Boundary Feedback Stabilization of a Vibrating String with an Interior Point Mass

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Abstract

We study the boundary feedback stabilization for a one-dimensional wave equation with an interior point mass. We show that if the initial data belong to a certain invariant subspace of the semigroup of operators that generates the solution of the system, then the energy will decay like C/time. This improves a result of Hansen and Zuazua [1] who consider decay of solutions belonging to the domain of a power of the infinitesimal generator of the semigroup.

1 Introduction.

In this paper we study the boundary feedback stabilization for a one-dimensional wave equation with an interior point mass. We show that if the initial data belong to a certain invariant subspace of the semigroup of operators that generates the solution of the system, then the energy will decay like C/time. This improves a result of Hansen and Zuazua [1] who consider decay of solutions belonging to the domain of a power of the infinitesimal generator of the semigroup. The system under investigation consists of two strings of length l_1 and l_2 respectively. In their rest states, the strings occupy the intervals $\Omega_1 = (-l_1, 0)$ and $\Omega_2 = (0, l_2)$ of the x-axis respectively. At the origin, each string is tied to a particle of mass M whose displacement away from the x-axis at time t is given by $z(t)$. The transverse displacements of the strings are given by u and v. In this model, the densities ρ_1, ρ_2 and the tensions σ_1, σ_2 are assumed constant.

The equations satisfied by the system are listed below (more details are given in [1].)

$$\begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1,\ t > 0, \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2,\ t > 0, \\ M z_{tt} + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & t > 0, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & x \in \Omega_1, \\ v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), & x \in \Omega_2. \end{cases} \tag{1}$$

We are interested in the problem of velocity feedback at one end. Note that Hansen and Zuazua [1] consider also the case of velocity feedback at both ends, and they show that energy decays uniformly exponentially. We assume that the velocity feedback occurs at the end $x = l_2$, while at the other end $x = -l_1$ we simply have a Dirichlet boundary condition. The boundary conditions are thus

$$\left\{ \begin{array}{ll} u(-l_1, t) = 0, & t > 0, \\ \sigma_2 v_x(l_2, t) + \gamma v_t(l_2, t) = 0, & t > 0, \end{array} \right. \tag{2}$$

where γ is positive.

2 A Representation of the Solution.

We can simplify the exposition by scaling the space variable x separately for $x < 0$ and for $x > 0$ so that the wave speed of each wave equation becomes unity. This is achieved by considering a new variable

$$\tilde{x} = \left\{ \begin{array}{ll} x(\rho_1/\sigma_1)^{1/2}, & x < 0, \\ x(\rho_2/\sigma_2)^{1/2}, & x \geq 0. \end{array} \right.$$

Thus, we may consider without loss of generality the following system (tildes have been removed)

$$\left\{ \begin{array}{ll} u_{tt} = u_{xx}, & x \in \Omega_1,\ t > 0, \\ v_{tt} = v_{xx}, & x \in \Omega_2,\ t > 0, \\ M z_{tt} + \mu_1 u_x(0, t) - \mu_2 v_x(0, t) = 0, & t > 0, \\ u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x), & x \in \Omega_1, \\ v(x, 0) = v^0(x),\ v_t(x, 0) = v^1(x), & x \in \Omega_1, \end{array} \right. \tag{3}$$

where the modified tensions are given by $\mu_1 = (\rho_1 \sigma_1)^{1/2}$, $\mu_2 = (\rho_2 \sigma_2)^{1/2}$. The boundary conditions now become

$$\left\{ \begin{array}{ll} u(-l_1, t) = 0, & t > 0, \\ \mu_2 v_x(l_2, t) + \gamma v_t(l_2, t) = 0, & t > 0, \end{array} \right. \tag{4}$$

and the total mechanical energy simplifies to

$$\begin{aligned} \mathcal{E}(t) &= \frac{M}{2}|z_t(t)|^2 + \frac{\mu_1}{2} \int_{-l_1}^{0} |u_t(x, t)|^2 + |u_x(x, t)|^2 \, dx \\ &+ \frac{\mu_2}{2} \int_{0}^{l_2} |v_t(x, t)|^2 + |v_x(x, t)|^2 \, dx. \end{aligned} \tag{5}$$

We define the finite energy space $\mathcal{H} = \{(U^0 \in H^1(\Omega_1),\ V^0 \in H^1(\Omega_2),\ Z^0 \in R,\ U^1 \in L^2(\Omega_1),\ V^1 \in L^2(\Omega_2),\ Z^1 \in R) : U^0(-l_1) = 0,\ U^0(0) = V^0(0) = Z^0\}$ and equip $\mathcal{H}$ with the norm

$$\begin{aligned} ||(U^0, V^0, Z^0, U^1, V^1, Z^1)|| &= \left(\mu_1 \int_{-l_1}^{0} |U^1(x)|^2 + |U_x^0(x)|^2 \, dx \right. \\ &+ \left. M|Z^1|^2 + \mu_2 \int_{0}^{l_2} |V^1(x)|^2 + |V_x^0(x)|^2 \, dx \right)^{1/2}. \end{aligned}$$

It is easy to see that $\mathcal{H}$ is a Hilbert space, and we define on $\mathcal{H}$ the operator $\mathcal{A}$, with domain $D(\mathcal{A}) = \{(U^0, V^0, Z^0, U^1, V^1, Z^1) \in \mathcal{H} \bigcap (H^2(\Omega_1) \times H^2(\Omega_2) \times R \times H^1(\Omega_1) \times H^1(\Omega_2) \times R) : U^1(-l_1) = 0,\, U^1(0) = V^1(0) = Z^1,\, \mu_2 V^0_x(l_2) + \gamma V^1(l_2) = 0\}$, given by

$$\mathcal{A}(U^0, V^0, Z^0, U^1, V^1, Z^1) = (U^1, V^1, Z^1, U^0_{xx}, V^0_{xx}, (\mu_2 V^0_x(0) - \mu_1 U_x(0))/M).$$

As is mentioned in [1], it is easy to check that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $T(t)$ on $\mathcal{H}$ (the Lumer Phillips Theorem, which is stated in [2], can be used to deduce this). The *finite energy solutions* of (3), (4) are then given by

$$(u(.,t), v(.,t), z(t), u_t(.,t), v_t(.,t), z_t(t)) = T(t)(u^0, v^0, z^0, u^1, v^1, z^1).$$

A convenient way to analyse the decay of solutions of (3), (4) is to use the fact that the solution of each of the wave equations is a sum of two waves, one moving to the left and the other to the right:

$$\left\{ \begin{array}{ll} u(x,t) = F(x+t+l_1) - G(t-x), & x \in \Omega_1,\ t \geq 0, \\ v(x,t) = H(x+t) - E(t-x+l_2), & x \in \Omega_2,\ t \geq 0. \end{array} \right. \tag{6}$$

The initial conditions of (3) imply that the functions F, G, H and E satisfy, modulo some irrelevant arbitrary constants,

$$\left\{ \begin{array}{ll} F(s) = \frac{1}{2}u^0(s-l_1) + \frac{1}{2}\int_0^{s-l_1} u^1(\sigma)\,d\sigma, & 0 < s < l_1, \\ G(s) = -\frac{1}{2}u^0(-s) + \frac{1}{2}\int_0^{-s} u^1(\sigma)\,d\sigma, & 0 < s < l_1, \\ H(s) = \frac{1}{2}v^0(s) + \frac{1}{2}\int_0^{s} v^1(\sigma)\,d\sigma, & 0 < s < l_2, \\ E(s) = -\frac{1}{2}v^0(l_2-s) + \frac{1}{2}\int_0^{l_2-s} v^1(\sigma)\,d\sigma, & 0 < s < l_2. \end{array} \right. \tag{7}$$

The values of these functions for other positive values of their arguments are found by solving a system of differential–delay equations, which are obtained from the conditions at $x = 0$ of (3) and the boundary conditions (4). Specifically, for $s > 0$,

$$\left\{ \begin{array}{lcl} G(s+l_1) & = & F(s), \\ F(s+l_1) + E(s+l_2) & = & G(s) + H(s), \\ H'(s+l_2) & = & qE'(s), \\ M(F(s+l_1) - G(s))_{ss} & = & \mu_2(H'(s) + E'(s+l_2)) \\ & - & \mu_1(F'(s+l_1) + G'(s)), \end{array} \right. \tag{8}$$

where $q = (\gamma - \mu_2)/(\gamma + \mu_2)$.

In terms of these functions, the energy (5) may be written

$$\begin{aligned} \mathcal{E}(t) &= \mu_1 \int_{-l_1}^{0} |F'(t-x)|^2 + |G'(t-x)|^2\,dx \\ &+ \mu_2 \int_0^{l_2} |E'(t+x)|^2 + |H'(t+x)|^2\,dx \\ &+ \frac{1}{2}M|\frac{d}{dt}(F(t+l_1) - G(t))|^2. \end{aligned} \tag{9}$$

We define

$$\begin{cases} f(\lambda) = \int_0^\infty e^{-\lambda t} F'(t)\,dt, & f^1(\lambda) = \int_0^{l_1} e^{-\lambda t} F'(t)\,dt, \\ g(\lambda) = \int_0^\infty e^{-\lambda t} G'(t)\,dt, & g^1(\lambda) = \int_0^{l_1} e^{-\lambda t} G'(t)\,dt, \\ h(\lambda) = \int_0^\infty e^{-\lambda t} H'(t)\,dt, & h^1(\lambda) = \int_0^{l_2} e^{-\lambda t} H'(t)\,dt, \\ e(\lambda) = \int_0^\infty e^{-\lambda t} E'(t)\,dt, & e^1(\lambda) = \int_0^{l_2} e^{-\lambda t} E'(t)\,dt. \end{cases} \tag{10}$$

Note that the Laplace transforms in Equations (10) should be interpreted in the sense of Fourier-Plancherel transforms. This is because for finite energy solutions the functions F', G', H' and E' are all locally $L^2(0,\infty)$ functions whose L^2 norms on any bounded subinterval of $(0,\infty)$ are bounded by constants that depend on the length, but not the location, of the subinterval. Thus, if $\lambda = \sigma + i\xi$ and $\sigma > 0$ then $f(\lambda)$ is just the Fourier transform of $e^{-\sigma t}F'(t)\chi_{[0,\infty]}(t)$. Here and in what follows we use the usual convention that χ_A denotes the characteristic function of a set A.

Formally, Equations (8) imply that

$$\begin{cases} 0 & = & g(\lambda) - g^1(\lambda) - f(\lambda)e^{-l_1\lambda}, \\ 0 & = & e^{l_1\lambda}(f(\lambda) - f^1(\lambda)) - g(\lambda) - h(\lambda) + e^{l_2\lambda}(e(\lambda) - e^1(\lambda)), \\ 0 & = & h(\lambda) - h^1(\lambda) - e^{-l_2\lambda} q e(\lambda), \\ 0 & = & M(\lambda e^{l_1\lambda}(f(\lambda) - f^1(\lambda)) - F'(l_1) + G'(0) - \lambda g(\lambda)) \\ & - & \mu_2(h(\lambda) + e^{l_2\lambda}(e(\lambda) - e^1(\lambda)) \\ & + & \mu_1(e^{l_1\lambda}(f(\lambda) - f^1(\lambda)) + g(\lambda)). \end{cases} \tag{11}$$

The solutions of these equations are easily found. First we define

$$\begin{aligned} S(\lambda) \quad = \quad & (1 - qe^{-2l_2\lambda})^{-1} \left(\left[M\lambda + \mu_1 + \left(\frac{1 + qe^{-2l_2\lambda}}{1 - qe^{-2l_2\lambda}} \right) \mu_2 \right] \right. \\ & \left. - e^{-2l_1\lambda} \left[M\lambda - \mu_1 + \left(\frac{1 + qe^{-2l_2\lambda}}{1 - qe^{-2l_2\lambda}} \right) \mu_2 \right] \right)^{-1}, \end{aligned} \tag{12}$$

and now we may write the solutions of System (11) as follows.

$$\begin{aligned} f(\lambda) \quad = \quad & \{ f^1(\lambda)[M\lambda + \mu_1 + \mu_2 - qe^{-2l_2\lambda}(M\lambda + \mu_1 - \mu_2)] \\ + \quad & g^1(\lambda)e^{-l_1\lambda}[M\lambda - \mu_1 + \mu_2 - qe^{-2l_2\lambda}(M\lambda - \mu_1 - \mu_2)] \\ + \quad & 2\mu_2 h^1(\lambda)e^{-l_1\lambda} + 2q\mu_2 e^1(\lambda)e^{-(l_1+l_2)\lambda} \\ + \quad & M(F'(l_1) - G'(0))e^{-l_1\lambda}(1 - qe^{-2l_2\lambda}) \} S(\lambda), \end{aligned} \tag{13}$$

$$\begin{aligned} g(\lambda) \quad = \quad & \{ g^1(\lambda)[M\lambda + \mu_1 + \mu_2 - qe^{-2l_2\lambda}(M\lambda + \mu_1 - \mu_2)] \\ + \quad & f^1(\lambda)e^{-l_1\lambda}[M\lambda + \mu_1 + \mu_2 - qe^{-2l_2\lambda}(M\lambda + \mu_1 - \mu_2)] \\ + \quad & 2\mu_2 h^1(\lambda)e^{-2l_1\lambda} + 2q\mu_2 e^1(\lambda)e^{-2l_1\lambda} \\ + \quad & M(F'(l_1) - G'(0))e^{-2l_1\lambda}(1 - qe^{-2l_2\lambda}) \} S(\lambda), \end{aligned} \tag{14}$$

$$
\begin{aligned}
h(\lambda) &= \{h^1(\lambda)[M\lambda+\mu_1+\mu_2-e^{-2l_1\lambda}(M\lambda-\mu_1+\mu_2)] \\
&+ e^1(\lambda)qe^{-l_2\lambda}[M\lambda+\mu_1+\mu_2-e^{-2l_1\lambda}(M\lambda-\mu_1+\mu_2)] \\
&+ 2\mu_1 qf^1(\lambda)e^{-(l_1+2l_2)\lambda}+2q\mu_1 g^1(\lambda)e^{-2l_2\lambda} \\
&- qM(F'(l_1)-G'(0))e^{-2l_2\lambda}(1-e^{-2l_1\lambda})\}S(\lambda), \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
e(\lambda) &= \{e^1(\lambda)[M\lambda+\mu_1+\mu_2-e^{-2l_1\lambda}(M\lambda-\mu_1+\mu_2)] \\
&+ h^1(\lambda)e^{-l_2\lambda}[M\lambda+\mu_1-\mu_2-e^{-2l_1\lambda}(M\lambda-\mu_1-\mu_2)] \\
&+ 2\mu_1 qf^1(\lambda)e^{-(l_1+l_2)\lambda}+2\mu_1 g^1(\lambda)e^{-l_2\lambda} \\
&- M(F'(l_1)-G'(0))e^{-l_2\lambda}(1-e^{-2l_1\lambda})\}S(\lambda). \qquad (16)
\end{aligned}
$$

Lemma 1 *$S(\lambda)$ has no poles in the closed right half plane.*

Proof. We let

$$Q(\lambda)=\frac{1+qe^{-2l_2\lambda}}{1-qe^{-2l_2\lambda}}.$$

A simple calculation shows that

$$\operatorname{Re} Q(\sigma+i\xi)=\frac{1-q^2e^{-4l_2\sigma}}{1-2qe^{-2l_2\sigma}\cos\xi+q^2e^{-4l_2\sigma}}.$$

Hence

$$\frac{1-qe^{-2l_2\sigma}}{1+qe^{-2l_2\sigma}}\le \operatorname{Re} Q(\sigma+i\xi)\le\frac{1+qe^{-2l_2\sigma}}{1-qe^{-2l_2\sigma}}.$$

But $|q|<1$, so $\operatorname{Re} Q(\sigma+i\xi)>0$ in the set $\sigma>-\frac{\log q}{2l_2}$, which contains the right half plane. Hence,

$$\operatorname{Re}(M\lambda+\mu_1+Q(\lambda)\mu_2)>\operatorname{Re}(M\lambda-\mu_1+Q(\lambda)\mu_2)$$

if $\operatorname{Re}\lambda\ge 0$. It is clear that

$$\operatorname{Im}(M\lambda+\mu_1+Q(\lambda)\mu_2)=\operatorname{Im}(M\lambda-\mu_1+Q(\lambda)\mu_2),$$

so

$$|M\lambda+\mu_1+Q(\lambda)\mu_2|>|M\lambda-\mu_1+Q(\lambda)\mu_2| \qquad (17)$$

if $\operatorname{Re}\lambda\ge 0$, and the assertion of the lemma follows immediately.

Lemma 2 *The following formulae hold.*

$$
\begin{aligned}
F'(t) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(\lambda)e^{\lambda t}\,d\lambda, \\
F(t) &= F(0)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{f(\lambda)}{\lambda}e^{\lambda t}\,d\lambda,
\end{aligned}
$$

where $\sigma>0$ and f is given by Equation (13). Similar formulae hold for G, H and E.

Sketch of Proof. Convergence of the integrals is in the sense of Fourier-Plancherel transforms. Indeed, we have

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(\lambda)e^{\lambda t}\,d\lambda = \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty} f(\sigma+i\xi)\,d\xi$$

Since $f^1(\sigma+i\xi)$ is just the Fourier transform of $e^{-\sigma t}F'(t)\chi_{[0,l_2]}(t)$, it follows from Plancherel's equality that $f^1(\sigma+i\xi)$ is an L^2 function of ξ. Similar statements hold for g^1, h^1 and e^1. By Lemma 1, the coefficients of $f^1(\sigma+i\xi)$, $g^1(\sigma+i\xi)$, etc. are bounded functions of ξ. Hence the terms involving $f^1(\sigma+i\xi)$, $g^1(\sigma+i\xi)$, etc., in f are all members of L^2 as function of ξ. The other terms in Equation (1) may be treated similarly.

The next step in the proof is to verify that Equations (8) are satisfied. We do not do that here because the details are simple and somewhat lengthy. This completes our sketch of the proof.

3 Energy Decay Estimates.

In this section we analyse the decay of energy of the string-mass system. We find that the energy of the string to the right of the particle (i.e. the part of the string corresponding to the interval $(0,l_2)$ of the x-axis) decays uniformly when the initial data have finite energy. A similar decay rate holds for the remainder of the energy, but this requires an extra derivative in the initial data for the part of the string to the left of the particle.

In order to prove our results, it is useful to consider for $m=1,2,\dots$, the following approximations $S_m(\lambda)$ to $S(\lambda)$.

$$S_m(\lambda) = (1-qe^{-2l_2\lambda})^{-1}\sum_{k=0}^{m}\frac{e^{-2kl_1\lambda}\left[M\lambda-\mu_1+\left(\frac{1+qe^{-2l_2\lambda}}{1-qe^{-2l_2\lambda}}\right)\mu_2\right]^k}{\left[M\lambda+\mu_1+\left(\frac{1+qe^{-2l_2\lambda}}{1-qe^{-2l_2\lambda}}\right)\mu_2\right]^{k+1}}. \tag{18}$$

Lemma 3 $\lim_{m\to\infty} S_m(\lambda) = S(\lambda)$, *with uniform convergence on sets of the form* $\{\lambda : \operatorname{Re}\lambda \geq \sigma\}$, *where* $\sigma > 0$.

Proof. The stated convergence properties of the geometric series are an immediate consequence of Inequality (17).

We now consider each part of the energy. Specifically, we write $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$, where

$$\left\{\begin{array}{lcl}
\mathcal{E}_1(t) & = & \frac{\mu_1}{2}\int_{-l_1}^{0} |u_t(x,t)|^2 + |u_x(x,t)|^2\,dx \\
 & = & \mu_1\int_{-l_1}^{0} |F'(t-x)|^2 + |G'(t-x)|^2\,dx, \\
\mathcal{E}_2(t) & = & \frac{M}{2}|z_t(t)|^2 \\
 & = & \frac{1}{2}M|\frac{d}{dt}(F(t+l_1)-G(t))|^2, \\
\mathcal{E}_3(t) & = & \frac{\mu_2}{2}\int_0^{l_2} |v_t(x,t)|^2 + |v_x(x,t)|^2\,dx \\
 & = & \mu_2\int_0^{l_2} |E'(t+x)|^2 + |H'(t+x)|^2\,dx.
\end{array}\right. \tag{19}$$

Theorem 1 *There exists a constant C such that all finite energy solutions of the string-mass system satisfy*

$$\mathcal{E}_3(t) \leq C\mathcal{E}(0)/t, \qquad t > 0.$$

Proof of Theorem 1 (Part 1). We start with the expression

$$H'(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} h(\lambda)e^{\lambda t}\, d\lambda$$

and rewrite it as a sum of "good" and "bad" parts, $H'(t) = H_g^1(t) + H_b^1(t)$, where

$$H_g^1(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} h_g(\lambda)e^{\lambda t}\, d\lambda, \qquad H_b^1(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} h_b(\lambda)e^{\lambda t}\, d\lambda,$$

$$h_g(\lambda) = \frac{h^1(\lambda) + qe^{-2l_2\lambda}e^1(\lambda) - qe^{-2l_2\lambda}M(F'(l_1) - G'(0))/(M\lambda + \mu_1)}{1 - qe^{-2l_2\lambda}}, \tag{20}$$

$$\left\{\begin{array}{lcl} h_b(\lambda) & = & S(\lambda)\{2q\mu_1 e^{-2l_2\lambda}(e^{-l_1\lambda}f^1(\lambda) + g^1(\lambda)) \\ & - & (1 - qe^{-2l_2\lambda})^{-1}(h^1(\lambda) + qe^{-2l_2\lambda}e^1(\lambda)) \\ & + & (1 - qe^{-2l_2\lambda})^{-1}(M\lambda + \mu_1)^{-1}qe^{-2l_2\lambda}M(F'(l_1) - G'(0)) \\ & & (\mu_2(1 - e^{-2l_1\lambda})(1 + qe^{-2l_2\lambda}) + 2\mu_1 e^{-2l_1\lambda}(1 - qe^{-2l_2\lambda}))\}. \end{array}\right. \tag{21}$$

We analyse the "good" functions in this part of the proof, and we leave the "bad" functions to Part 2 of the proof. We deform the contour defining $H_g'(t)$ so that it becomes the line $\operatorname{Re}\lambda = -\gamma$, where

$$\gamma = \min(\mu_1/(2M), -\log(q)/(4l_2)).$$

But

$$f^1(-\gamma + i\xi) = \int_0^{l_1} F'(t)e^{\gamma t}e^{-i\xi t}\, dt,$$

and thus by the Plancherel equality, the L^2 norm of $f^1(-\gamma + i\xi)$, as a function of ξ is no greater than $e^{\gamma l_1}(\mathcal{E}(0)/\mu_1)^{1/2}$. All of the other terms in the expression for $h_g(-\gamma+i\xi)$ may be estimated similarly and thus the L^2 norm of $h_g(-\gamma+i\xi)$ as a function of ξ is no greater than a constant times $(\mathcal{E}(0))^{1/2}$. But again by the Plancherel equality, this implies that $H_g^1(t)$ is equal to $e^{-\gamma t}$ times a function whose L^2 norm is no greater than a constant times $(\mathcal{E}(0))^{1/2}$. Thus the term in the energy corresponding to H_g^1 decays exponentially with time. This completes Part 1 of the proof.

Before preceding to Part 2 of the proof of Theorem 1, we state and prove an inequality that we will need in the proof.

Lemma 4 *If γ, θ, μ and k are real and $k > 0$, $\gamma > \mu > 0$ then*

$$\left|\frac{(\gamma + i\theta)^k}{(\mu + i\theta)^{k+1}}\right|^2 \leq \frac{1}{(k+1)(\mu^2 - \gamma^2)}$$

Proof. Differentiating the expression with respect to θ shows that a maximum occurs at $\theta = 0$ if $k\mu^2 - (k+1)\gamma^2 \leq 0$ and at $\theta^2 = k\mu^2 - (k+1)\gamma^2$ if $k\mu^2 - (k+1)\gamma^2 > 0$. In both cases, one easily sees that the inequality is satisfied. this completes the proof of the lemma.

Proof of Theorem 1 (Part 2).

We would like to deform the contour defining H_b^1 so that it ends up as a line parallel to the imaginary axis in the left half plane. But unfortunately this is impossible because, unlike h_g, h_b has a sequence of poles converging to the imaginary axis. These poles in fact correspond to the sequence of eigenvalues converging to the imaginary axis, and we note that this sequence has been investigated in [1]. However, there is a way to get around this problem. We define $h_{b,m}(\lambda)$ by the same formula (21) defining h_b, the only difference being that $S(\lambda)$ is replaced by $S_m(\lambda)$. Next, we note that

$$S_m(\lambda) - S(\lambda) = e^{-2(m+1)\lambda l_1} \left[\frac{M\lambda - \mu_1 + Q(\lambda)\mu_2}{M\lambda + \mu_1 + Q(\lambda)\mu_2} \right]^{m+1} S(\lambda)$$

This shows that if $t < 2(m+1)l_1$ then

$$\int_{\sigma - i\infty}^{\sigma + i\infty} (h_{b,m}(\lambda) - h_b(\lambda)) e^{\lambda t} \, d\lambda \to 0$$

as $\sigma \to \infty$ Since the expression is independent of σ, it must be zero for $t < 2(m+1)l_1$. Thus, for t in this range, we may use $h_{b,m}(\lambda)$ instead of $h(\lambda)$ in the formula for $H_b^1(t)$.

We now consider $0 < \gamma < \min(\mu_1/(2M), -\log(q)/(4l_2))$ and use the contour $\operatorname{Re}\lambda = -\gamma$. Applying Lemma 4, we see that the kth term of the integrand is bounded by an L^2 function multiplied by

$$[(k+1)(\alpha\mu_2 - M\gamma)]^{-1/2},$$

where $\alpha = \operatorname{Re} Q(-\gamma + i\xi)$. Note that, by the proof of Lemma 1, we can always choose γ so that $\alpha\mu_2 - M\gamma$ is bounded below by a positive constant. Finally, we note that if we take t in the interval $(2ml_1, 2(m+1)l_1)$, and if m is sufficiently large, we obtain the required estimate for H_b^1. The analysis for E' is similar.

Theorem 2 *If, in the interval $(-l_1, 0)$, the initial displacement is in H^2 and the initial velocity is in H^1, and all of the initial data has finite energy, then the energy at time t satisfies an estimate*

$$\mathcal{E}(t) \leq C(\mathcal{E}(0) + \mathcal{J}(0))/t, \qquad t > 0,$$

where $\mathcal{J}(t)$ denotes the sum of the $H^2(-l_1, 0)$ norm of the initial displacement and the $H^1(-l_1, 0)$ of the initial velocity.

Proof (Outline). The proof is similar to the proof of the preceding theorem. We start by integrating the formulae for f^1 and g^1 by parts to make use of the extra smoothness.

References

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