On the Interaction between a Group of Unitary Operators and a Projection

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1 Introduction

We consider the following objects: $\mathcal{S}$ is a closed subspace of a Hilbert Space $\mathcal{H}$, $\mathcal{P}$ is the projection operator for $\mathcal{S}$, and $\mathcal{U}(t)$ is a strongly continuous group of unitary operators on $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. We let $U = \mathcal{U}(T)$, where $T > 0$ is fixed. The questions that we ask are

- Under what conditions is $\mathcal{P}U\mathcal{P}$ a contraction?
- Under what conditions can we steer $g \in \mathcal{S}$ to $h \in \mathcal{S}$ in the sense that we can find $f \in \mathcal{H}$ such that $\mathcal{P}f = g$ and $\mathcal{P}Uf = h$?

These considerations grew out of an attempt to obtain boundary controllability results for PDEs from certain smoothing estimates of the PDEs ([5, 6, 9, 8, 7, 10, 1]. The authors soon found that the analysis that they used could be put comfortably in a completely abstract setting in which the results will hopefully be of interest in their own right, even outside the area of control theory. The purpose of this paper is to present the theory in the more abstract setting and extend it.

2 The Main Results.

Lemma 1 *Suppose that $\mathcal{S}$ is a closed subspace of a Hilbert Space $\mathcal{H}$, $\mathcal{P}$ is the projection operator for $\mathcal{S}$, and W is a unitary operator on $\mathcal{H}$ such that $\mathcal{P}W\mathcal{P}$ is compact and $||(\mathcal{P}W\mathcal{P})^n|| = 1$ for some positive integer n. Then the set*

$$V = \{z \in \mathcal{S} : Wz \in \mathcal{S}, W^2 z \in \mathcal{S}, \ldots, W^n z \in \mathcal{S}\}$$

is a non-trivial finite dimensional subspace of $\mathcal{S}$.

Proof. The kernel K of $I - \mathcal{P}W^{-1}\mathcal{P}W\mathcal{P}$ is finite dimensional because $\mathcal{P}W\mathcal{P}$ is compact. Thus, V is a finite dimensional subspace because it is contained in K.

Next, we show that V is non-trivial. We can find a sequence $\{z_m\}$ contained in $\mathcal{S}$ such that $||z_m|| \leq 1$ and $||(\mathcal{P}W\mathcal{P})^n z_m|| \to 1$ as $m \to \infty$. Since $\mathcal{P}W\mathcal{P}$ is compact, we can even arrange to have the sequence $\{\mathcal{P}W\mathcal{P}z_m\}$ convergent. Let r be the limit of this convergent sequence and set $z = W^{-1}r$. It is clear that r is contained in $\mathcal{S}$. We must have $||(\mathcal{P}W\mathcal{P})^j z_m|| \to 1$ as $m \to \infty$ for all $1 \leq j \leq n$ because $||(\mathcal{P}W\mathcal{P})^j z_m|| \geq ||(\mathcal{P}W\mathcal{P})^n z_m||$. Further, we have

$$\begin{aligned}
||z - z_m||^2 &= ||W^{-1}r - z_m||^2 \\
&= ||r - Wz_m||^2 \\
&= ||r - W\mathcal{P}z_m||^2 \\
&= ||\mathcal{P}r - \mathcal{P}W\mathcal{P}z_m||^2 + ||(I-\mathcal{P})r - (I-\mathcal{P})W\mathcal{P}z_m||^2 \\
&= ||r - \mathcal{P}W\mathcal{P}z_m||^2 + ||(I-\mathcal{P})W\mathcal{P}z_m||^2 \\
&= ||r - \mathcal{P}W\mathcal{P}z_m||^2 + ||W\mathcal{P}z_m||^2 - ||\mathcal{P}W\mathcal{P}z_m||^2 \\
&\leq ||r - \mathcal{P}W\mathcal{P}z_m||^2 + 1 - ||\mathcal{P}W\mathcal{P}z_m||^2.
\end{aligned}$$

But since $||r|| = 1$ and $\mathcal{P}W\mathcal{P}z_m \to r$ as $m \to \infty$, we see that $z_m \to z$ as $m \to \infty$. Hence $z \in \mathcal{S}$. Thus, the subspace $\{z \in \mathcal{S} : Wz \in \mathcal{S}\}$ is non-trivial. Finally,

$$||(\mathcal{P}W\mathcal{P})^j z|| = \lim_{m\to\infty} ||(\mathcal{P}W\mathcal{P})^j z_m|| = 1$$

for $1 \leq j \leq n$. Hence $z \in V$.

Theorem 1 *Suppose that $\mathcal{S}$ is a closed subspace of a Hilbert Space $\mathcal{H}$, $\mathcal{P}$ is the projection operator for $\mathcal{S}$, and $\mathcal{U}(t)$ is a strongly continuous group of unitary operators on $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. We let $U = \mathcal{U}(T)$, where $T > 0$ is fixed. Suppose that the following conditions hold:*

1. *U has the smoothing property. i.e. if $f \in \mathcal{S}$ and $Uf \in \mathcal{S}$ then $Uf \in \mathcal{D}(\mathcal{A})$.*
2. *$\mathcal{P}U\mathcal{P}$ is a compact mapping.*
3. *$\mathcal{S}$ is compatible with $\mathcal{A}$ in the sense that if $f \in \mathcal{S} \cap \mathcal{D}(\mathcal{A})$ then $\mathcal{A}f \in \mathcal{S}$.*

Then $\mathcal{P}U\mathcal{P}$ is a contraction if and only if $\mathcal{S}$ contains no eigenvectors of $\mathcal{A}$.

Proof. It is clear that $||\mathcal{P}U\mathcal{P}|| \leq 1$. Suppose that $||\mathcal{P}U\mathcal{P}|| = 1$. Then by Lemma 1 the subspace

$$V = \{z \in \mathcal{S} : Uz \in \mathcal{S}\}$$

is non-trivial and finite dimensional. Further, by the smoothing property and because both U and U^{-1} preserve $\mathcal{D}(\mathcal{A})$, it follows that V is contained in $\mathcal{D}(\mathcal{A})$. Thus, $\mathcal{A}$ is a bounded operator on the finite dimensional space V and must possess at least one eigenvector belonging to V.

Conversely, one easily sees that if $\mathcal{A}$ has an eigenvector in $\mathcal{S}$ then $||\mathcal{P}U\mathcal{P}|| = 1$. This completes the proof.

Theorem 2 *Let W be a unitary operator and suppose that $\mathcal{P}W\mathcal{P}$ is a contraction. Then the bounded linear operators*

$$Q = (W^{-1}\mathcal{P} - \mathcal{P}W^{-1}\mathcal{P})(I - \mathcal{P}W\mathcal{P}W^{-1}\mathcal{P})^{-1}$$

$$R = (\mathcal{P} - W^{-1}\mathcal{P}W\mathcal{P})(I - \mathcal{P}W^{-1}\mathcal{P}W\mathcal{P})^{-1}$$

are such that if $g \in \mathcal{S}$ and $h \in \mathcal{S}$ then $f = Rg + Qh$ satisfies $\mathcal{P}f = g$ and $\mathcal{P}Wf = h$. Moreover, R and Q are optimal in the sense that if $\tilde{f}$ also satisfies $\mathcal{P}\tilde{f} = g$ and $\mathcal{P}W\tilde{f} = h$ then $||\tilde{f}|| \geq f$, with equality holding if and only if $\tilde{f} = f$.

Proof. By expanding $(I - \mathcal{P}W\mathcal{P}W^{-1}\mathcal{P})^{-1}$ and $(I - \mathcal{P}W^{-1}\mathcal{P}W\mathcal{P})^{-1}$ as geometric series, it is easy to verify the identities

$$\begin{array}{rl} \mathcal{P}R\mathcal{P} = \mathcal{P}, & \mathcal{P}WR\mathcal{P} = 0, \\ \mathcal{P}Q\mathcal{P} = 0, & \mathcal{P}WQ\mathcal{P} = \mathcal{P}. \end{array}$$

This proves the stated algebraic properties of R and Q.

Suppose now that $\tilde{f}$ has the properties stated. Then

$$\begin{array}{rcl} (f, \tilde{f} - f) & = & (Rg + Qh, \tilde{f} - f) \\ & = & ((\mathcal{P} - W^{-1}\mathcal{P}W\mathcal{P})(I - \mathcal{P}W^{-1}\mathcal{P}W\mathcal{P})^{-1}g, \tilde{f} - f) \\ & + & ((W^{-1}\mathcal{P} - \mathcal{P}W^{-1}\mathcal{P})(I - \mathcal{P}W\mathcal{P}W^{-1}\mathcal{P})^{-1}h, \tilde{f} - f) \\ & = & (-W^{-1}\mathcal{P}W\mathcal{P}(I - \mathcal{P}W^{-1}\mathcal{P}W\mathcal{P})^{-1}g, \tilde{f} - f) \\ & + & (W^{-1}\mathcal{P}(I - \mathcal{P}W\mathcal{P}W^{-1}\mathcal{P})^{-1}h, \tilde{f} - f) \\ & = & (-W\mathcal{P}(I - \mathcal{P}W^{-1}\mathcal{P}W\mathcal{P})^{-1}g, \mathcal{P}W(\tilde{f} - f)) \\ & + & (W^{-1}\mathcal{P}(I - \mathcal{P}W\mathcal{P}W^{-1}\mathcal{P})^{-1}h, \mathcal{P}W(\tilde{f} - f)) \\ & = & 0. \end{array}$$

Thus

$$||\tilde{f}||^2 = ||\tilde{f} - f||^2 + ||f||^2 + 2\mathrm{Re}(f, \tilde{f} - f) = ||\tilde{f} - f||^2 + ||f||^2,$$

which completes the proof.

Remark. One might think that the assumptions of Theorem 2 could be weakened by replacing the requirement that $\mathcal{P}W\mathcal{P}$ is a contraction with the requirement that $\mathcal{P}W^{-1}\mathcal{P}W\mathcal{P}$ is a contraction. However, nothing is gained by this because $(\mathcal{P}W^{-1}\mathcal{P})^* = \mathcal{P}W\mathcal{P}$ and thus

$$||\mathcal{P}W^{-1}\mathcal{P}||^2 = ||\mathcal{P}W\mathcal{P}||^2 = ||\mathcal{P}W^{-1}\mathcal{P}W\mathcal{P}||$$

Example. This simple example illustrates the notation and theory considered so far. We consider the Cauchy Problem for the Schrödinger Equation:

$$\begin{cases} \frac{\partial u}{\partial t} = i\Delta_x u, & x \in \mathbb{R}^n, \ t > 0, \\ u(x, 0) = \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

If ϕ is nice enough, the solution of this is

$$u(x,t) = (4\pi it)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{(x-y)^2}{4it}} \phi(y)\, dy. \tag{2}$$

We define $\mathcal{H} = L^2(\mathbb{R}^n)$ with the usual inner product (,) given by

$$(f,g) = \int_{\mathbb{R}^n} \bar{f}(x) g(x)\, dx.$$

The solution operator $\phi \to u(\cdot,t)$ preserves the L^2 norm, as can be seen by formally differentiating $||u(\cdot,t)||^2$ with respect to t. In fact, solutions of System (1) are given by a strongly continuous unitary group of operators $\mathcal{U}(t)$ on $\mathcal{H}$. The infinitesimal generator $\mathcal{A}$ of this unitary group has domain the Sobolev Space $H^2(\mathbb{R}^n)$ and is given by $\mathcal{A}u = i\Delta u$.

Suppose that Ω is a closed bounded subset of $\mathbb{R}^n$ and consider the closed subspace $\mathcal{S}$ of $\mathcal{H}$ consisting of those $f \in \mathcal{H}$ such that the support of f is contained in Ω. In this setting, the projection operator $\mathcal{P}$ is simply the operation of multiplication by χ_Ω, the characteristic function of Ω. It is clear that if the initial data are in $\mathcal{S}$ then the solution of the PDE is infinitely differentiable and that the conditions of Theorem 1 are satisfied. Note that if $\mathcal{A}$ had an eigenfunction in $\mathcal{S}$, then the eigenfunction would be infinitely differentiable and identically zero outside Ω. By Holmgren's uniqueness theorem, no such eigenfunction can exist. Hence $\mathcal{P}U\mathcal{P}$ must be a contraction. Overdetermined eigenvalue problems like this are typical of this method when applied to PDEs.

We note the quantum mechanical interpretation of this is that if a particle is in Ω with probably 1 at some time, then it is impossible for it to be in Ω with probably 1 at some later time.

Note also that more general Schrödinger equations may be treated (see [1]).

Theorem 3 *Suppose that $\mathcal{S}$ is a closed subspace of a Hilbert Space $\mathcal{H}$, $\mathcal{P}$ is the projection operator for $\mathcal{S}$, and W is a unitary operator on $\mathcal{H}$ such that $\mathcal{P}W\mathcal{P}$ is a compact. Then there is a positive integer n such that $(PWP)^n$ is a contraction if and only if W has no eigenvectors belonging to $\mathcal{S}$.*

Proof. Suppose that W has no eigenvectors in $\mathcal{S}$ and that $||(PWP)^n|| = 1$ for all positive integers n. By Lemma 1, for each n we can find $z_n \in \mathcal{S}$ such that $||z_n|| = 1$ and $W^j z_n \in \mathcal{S}$ for $1 \le j \le n$. Because of the compactness, we can extract a convergent subsquence with limit z such that $W^j z \in \mathcal{S}$ for $1 \le j < \infty$. Thus the set

$$V_\infty = \{z \in \mathcal{S} : W^j z \in \mathcal{S} \text{ for all } j\}$$

is a non-trivial subspace of $\mathcal{S}$ and W is a bounded operator on the finite dimensional space V_∞. This contradicts the fact that W has no eigenvectors in $\mathcal{S}$. Hence $||(PWP)^n|| < 1$ for some n.

Conversely, it is clear that if W has an eigenvector in $\mathcal{S}$ then $||(\mathcal{P}W\mathcal{P})^n|| = 1$ for all n.

Corollary 1 *Suppose that $\mathcal{S}$ is a closed subspace of a Hilbert Space $\mathcal{H}$, $\mathcal{P}$ is the projection operator for $\mathcal{S}$, and $\mathcal{U}(t)$ is a strongly continuous group of unitary operators on $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. We let $U = \mathcal{U}(T)$, where $T > 0$ is fixed. Suppose also that $\mathcal{P}U\mathcal{P}$ is a compact and that $\mathcal{A}$ has no eigenvalues. Then there is a positive integer n such that $(PUP)^n$ is a contraction.*

Proof. By Theorem 3, it is sufficient to prove that U has no eigenvalues. Suppose that U has an eigenvalue λ. Then $|\lambda| = 1$ because U is unitary. But, by a well-known theorem in semi-group theory, there is an eigenvalue μ of $\mathcal{A}$ such that $\lambda = e^{T\mu}$, which is impossible.

Theorem 4 *Suppose that W is unitary and $(\mathcal{P}W\mathcal{P})^n$ is a contraction for some positive integer n. Then for each $k = 1, 2, \ldots, n$, the bounded linear operators*

$$\begin{aligned} Q_k &= (W^{-1}(\mathcal{P}W^{-1}\mathcal{P})^{n-k} - (\mathcal{P}W\mathcal{P})^{k-1}(\mathcal{P}W^{-1}\mathcal{P})^n)(I - (\mathcal{P}W\mathcal{P})^n(\mathcal{P}W^{-1}\mathcal{P})^n)^{-1}, \\ R_k &= ((\mathcal{P}W\mathcal{P})^{k-1} - W^{-1}(\mathcal{P}W\mathcal{P})^{n-k}(\mathcal{P}W\mathcal{P})^n)(I - (\mathcal{P}W^{-1}\mathcal{P})^n(\mathcal{P}W\mathcal{P})^n)^{-1}, \end{aligned}$$

are such that if $g \in \mathcal{S}$ and $h \in \mathcal{S}$ then

$$f_k = R_k g + Q_k h, \qquad k = 1, 2, \ldots, n$$

satisfy

$$\begin{cases} \mathcal{P}f_1 = g, \\ \mathcal{P}Wf_k = \mathcal{P}f_{k+1}, \qquad k = 1, 2, \ldots, n-1, \\ \mathcal{P}Wf_n = h. \end{cases} \tag{3}$$

Moreover, the operators R_k and Q_k are optimal in the sense that if $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ also satisfy (3) then

$$\sum_{k=1}^{n} ||(I - \mathcal{P})f_k||^2 \leq \sum_{k=1}^{n} ||(I - \mathcal{P})\tilde{f}_k||^2$$

with equality holding if and only if $\tilde{f}_k = f_k$, $k = 1, 2, \ldots, n$.

Proof. The proof is similar to the proof of Theorem 2, so we omit it.

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