$\frac{C_3}{|\lambda|^{\epsilon}}\left[\int |(q_t^2+p_{st}^2-\lambda|^{\epsilon}|f_t(q_t+p_{st})|^2 dq_t\ dp_{st}+\right.$$

$$\left.\int |q_s^2+p_{st}^2-\lambda|^{\epsilon}|f_s(q_s+p_{st})|^2 dq_s\ dp_{st}\right]$$

for each positive ϵ. This is not sufficient for our aims.

Summarizing results of the last Lemma in general case of few two-body clusters and previous analysis of equations (21), (22), (23) which also can be done in general case we may derive the following statement for M- body scattering problem with generalized point interaction switched on between pairs (in two-body clusters):

Theorem 4.1 *The Hamiltonian of M distinguishable quantum particles with point interaction is semibounded if at least for $\frac{M(M-1)}{2}-1$ interaction planes of 2-body clusters the condition $\gamma_s+P_m^s b_s P_m^s=0$ is satisfied.*

This statement is a natural generalization of the result [27,28] for M particles.

1. Akhiezer, N. I. and Glazman, I. M., *Theory of linear operators in Hilbert space. Vol. II*, Pitman, Boston, 1981.

2. Albeverio, S., Hunziker, W., Schneider, W. and Schrader, R., *A note on L. D. Faddeev's three-particle theory*, Helv. Phys. Acta, **40** (1967) 745–748.

3. Albeverio, S., Høegh-Krohn, R. and Streit, L., *Energy forms, Hamiltonians, and distorted Brownian paths*, J. Math. Phys., **18** (1977) 907–917.

4. Albeverio, S., Høegh-Krohn, R. and Wu, T. T., *A class of exactly solvable three-body quantum mechanical problems and the universal low energy behavior*, Phys. Lett. A, **83** (1981) 105–109.

5. Albeverio, S., Gesztesy, F., Høegh-Krohn, R. and Holden, H., *Solvable models in quantum mechanics*, Springer-Verlag, New York, 1988.

6. Albeverio, S. and Kurasov, P., *Singular perturbations of differential operators*, Cambridge Univ. Press, to appear.

7. Altshuler, B.L., Lee, P.A., and Webb, R.A., edds., *Mesoscopic Phenomena in Solids*, North Holland, New-York, 1991.

8. Amrein, W., Jauch, J., Sinha, K. *Scattering theory in quantum mechanics*, Benjamin, Reading, Mass. (1977).

9. Atkinson, F.V., Langer, H., Mennicken, R. and Shkalikov, A., *The essential spectrum of some matrix operators*, Math. Nachr., **167** (1994) 5-20.

10. Berezin, F. A. and Faddeev, L. D., *Remark on the Schrödinger equation with singular potential*, Dokl. Akad. Nauk SSSR, **137** (1961) 1011–1014.

11. Demkov, Yu. N. and Ostrovsky, V.N., *Zero-range potentials and their applications in atomic physics*, Plenum Press, New York and London, 1988.

12. Derkach, V. A. and Malamud, M. M., *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal., **95**, (1991) 1–95.

13. Faddeev, L. D. and Merkuriev, S. P., *Quantum scattering theory for several particle systems*, Kluwer, Dordrecht, 1993.

14. Fermi, E., *Sul moto dei neutroni nelle sostanze idrogenate*, Ricerce Scientifica, 7:13-52, 1936 (in Italian), English translation in E.Fermi, Collected papers, vol. I, Italy 1921-1938, Univ. of Chicago Press, Chicago-London, 1962, pp. 980-1016.

15. Gorbachuk, V.I. and Gorbachuk, M.L., *Boundary value problems for operator differential equations*, Kluwer, Dordrecht, 1991.

16. S.Hassi, H. de Snoo, *On rank one perturbations of selfadjoint operators*, Integr. Eq. Oper. Theory, **29** (1997), 288-300.

17. S.Hassi, H.Langer, H. de Snoo, *Selfadjoint extensions for a class of symmetric operators with defect numbers* $(1,1)$. In Topics in operator theory, operator algebras and applications (Timisoara, 1994), pp. 115-145. Rom. Acad., Bucharest, 1995.

18. G.R.Kirchoff.*Gesammelte Abhandlungen* Publ. Leipzig: Barth, 1882, 641p.

19. Krein, M., *Concerning the resolvents of an Hermitian operator with the deficiency-index* (m,m), Comptes Rendue (Doklady) Acad. Sci. URSS (N.S.), **52** (1946) 651–654.

20 . M. Naimark.*Self - adjoint extensions of the second kind of a symmetric operatots* Bull.Acad.Sci.URSS.,Ser.Math.4,1940, 53-104.

21. V. Kostrykin and R. Schrader *Kirchhoff's rule for quantum wires* J. Phys. A: Math. Gen. 32 (1999) 595-630;

22. Kuperin, Yu. A. and Merkurev, S. P., *Selfadjoint extensions and scattering theory for several-body systems*, in: Spectral theory of operators (Novgorod, 1989), Amer. Math. Soc. Transl. Ser. 2, **150**, Amer. Math. Soc., Providence, RI, 1992, 141–176.

23. Kurasov, P.B. and Pavlov, B.S., *Scattering problem with physical behavior of scattering matrix and operator relations*, to be published in Vinogradov's memorial volume.

24. Kurasov, P.B. and Pavlov, B.S., *Generalized perturbations and operator relations*, to be published in Proc. Conf. "Hyperfunctions, Operator and Dynamical Systems", Brussel, 1997.

25. Kurasov, P.B., *Energy dependent boundary conditions and few-body scattering problem*, Rev. Math. Phys. **9** (1997) 853-906.

26. Makarov, K. A., *Semiboundedness of the energy operator of a system of three particles with paired interactions of δ-function type*, Algebra i Analiz, **4** (1992) 155–171.

27. Makarov, K. A. and Melezhik, V. V. and Motovilov, A. K., *Point interactions in the problem of three quantum particles with internal structure*, Teoret. Mat. Fiz., **102** (1995) 258–282.

28. Makarov, K. A. and Melezhik, V. V., *Two sides of a coin: the Efimov effect and collapse in a three-body system with point interactions. I*, Teoret. Mat. Fiz., **107** (1996) 415–432.

29. R.Mennicken,A.Shkalikov. *Spectral Decomposition of Symmetric Operator Matrices.* Mathematische Nachrichten. **179**(1996),259-273.

30. Merkuriev, S. P. and Yakovlev, S. L., *On the quantum scattering problem for four identical particles interacting in the S state*, Soviet J. Nuclear Phys. **39** (1984) 1002–1006.

31. S.P. Novikov, *Schr"odinger operators on graphs and symplectic geometry*, in: The Arnol'dfest (Proceedings of the Fields Institute Conference in Honour of the 60th Birthday of Vladimir I. Arnol'd), eds. E. Bierstone, B. Khesin, A. Khovanskii, and J. Marsden, to appear in the Fields Institute Communications Series.

32. B.Pavlov *A model of zero-radius potential with internal structure* Teor.Mat.Fiz. v59,3,1984,345-354 (English translation: Theoret. and Math. Phys **59** (1984), no 3, 544-550).

33. Pavlov, B. S., *Operator Extension Theory and explicitly solvable models*, Uspekhi Mat. Nauk, **42** (1987) 99–131, 247.

34. Pavlov, B. S., *Boundary conditions on thin manifolds and the semiboundedness of the three-body Schrödinger operator with point potential*, Mat. Sb. (N.S.), **136(178)** (1988) 163–177, 301.

35. Popov, I. Yu., *Theory of extensions and the localization of resonances for domains of trap type*, Mat. Sb., **181** (1990) 1366–1390.

36. Popov, I. Yu., *The Helmholtz resonator and operator extension theory in a space with indefinite metric*, Mat. Sb., **183** (1992) 3–27.

37. Shondin, Yu. G., *On the three-particle problem with δ-potentials*, Teoret. Mat. Fiz., **51** (1982) 181–191.

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