BRACHISTOCHRONES FOR REPULSIVE LOGARITHMIC POTENTIAL

Garry J. Tee
Department of Mathematics, University of Auckland
Auckland, New Zealand **tee@math.auckland.ac.nz**

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Abstract

Brachistochrones are constructed for repulsive central force, with logarithmic potential. Each pair of points is connected by infinitely many brachistochrones, on each of which the passage time is a local minimum, and the global minimum passage time is attained on one (or on infinitely many) of those curves. The bounded brachistochrones starting at a fixed point are separated from the unbounded brachistochrones by a Critical Brachistochrone.

Short title: REPULSIVE LOGARITHMIC POTENTIAL

1 Repulsive Logarithmic Potential

Brachistochrones for central forces have been constructed [1] as the orbits of associated free particles under transformed central forces. Full details were given for inverse square attraction [1], for inverse square repulsion [2] and for attractive logarithmic potential [3]. Full details are given here for repulsive logarithmic potential

$$\varphi(r) \;=\; -\log r. \tag{1}$$

At radius $r \geq 1$ the constrained particle has (cf. [3], (3)) the speed $v(r)$, where

$$v(r)^2 \;=\; -2\varphi \;=\; 2\;\log r, \tag{2}$$

and so $v(r)$ increases unboundedly as r increases, with

$$r \;=\; e^{-\varphi} \;=\; e^{v^2/2}. \tag{3}$$

The family of brachistochrones starting at a fixed point A (whose polar coordinates are taken as $r = 1,\ \vartheta = 0$) can be parametrized by the angular momentum K of the associated free particle. Here, it is more convenient to parametrize them by $J = 2K^2$.

Equations (10), (11) and (12) in [3] then give angle ϑ, arclength s and time t, each as a definite integral between radius 1 and r:

$$\vartheta \;=\; \int_1^r \frac{\mathrm{d}x}{\pm x\sqrt{\frac{x^2}{J\ \log x}-1}}\ , \tag{4}$$

$$s \;=\; \int_1^r \frac{\mathrm{d}x}{\pm\sqrt{1-\frac{J\ \log x}{x^2}}}\ . \tag{5}$$

$$t \;=\; \int_1^r \frac{\mathrm{d}x}{\pm\sqrt{2\ \log x\left(1-\frac{J\ \log x}{x^2}\right)}}\ . \tag{6}$$

In terms of potential φ and speed v, these become:

$$\vartheta \;=\; \int_{-\log r}^{0} \pm\sqrt{\frac{-J\varphi}{e^{-2\varphi}+J\varphi}}\ \mathrm{d}\varphi \;=\; \int_0^{\sqrt{2\log r}} \pm v^2\sqrt{\frac{J}{2e^{v^2}-Jv^2}}\ \mathrm{d}v\ , \tag{7}$$

$$s \;=\; \int_{-\log r}^{0} \frac{e^{-2\varphi}}{\pm\sqrt{e^{-2\varphi}+J\varphi}}\ \mathrm{d}\varphi \;=\; \int_0^{\sqrt{2\log r}} \frac{\pm v\ e^{v^2/2}}{\sqrt{1-\frac{1}{2}Jv^2e^{-v^2}}}\ \mathrm{d}v\ , \tag{8}$$

$$t \;=\; \int_{-\log r}^{0} \frac{e^{-2\varphi}}{\pm\sqrt{-2\varphi\left(e^{-2\varphi}+J\varphi\right)}}\ \mathrm{d}\varphi \;=\; \int_0^{\sqrt{2\log r}} \frac{\pm e^{v^2/2}}{\sqrt{1-\frac{1}{2}Jv^2e^{-v^2}}}\ \mathrm{d}v\ . \tag{9}$$

The particle released from rest at A always has $\varphi \le 0$ and $v \ge 0$. In each of these 9 integrals the positive square root is to be taken where radius increases with t, and the negative square root is to be taken where radius decreases with t.

In each of these 9 integrals, the denominator has the factor $\sqrt{\mathcal{G}(\varphi)}$ or $\sqrt{\mathcal{H}(x)}$ or $\sqrt{\mathcal{L}(v)}$, where

$$\mathcal{G}(\varphi) \;=\; e^{-2\varphi}+J\varphi \;=\; \mathcal{H}(x) \;=\; x^2-J\ \log x \;=\; \mathcal{L}(v) \;=\; e^{v^2}-\tfrac{1}{2}Jv^2. \tag{10}$$

1.1 Equation for Apsidal Radius R

The orbit of the associated free particle with angular momentum K has an apsidal radius R, if and only if $K = R/v(R)$ (cf. [1] (25)). In terms of $J = 2K^2$,

$$J \;=\; \frac{2R^2}{v(R)^2} \;=\; \frac{R^2}{\log R}, \tag{11}$$

in view of (2), and so

$$\frac{\mathrm{d}J}{\mathrm{d}R} \;=\; \frac{R(2\ \log R-1)}{(\log R)^2}\ . \tag{12}$$

Hence for real $R > 1$, J attains its minimum value $J = 2e$ at

$$R \;=\; \sqrt{e} \;=\; 1{\cdot}6487212707001281468\ . \tag{13}$$

Therefore, orbits with $J < 2e$ do not have any apse — e.g. with $J = 0$ the constrained particle accelerates radially outwards, with speed $v(r)$ increasing unboundedly as r increases (cf. [3], (9)).

Hereafter, we consider only $J > 0$.

At any apse,

$$0 \;=\; \frac{\mathrm{d}r}{\mathrm{d}\vartheta} \;=\; \frac{\mathrm{d}r}{\mathrm{d}s} \;=\; \frac{\mathrm{d}r}{\mathrm{d}t}\;. \tag{14}$$

The integrands in (4), (5) and (6) are the reciprocals of these derivatives. Therefore, each of those 3 integrands has an integrable singularity at the apse $x = R$. Similarly, each of the 6 integrands in (7), (8) and (9) has an integrable singularity at the apse $\varphi = -\log\, R,\;\, v = \sqrt{2\, \log\, R}$.

2 Bounded Brachistochrones

For fixed J, equation (11) gives an equation in R:

$$\mathcal{H}(R) \;=\; R^2 - J\;\log\, R \;=\; 0. \tag{15}$$

Therefore, at an apse the potential $\varphi(R) = -\log\, R$ satifies the equation $\mathcal{G}(\varphi) = 0$.

With $0 < J < 2e$ there is no real root of (15).

With $J = 2e$, the osculation of the graphs of $y = R^2$ and $y = J\;\log\, R$ gives the double root $R = \sqrt{e}$.

With $J > 2e$, the 2 intersections of those 2 graphs shew that the equation (15) in R has 2 real roots (α, β), with $1 < \alpha < \sqrt{e} < \beta$. As J increases from $2e$ then β increases unboundedly and α decreases, with $\log\, \alpha = \alpha^2/J \sim 1/J$ so that $\alpha \sim e^{1/J}$. As $J \searrow 2e$ then α increases and β decreases, both to $\sqrt{e}$. That may conveniently be written as $\alpha \nearrow \sqrt{e} \swarrow \beta$.

For any $J > 2e$, the apsidal radius R can be found by solving the equation $\mathcal{H}(R) = 0$ for the root in $(1, \sqrt{e})$. Since $\mathcal{H}(1) = 1 > 0$ and $\mathcal{H}(\sqrt{e}) = \frac{1}{2}(2e - J) < 0$, the bisection method can be applied, supplemented by the Newton–Raphson method for high accuracy.

When a particle with $J > 2e$, starting from rest at $r = 1$, reaches the apsidal radius $r = \alpha < \sqrt{e}$, then it must return to $r = 1$ without reaching any radius $r > \alpha$; since each brachistochrone under central force is symmetric about each apsidal line ([1] §2.1). Hence, the root $\beta > \sqrt{e}$ of the equation (15) cannot be the radius of any point on that brachistochrone, with $J > 2e$. Therefore, the only apsidal radius is the smaller positive root $R = \alpha < \sqrt{e}$.

Hence the brachistochrone has an apsidal radius R if and only if $J > 2e$; and $1 < R < \sqrt{e}$. We call this a *bounded brachistochrone*, and it can be parametrized by the maximum radius R, as an alternative to J.

Everywhere on a bounded brachistochrone the potential is

$$\varphi \;=\; -\log\, r \geq -\log\, R > -\tfrac{1}{2}\;, \tag{16}$$

and the speed is $v < 1$.

2.1 Angle as a Function of Radius for Bounded Brachistochrones

Taking the positive square roots in (7) (8) and (9), we get integral expressions for angle, arclength and time as functions of radius on the first half-arch; each with the minimum radius R as a parameter:

$$
\begin{aligned}
\Theta(r,R) &= \int_{-\log r}^{0} \sqrt{\frac{-R^2\varphi}{\log\, R\; e^{-2\varphi}+R^2\varphi}}\;\mathrm{d}\varphi && (17)\\
S(r,R) &= \int_{-\log r}^{0} \frac{e^{-2\varphi}}{\sqrt{e^{-2\varphi}+\frac{R^2}{\log\, R}\,\varphi}}\;\mathrm{d}\varphi\ , && (18)\\
T(r,R) &= \int_{-\log r}^{0} \frac{e^{-2\varphi}}{\sqrt{-2\varphi\left(\frac{R^2}{\log\, R}\,\varphi+e^{-2\varphi}\right)}}\;\mathrm{d}\varphi\ . && (19)
\end{aligned}
$$

Each brachistochrone is symmetric about each apsidal line; and hence on the second half–arch the negative square roots are taken in (7), (8) and (9), giving

$$
\vartheta = 2\Theta(R,R)-\Theta(r,R),\ \ s = 2S(R,R)-S(r,R),\ \ t = 2T(R,R)-T(r,R). \qquad (20)
$$

Or, if angle ω, arclength σ and time τ are measured from the apse, then the complete brachistochrone is given in terms of r by the integrals

$$
\begin{aligned}
\omega &= \pm\int_{-\log R}^{-\log r} \sqrt{\frac{-R^2\varphi}{\log\, R\; e^{-2\varphi}+R^2\varphi}}\;\mathrm{d}\varphi\ , && (21)\\
\sigma &= \pm\int_{-\log R}^{-\log r} \frac{e^{-2\varphi}}{\sqrt{e^{-2\varphi}+\frac{R^2}{\log\, R}\,\varphi}}\;\mathrm{d}\varphi\ , && (22)\\
\tau &= \pm\int_{-\log R}^{-\log r} \frac{e^{-2\varphi}}{\sqrt{-2\varphi\left(e^{-2\varphi}+\frac{R^2}{\log\, R}\,\varphi\right)}}\;\mathrm{d}\varphi\ . && (23)
\end{aligned}
$$

2.1.1 Computation of Brachistochrones

The integral (17) for ϑ has a singularity in the derivative of the integrand at $\varphi = 0$, and when $r = R$ it has an integrable singularity at $\varphi = -\log\, R$. Those singularities may be eliminated by an appropriate change of variable, to produce a smooth integrand which is suitable for quadrature by Romberg integration.

In (17), substitute

$$
-\log\, r = \varphi(\kappa)\ = -\tfrac{1}{2}\log\, R\ (1-\cos\,\kappa)\ =\ -\log\, R\ \sin^2(\kappa/2), \qquad (24)
$$

so that as φ decreases from 0 to $-\log\, R > -\frac{1}{2}$, κ increases from 0 to π, and

$$
\kappa\ =\ \kappa(\varphi)\ =\ \arccos\left(1+\frac{2\varphi}{\log\, R}\right)\ . \qquad (25)
$$

Now,

$$
\cos^2(\kappa/2)\ =\ 1-\sin^2(\kappa/2)\ =\ 1-\frac{\log\, r}{\log\, R}\ =\ \frac{\log\, R+\varphi}{\log\, R}, \qquad (26)
$$

and therefore

$$
\begin{aligned}
\mathrm{d}\varphi &= -\log\, R\ \sin(\kappa/2)\ \cos(\kappa/2)\ \mathrm{d}\kappa\\
&= -\sin(\kappa/2)\ \sqrt{\log\, R\ (\log\, R+\varphi)}\ \mathrm{d}\kappa. && (27)
\end{aligned}
$$

Then the integral (17) becomes

$$\Theta(r,R) \;=\; R\ \log\,R \int_0^{\kappa(-\log\,r)} \sin^2(\kappa/2)\,\sqrt{\frac{\varphi+\log\,R}{R^2\varphi+\log\,R\ e^{-2\varphi}}}\ \mathrm{d}\kappa, \tag{28}$$

where the function $\kappa(\varphi)$ is given by (25) and the inverse function $\varphi(\kappa)$ is given by (24).

In (28) the integrand is smooth for all $r \in (R,1]$. But, as $r \nearrow R$, $\varphi \searrow -\log\,R$, and hence the argument of the square root approaches the form $0/0$.

In order to avoid that indeterminate expression, substitute

$$w(\kappa) \;=\; \varphi+\log\,R \;=\; \log\,R\ \cos^2(\kappa/2), \tag{29}$$

so that

$$x \;=\; e^{-\varphi} \;=\; e^{\log\,R-w} \;=\; Re^{-w}. \tag{30}$$

Then, (28) becomes

$$\Theta(r,R) \;=\; \int_0^{\kappa(-\log\,r)} \frac{\log\,R\ \sin^2(\kappa/2)}{\sqrt{1-\log\,R\,\mathcal{F}(-\log\,R\ \cos^2(\kappa/2))}}\ \mathrm{d}\kappa\ , \tag{31}$$

where

$$\mathcal{F}(x) \;\stackrel{\text{def}}{=}\; \frac{e^{2x}-1}{x} \qquad (w \neq 0), \tag{32}$$

and $\mathcal{F}(0) \;\stackrel{\text{def}}{=}\; 2$ (for continuity).

For $|x| \geq \delta$ for some suitable $\delta < 1$ (e.g. $\delta = 10^{-3}$), $\mathcal{F}(x)$ can safely be evaluated directly from (32). But for $|x| < \delta$, $\mathcal{F}(x)$ should be evaluated from the rapidly convergent power series:

$$\mathcal{F}(x) \;=\; \frac{2}{1!}+\frac{2(2x)}{2!}+\frac{2(2x)^2}{3!}+\frac{2(2x)^3}{4!}+\ldots\ . \tag{33}$$

In this manner, the integral expression (31) for $\Theta(r,R)$ can be computed by Romberg integration with respect to κ. In particular, the apsidal angle

$$\Theta(R,R) \;=\; \int_0^{\pi} \frac{\log\,R\ \sin^2(\kappa/2)}{\sqrt{1-\log\,R\,\mathcal{F}(-\log\,R\ \cos^2(\kappa/2))}}\ \mathrm{d}\kappa\ , \tag{34}$$

can be computed by Romberg integration with respect to κ.

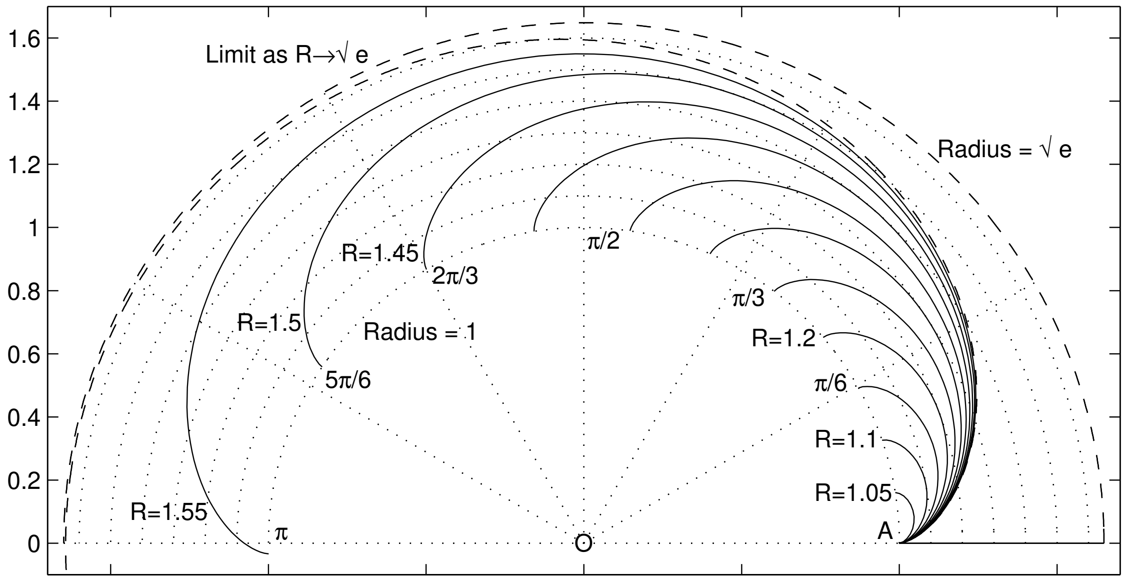
Apsidal Locus The curve $\vartheta = \Theta(R,R)$, which we shall call the *apsidal locus*, separates those points C for which the brachistochrone AC is part of the (open) first half–arch (17) of a complete bounded brachistochrone AB, from those points C for which the brachistochrone AC includes part of the (open) second half–arch (20).

2.1.2 Pictures of Bounded Brachistochrones

Each bounded brachistochrone is symmetric about its apsidal radius.

Figure 1 shews the complete brachistochrones for $R = 1{\cdot}05\ (0{\cdot}05)\ 1{\cdot}55$, with the total angle for each complete brachistochrone here being less than π; except for $R = 1{\cdot}55$, with total angle $\Gamma = 3{\cdot}175603356982154$, which is slightly greater than π. Note that the small brachistochrones do indeed resemble cycloids, as expected. Also, there is shewn the limit of those curves as $R \nearrow \sqrt{e}$ (and ($J \searrow 2e$), which is the Critical Brachistochrone with $J = 2e$ (cf. §3). That limit is not symmetric.

Figure 1. Bounded Brachistochrones, R = 1.05 (0.05) 1.55



Critical radius = √ e = 1.6487212707001281468

Figure 2 shews the first and fifth quadrants of the Critical Brachistochrone and of the apsidal locus, which is computed from (34). All first half–arches of bounded brachistochrones are confined between the apsidal locus and the Critical Brachistochrone.

Both the apsidal locus and the Critical Brachistichrone have unbounded total angle ϑ, each spiralling out asymptotically towards the critical radius $r = \sqrt{e}$, without reaching it. After winding once around O, in the 5th quadrant both the apsidal locus and the Critical Brachistochrone are very close to the critical radius $r = \sqrt{e}$.

Figure 3 shews the complete brachistochrones for $R = 1{\cdot}6\ (0{\cdot}005)\ 1{\cdot}64$, with the total angle for complete brachistochrone ranging from $4{\cdot}221834463547582$ to $\Delta = 6{\cdot}693666861152652$ (slightly more than 2π), after winding once around O.

The curve with $R = 1{\cdot}55$ is the unique complete brachistochrone with total angle of Γ. But the least time for the particle to get from A to that point is given by the complete brachistochrone with total angle $2\pi - \Gamma = 3{\cdot}107581950197432 < \pi$. The curve with $R = 1{\cdot}64$ is the unique complete brachistochrone with total angle of Δ. But the least time for the particle to get from A to that point is given by the complete brachistochrone with total angle $\Delta - 2\pi = 0{\cdot}310481553973066$. And similarly for all brachistochrones with total angle greater than π.

Note that the complete brachistochrone with $R = 1{\cdot}64$ intersects itself, at $\vartheta = 0{\cdot}155240776986533 \equiv 2\pi + 0{\cdot}155240776986533 \pmod{2\pi}$.

Figure 2. Apsidal Locus and Critical Brachistochrone, 1st and 5th Quadrant

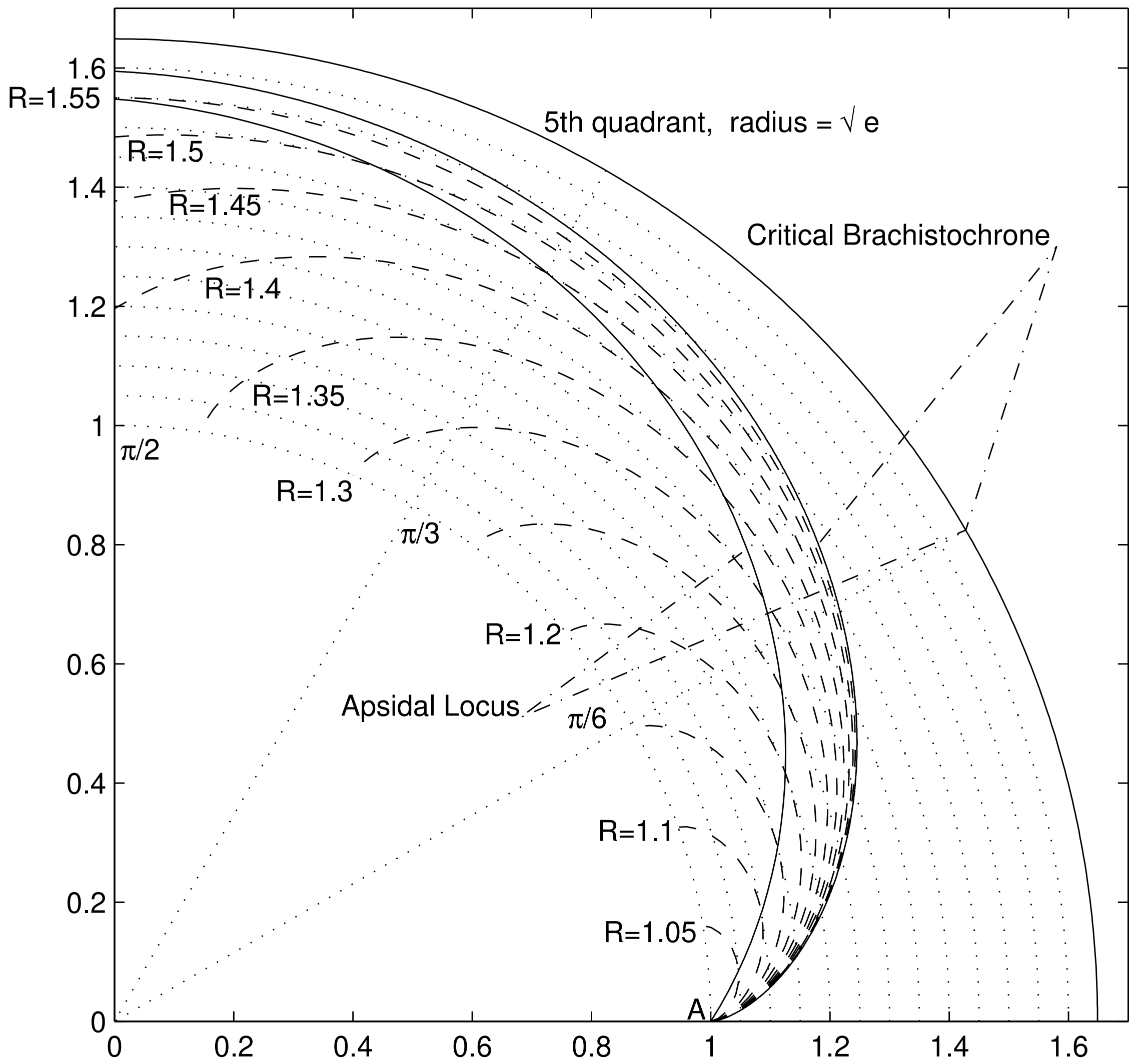
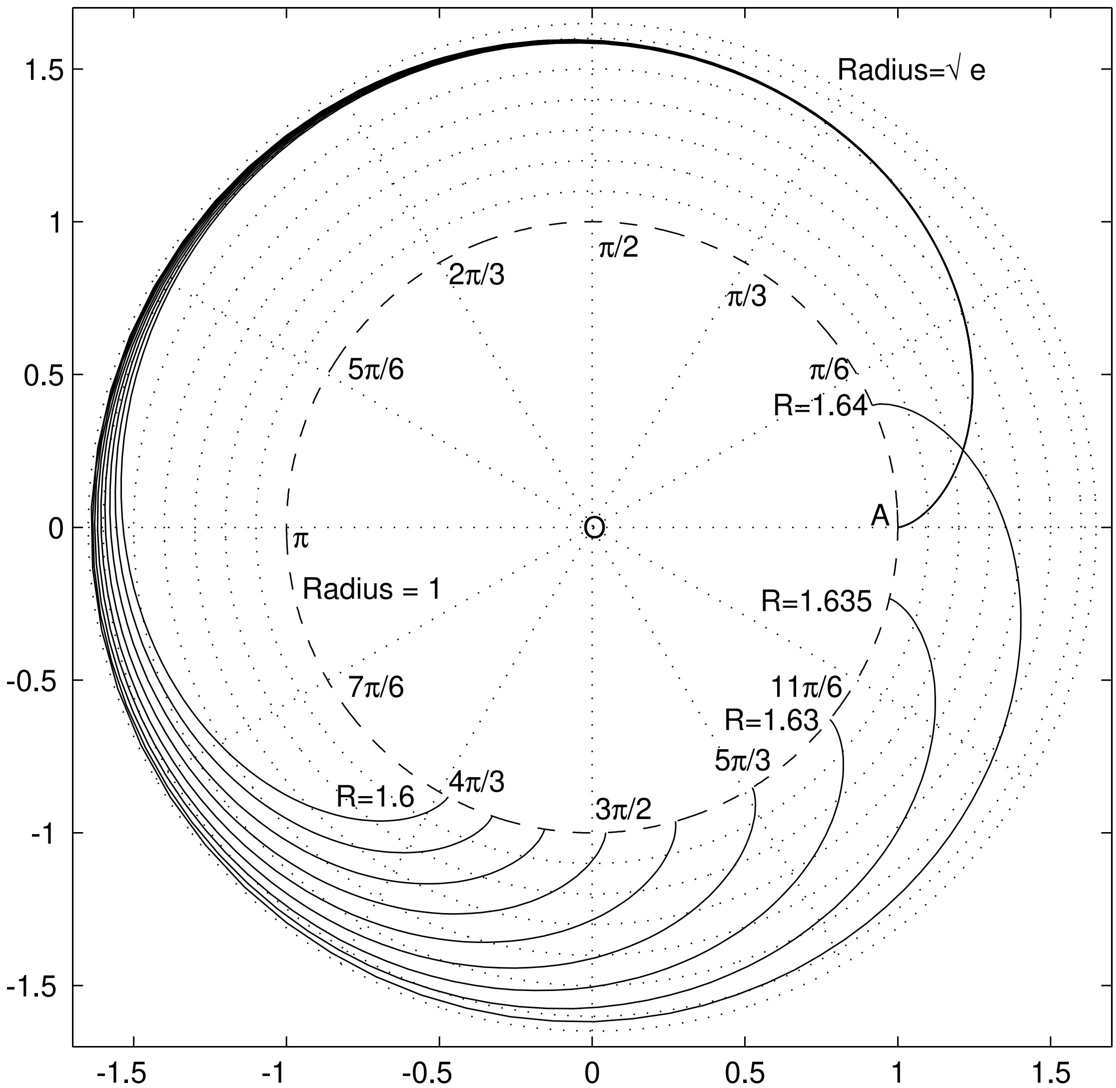


Figure 4 shews the complete brachistochrones (computed similarly to Table 1 in [1]) for $R = 1{\cdot}547748649853155$ with total angle π from A to D; and for $R = \tilde{R} = 1{\cdot}637086621902548$, winding once around O with total angle 2π from A to A. Hence, at $\vartheta = \frac{1}{2}\pi$, the apsidal locus (Figure 2) has radius $1{\cdot}547748649853155$.

For the case of a radial 3–dimensional field (rather than plane repulsive logarithmic potential), the curve with total angle π could be rotated through any angle about the line OA, producing infinitely many complete brachistochrones connecting the diametrically opposite points A and D; and similarly for brachistochrones with total angle which is any nonzero integer multiple of π. The curve of least time with total angle 2π is the complete brachistochrone with angle of 2π, and it is unique apart from being rotated through any angle around OA. But the least time for the particle to get from A to A is <u>zero</u>, which is achieved by the particle on the complete brachistochrone consisting of the single point A, with zero total angle.

Figure 5 shews the complete brachistochrone for $R = 1{\cdot}6486$, with total angle $\Phi = 12{\cdot}748671498770976$, after winding twice around O. That is the unique curve of least time with total angle of Φ.

Figure 3. Bounded Brachistochrones, for maximum radius R = 1.6 (0.005) 1.64



For R = 1.64, self-intersection at ϑ = 0.155240776986533 (mod 2π)

But the least time for the particle to get from A to that point is given by the complete brachistochrone with total angle $\Phi - 4\pi = 0{\cdot}182300884411803$.

Note that the complete brachistochrone with $R = 1{\cdot}6486$ twice intersects itself, at $\vartheta = 0{\cdot}091150442205901 \equiv 4\pi + 0{\cdot}091150442205901$ and also at $\vartheta = \pi + 0{\cdot}091150442205901 \equiv 3\pi + 0{\cdot}091150442205901$.

2.2 Arclength as a Function of Radius

The integral (8) for s has an integrable singularity at the apse $v = \sqrt{2 \log R}$, when $r = R$. Apply the substitution

$$v^2 \ = \ 2 \ \log R - w^2, \tag{35}$$

so that

$$w^2 \ = \ 2 \ \log R - v^2 \ = \ 2 \ \log(R/r), \tag{36}$$

Figure 4. Bounded Brachistochrones, with Total Angles π and 2π

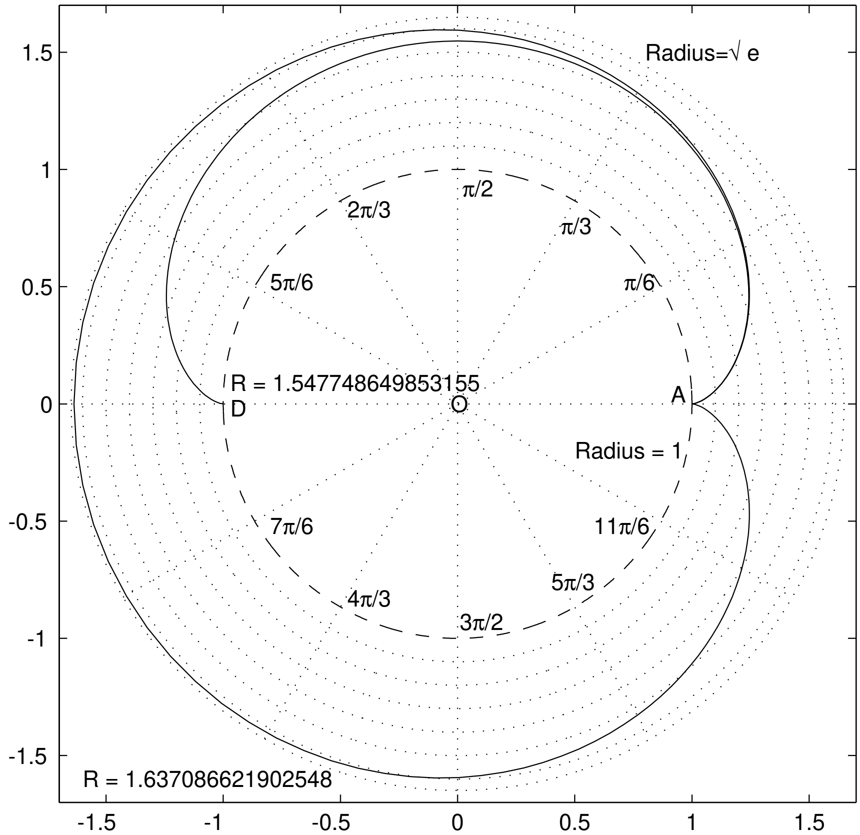
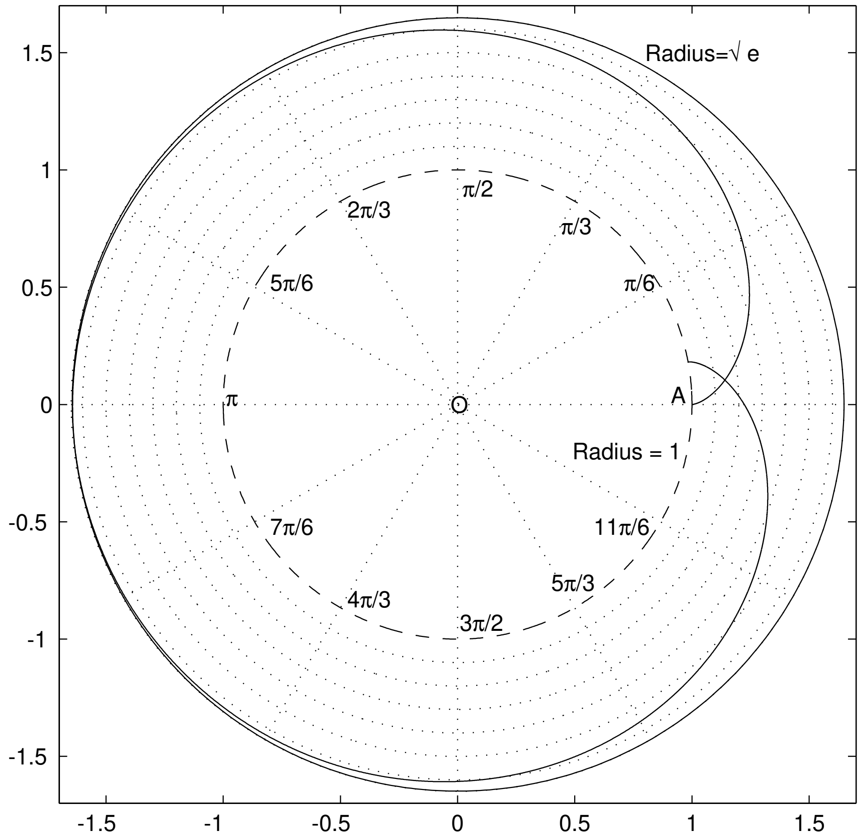


Figure 5. Bounded Brachistochrone, for maximum radius R = 1.6486



Two self-intersections, at $\vartheta = 0.091150442205901$ (mod π)

and

$$e^{v^2} \;=\; R^2 e^{-w^2}. \tag{37}$$

Then the equation (8) for s becomes:

$$S(r,R) \;=\; \int_{\sqrt{2\,\log(R/r)}}^{\sqrt{2\,\log R}} \frac{R e^{-w^2}}{\sqrt{\left(\frac{1}{\log R} - \mathcal{F}(\,-\,\frac{1}{2}w^2)\right)/2}}\;\mathrm{d}w\;, \tag{38}$$

where the function $\mathcal{F}$ is to be evaluated as with (32).

In particular, the arclength to the apse is

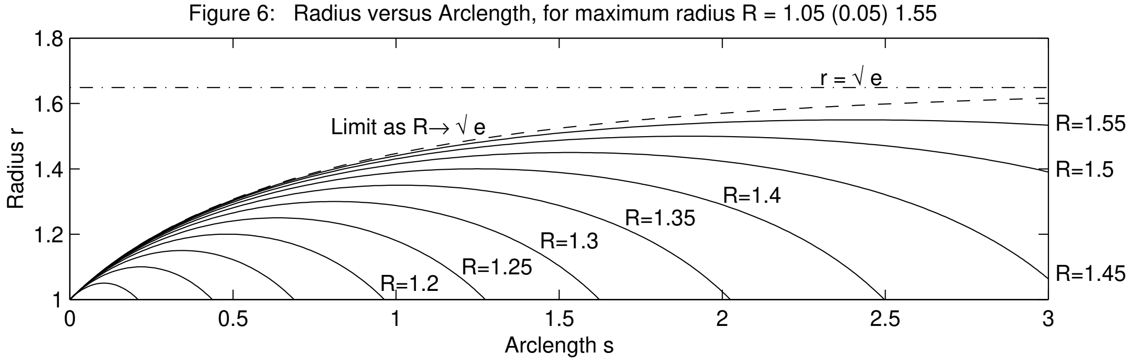
$$S(R,R) \;=\; \int_{0}^{\sqrt{2\,\log R}} \frac{R e^{-w^2}}{\sqrt{\left(\frac{1}{\log R} - \mathcal{F}(\,-\,\frac{1}{2}w^2)\right)/2}}\;\mathrm{d}w\;. \tag{39}$$

And these transformed integrals are suitable for Romberg integration with respect to w.

2.2.1 Graphs of Radius versus Arclength

Figure 6 shews graphs of r versus s for $R = 1{\cdot}05$ $(0{\cdot}05)$ $1{\cdot}55$, with s computed for numerous values of r by Romberg integration of (38). Each graph is symmetric about the line $s = S(R,R)$.

Figure 6: Radius versus Arclength, for maximum radius R = 1.05 (0.05) 1.55



Also, there is shewn that the limit of those graphs as $R \nearrow \sqrt{e}$ (and $J \searrow 2e$), which is the graph for the brachistochrone with $J = 2e$ (cf. §3). That limit is not symmetric.

2.3 Time as a Function of Radius

The integral (19) for t has an integrable singularity in the integrand at $\varphi = 0$, and when $r = R$ it has an integrable singularity at $\varphi = \log R$. Those singularities may be eliminated by an appropriate change of variable, to produce a smooth integrand which is suitable for quadrature by Romberg integration.

With the substitution (24), the equation (19) becomes

$$T(r,R) \;=\; \int_{0}^{\kappa(-\log r)} \frac{\mathrm{d}\kappa}{R^{\cos\kappa}\;\sqrt{2\left(\frac{1}{\log R} - \mathcal{F}(\,-\,\log R\;\cos^2(\kappa/2))\right)}}\;, \tag{40}$$

where the function $\kappa = \kappa(\varphi)$ is given by (24). In particular, the time at the apse is

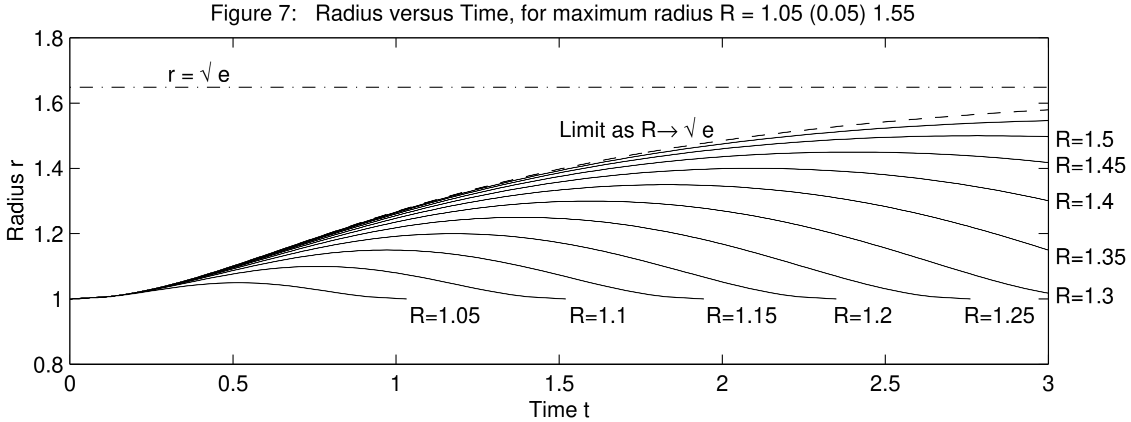
$$T(R,R) \;=\; \int_0^{\pi} \frac{\mathrm{d}\kappa}{R^{\cos\kappa}\,\sqrt{2\left(\frac{1}{\log R} - \mathcal{F}(\,-\log\,R\;\cos^2(\kappa/2))\right)}}\;. \tag{41}$$

And these transformed integrals are suitable for Romberg integration with respect to κ.

2.3.1 Graphs of Radius versus Time

Figure 7 shews graphs of r versus t, for $R = 1{\cdot}05\ (0{\cdot}05)\ 1{\cdot}55$, with t computed for numerous values of r by Romberg integration of (40). Each graph is symmetric about the line $t = T(R,R)$.

Figure 7: Radius versus Time, for maximum radius R = 1.05 (0.05) 1.55



Also, there is shewn that the limit of those graphs as $R \nearrow \sqrt{e}$ (and $J \searrow 2e$), which is the graph for the brachistochrone with $J = 2e$ (cf. §3). That limit is not symmetric.

2.4 Angle, Arclength and Time for Bounded Brachistochrones

Each of ϑ, s and t is a computable function of r. Accordingly, for various values of R, each of the three relations between pairs of these variables can be represented by a graph, with r as the parameter for computing values of the chosen pair of variables. Since each of angle, arclength and time, measured from the apse, is an even function of r, each of these graphs (for each R) is symmetric about its midpoint, corresponding to the apse.

Figure 8 shews graphs of ϑ versus s, Figure 9 shews graphs of ϑ versus t, and Figure 10 shews graphs of s versus t; each for $R = 1{\cdot}05\ (0{\cdot}05)\ 1{\cdot}45$, with the radius everywhere less than $\sqrt{e}$. Also, in each figure there is shewn the limit of those graphs as $R \nearrow \sqrt{e}$ (and $J \searrow 2e$), which is the graph for the brachistochrone with $J = 2e$

(cf. §3). That limit is not symmetric.

Figure 8: Angle versus Arclength, for maximum radius R = 1.05 (0.05) 1.55

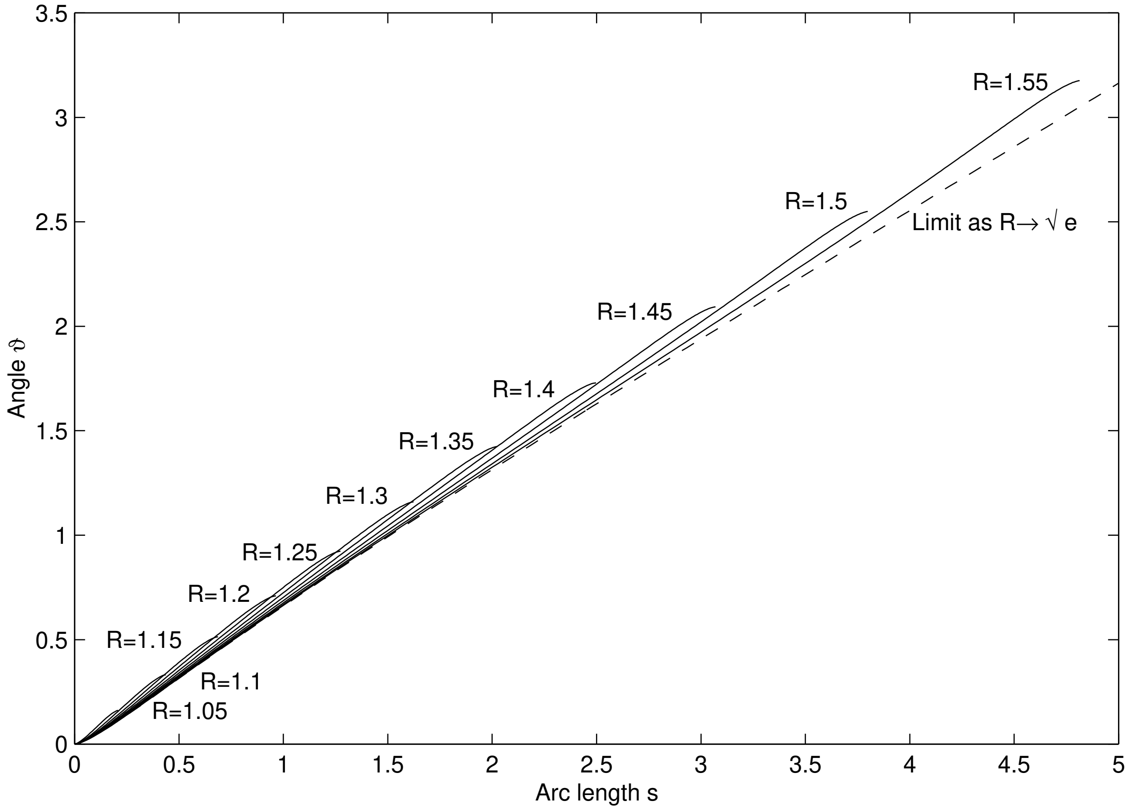


Figure 9: Angle versus Time, for maximum radius R = 1.05 (0.05) 1.55

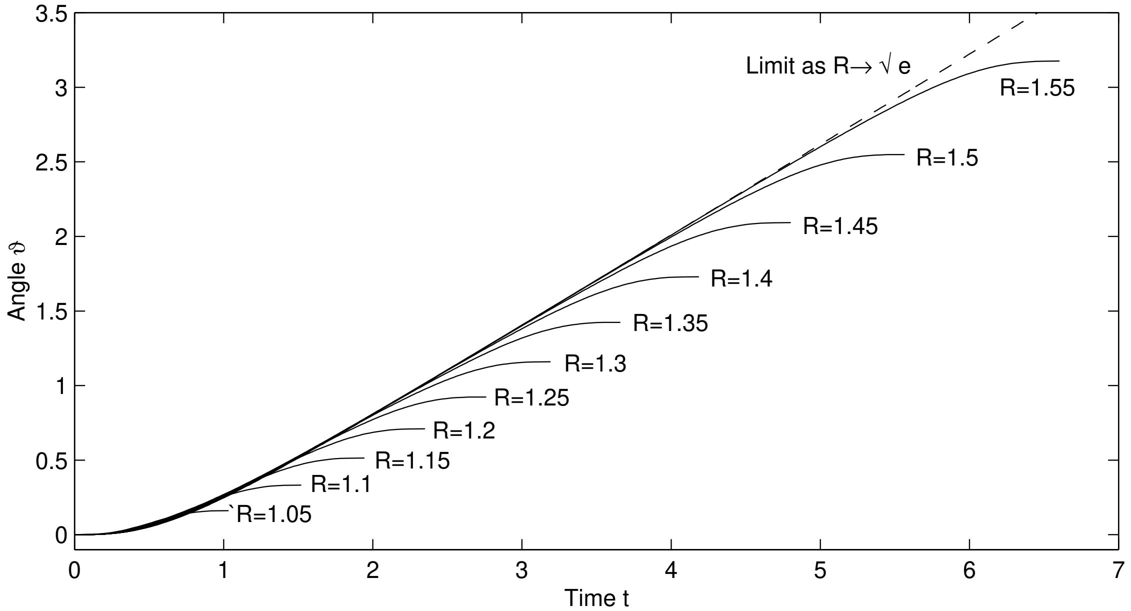
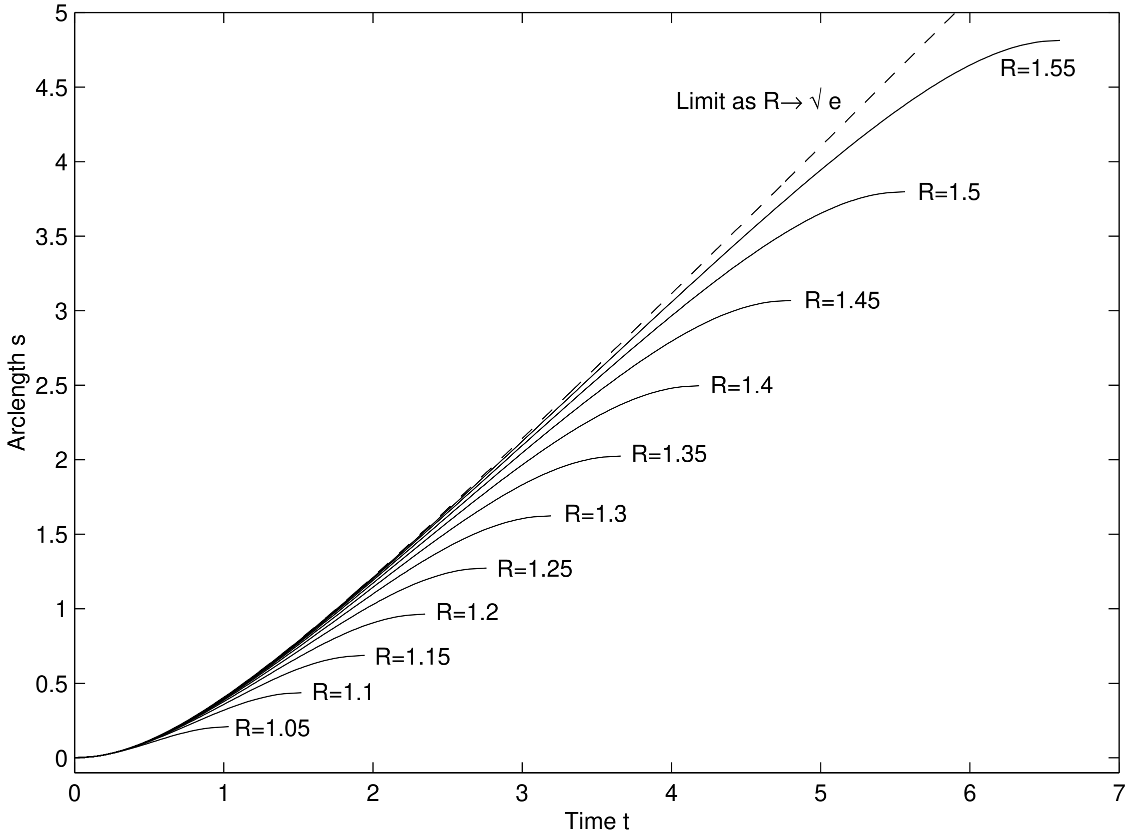


Figure 10: Arclength versus Time, for maximum radius R = 1.05 (0.05) 1.55



3 The Critical Brachistochrone

Equation (15) for the apsidal radius R has no real root if $0 < J < 2e$, if $J > 2e$ it has 2 real roots (greater than 1), and if $J = 2e$ it has the double root $R = \sqrt{e}$. The brachistochrone with $J = 2e$ is here called the *Critical Brachistochrone*.

With $J \geq 2e$, the integrands (7) in terms of v for ϑ and (9), for t are smooth at $v = 0$; and the integrands (5) and (8) for s are smooth at $r = 1$ and at $v = 0$.

When the particle starts from A at time $t = 0$, the repulsive force will make r increase with t (initially, at least); and hence the positive square root is to be used in each of these integrals. If $\mathrm{d}r/\mathrm{d}t$ ever became negative then it would pass through 0 at an apse at radius R, satisfying equation (15); and with $J = 2e$ that equation is satisfied only by $R = \sqrt{e}$.

With $J = 2e,\ R = \sqrt{e}$ is a double root of (15), and (7) becomes (with positive square root):

$$\vartheta \;=\; \int_0^{\sqrt{2\,\log r}} \frac{v^2}{\sqrt{e^{v^2-1} - v^2}}\,\mathrm{d}v. \tag{42}$$

Substitute

$$v^2 \;=\; 1 - w^2, \tag{43}$$

so that

$$w^2 \;=\; 1 - 2\,\log r \;=\; \log\left(e/r^2\right), \tag{44}$$

$$v\,\mathrm{d}v \;=\; -w\,\mathrm{d}w; \tag{45}$$

and

$$e^{v^2-1} - v^2 \;=\; e^{-w^2} - 1 + w^2 \;=\; \frac{w^4}{2} - \frac{w^6}{6} + O(w^8), \tag{46}$$

as $r \nearrow \sqrt{e}$ and $w \searrow 0$.

Then (42) becomes:

$$\begin{aligned}
\vartheta \;&=\; \int_{\sqrt{\log(e/r^2)}}^{1} w \sqrt{\frac{1-w^2}{w^4\left(\frac{1}{2} - \frac{1}{6}w^2 + O(w^4)\right)}}\, \mathrm{d}w \\
&=\; \int_{\sqrt{\log(e/r^2)}}^{1} \sqrt{\frac{2(1-w^2)}{1-\frac{1}{3}w^2+O(w^4)}}\, \frac{\mathrm{d}w}{w} \;=\; \int_{\sqrt{\log(e/r^2)}}^{1} \sqrt{2}\left[1-\tfrac{1}{3}w^2+O(w^4)\right]\frac{\mathrm{d}w}{w} \\
&=\; \int_{\sqrt{\log(e/r^2)}}^{1} \sqrt{2}\left[\frac{1}{w} - \frac{w}{3} + O(w^3)\right]\mathrm{d}w = \sqrt{2}\left[\log\, w - O(w^2)\right]_{\sqrt{\log(e/r^2)}}^{1}.
\end{aligned} \tag{47}$$

Hence,

$$\vartheta \;=\; -2^{-1/2}\log\,\log(e/r^2) - O(1). \tag{48}$$

Therefore, on the Critical Brachistochrone, as $r \nearrow \sqrt{e}$ and $w \searrow 0$, the particle spirals outwards, with angle asymptotically

$$\vartheta \;\sim\; \frac{\log\,\log(e/r^2)}{\sqrt{2}}, \tag{49}$$

which increases unboundedly.

Thus, the critical radius $R = \sqrt{e}$ is never attained by the Critical Brachistochrone, and so $\mathrm{d}r/\mathrm{d}t$ never changes sign from + to -. Therefore, in the integrals (4), (5), (6), (7), (8) and (9), the positive square roots are always to be used. Unlike those bounded total brachistochrones with total angle $\geq 2\pi$, the Critical Brachistochrone does not intersect itself.

Thus, the Critical Brachistochrone spirals endlessly out from $r = 1$, winding infinitely often around the centre and approaching asymptotically the critical radius $r = \sqrt{e}$, but never reaching it.

The motion on the particle approaches uniform circular motion with speed $v = 1$ on the critical circle with radius $\sqrt{e}$, so that $\mathrm{d}\vartheta/\mathrm{d}s$ and $\mathrm{d}\vartheta/\mathrm{d}t$ both increase asymptotically towards the limit $1/\sqrt{e}$, and $s/\vartheta \searrow \sqrt{e}$ and $t/\vartheta \searrow \sqrt{e}$. Hence, as the particle spirals outwards on the Critical Brachistochrone,

$$s \;\sim\; t \;\sim\; -\sqrt{\tfrac{e}{2}}\;\log\,\log\left(\tfrac{e}{r^2}\right). \tag{50}$$

Thus the Critical Brachistochrone is bounded in radius, but it is unbounded in angle, in arclength and in time.

3.1 Computation of the Critical Brachistochrone

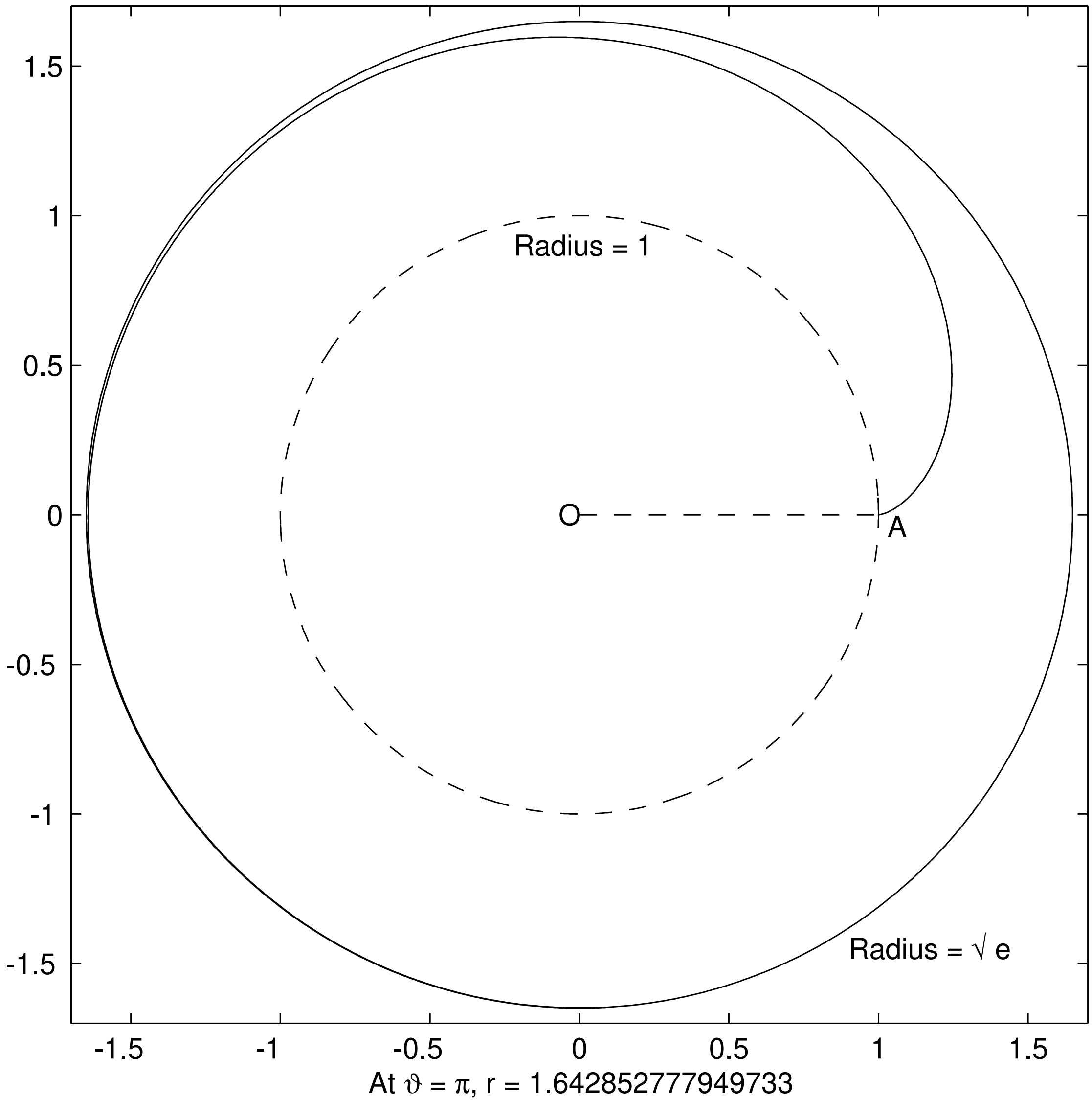
The positive square roots are to be taken in the integrals (4), (5), (6), (7), (8) and (9), with $J = 2e$. Each of those integrals diverges, as $r \nearrow \sqrt{e}$.

In the equations (7) for ϑ and (9) for t, the integrals are then suitable for Romberg integration with respect to v, and the integral (5) for s is then suitable for Romberg integration with respect to x.

3.2 Picture of the Critical Brachistochrone

The Critical Brachistochrone is constructed in Figure 11, from the integral expression (7) for ϑ as a function of r. Note that, for $\vartheta \geq \pi$, the Critical Brachistochrone is very close to the critical radius $r = \sqrt{e} = 1{\cdot}64872127070001281468$. Indeed, solving equation (7) for r with $\vartheta = \pi$, by using the Newton–Raphson method, we get that $r = \tilde{r} = 1{\cdot}642852777949733$ at $\vartheta = \pi$.

Figure 11. Critical Brachistochrone for Repulsive Logarithmic Potential



In each of Figures 1, 2 and 6 to 10 with graphs of bounded brachistochrones for $J > 2e$, the Critical Brachistochrone is included as the graph with $J = 2e$, which is the limit of those graphs as $R \nearrow \sqrt{e}$ and $J \searrow 2e$. Likewise, in Figure 12 and in each of Figures 18 to 23, with graphs of unbounded brachistochrones for $J < 2e$, the Critical Brachistochrone is represented by the graph with $J = 2e$, which is the limit of those graphs as $J \nearrow 2e$. But the Critical Brachistochrone is not shewn in Figures 3, 4 or 5, since each of the first half–arches of the bounded brachistochrones, for R close to $\sqrt{e}$, would be barely distinguishable from the Critical Brachistochrone.

4 Unbounded Brachistochrones

With $0 < J < 2e$ there is no apse at which $\mathrm{d}r/\mathrm{d}s = 0$, and hence $\vartheta,\ s$ and t each increase monotonically with r; and conversely r increases monotonically with ϑ, with s and with t. Hence, no unbounded brachistochrone intersects itself. In the integrals (4), (5), (6), (7), (8) and (9), the positive sign is to be taken everywhere.

For $x > 1$, the minimum value of $\mathcal{G}(\varphi) = \mathcal{G}(-\log x) = x^2 - J\ \log x$ is attained at $x = \sqrt{J/2}$, so that $x < \sqrt{e}$ for $J < 2e$. That minimum value is

$$\mathcal{G}\left(-\log\sqrt{J/2}\right) \ = \ \frac{J}{2}(1 + \log 2 - \log J) \ > \ 0, \tag{51}$$

for all $0 < J < 2e$. Therefore, the factor $\mathcal{G}(\varphi)$ in the denominators of (7), (8) and (9) is positive for all $v > 0,\ \varphi < 0$.

4.1 Form of the Curve

As $r \nearrow \infty$ (and so $s > r - 1 \nearrow \infty$) then $v = \sqrt{2\ \log r}$, and hence

$$t \ = \ \int \frac{\mathrm{d}s}{v} \ \geq \ \int \frac{\mathrm{d}r}{v(r)} \ = \ \int \frac{\mathrm{d}r}{\sqrt{2\ \log r}} \ > \ \int \frac{\mathrm{d}r}{2\ \log r}, \tag{52}$$

which diverges. As $x \nearrow \infty$ the integrand in (4) is asymptotically $\sqrt{J\ \log x}/x^2$, and hence the integral converges as $r \nearrow \infty$. Hence, for any fixed $J < 2e$, ϑ converges to a least upper bound, which we call the *asymptotic angle*. With $J = 0$ the asymptotic angle is 0; and as $J \nearrow 2e$, inside $r = \sqrt{e}$ the unbounded brachistochrone converges to the Critical Brachistochrone, which winds infinitely many times around the centre O. Hence, as $J \nearrow 2e$, the asymptotic angle increases unboundedly, with the path for $r < \sqrt{e}$ (and close to that critical radius) winding unboundedly many times around the centre O.

Reasoning similar to that in ([2], §4.1) shews that, after the particle penetrates the critical radius $r = \sqrt{e}$, the path outside that critical radius (but close to it) also winds unboundedly many times around the centre, as $J \nearrow 2e$.

4.1.1 Angle Between Tangent and Radius Vector

Denote by $\xi(r)$ the angle between the tangent and the radius vector, so that (cf. [1] (18))

$$\sin\xi \ = \ \frac{r\ \mathrm{d}\vartheta}{\mathrm{d}s} \ = \ \frac{Kv}{r} \ = \ \frac{K\sqrt{2\ \log r}}{r} \ = \ \sqrt{J\log r/r^2}\ . \tag{53}$$

At the startpoint with $r = 1,\ \xi(1) = 0$.

The particle never moves at a right–angle to the radius vector for an unbounded brachistochrone — but (for each $J < 2e$, ξ will be maximized at that radius r for which $\log r/r^2$ is maximized. Now,

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^{-2}\ \log r) \ = \ -2r^{-3}\log r + r^{-2}/r \ = \ \frac{1 - 2\ \log r}{r^3}\ . \tag{54}$$

Therefore, as r increases from 1 to $\sqrt{e}$, $(\log r/r^2)$ increases from 0 to its maximum value of $1/(2e)$; and as r increases from $\sqrt{e}$ then $(\log r/r^2)$ decreases.

Hence the maximum angle between the tangent and the radius vector is

$$\xi(\sqrt{e}) \;=\; \arcsin\sqrt{\tfrac{J}{2e}}\ . \tag{55}$$

As $J \nearrow 2e$, then $\xi(\sqrt{e}) \nearrow \frac{1}{2}\pi$.

Thus, for each unbounded brachistochrone the path spiralling outwards will be closest to being circular at the critical radius $r = \sqrt{e}$. For J slightly less than $2e$, with the path winding around O more than once, the successive turns of the spiral are there closest together, as in Figure 16 with $J = 5{\cdot}43655$.

As $r \nearrow \infty$ then $\xi \searrow 0$. Hence, as the particle moves outwards, its direction becomes ever closer to radial, and the brachistochrone converges asymptotically to the radius vector at its asymptotic angle.

4.2 Inversion of Unbounded Brachistochrones

As in [2], unbounded brachistochrones for $r \gg 1$ can conveniently be represented by their inverse images, with each point (r, ϑ) being mapped onto $(1/r, \vartheta)$. As $r \nearrow \infty$ the brachistochrone converges asymptotically to the radius vector at its asymptotic angle. Hence, that radius vector is the tangent to the inverse image at the centre.

4.3 Angle as a Function of Radius for Unbounded Brachistochrones

For unbounded brachistochrones (and for the Critical Brachistochrone), positive signs are to be taken in the integrals (4), (5), (6), (7), (8) and (9), with $J \leq 2e$.

The integral (7) is then suitable for Romberg integration with respect to v, for r up to some finite value η (e.g. $\eta = 1{\cdot}6$).

4.3.1 Unbounded Radius

For $r > \eta$, substitute

$$y \;=\; \frac{1}{\log x}, \tag{56}$$

so that (4) becomes

$$\vartheta(r) \;=\; \int_{1/\log r}^{\infty} \frac{\sqrt{J}e^{-1/y}}{y^2\,\sqrt{y - Je^{-2/y}}}\;\mathrm{d}y. \tag{57}$$

Thus, the asymptotic angle ϑ may conveniently be computed by (7) over $r = [0, \eta]$, and by (57) for $r > \eta$:

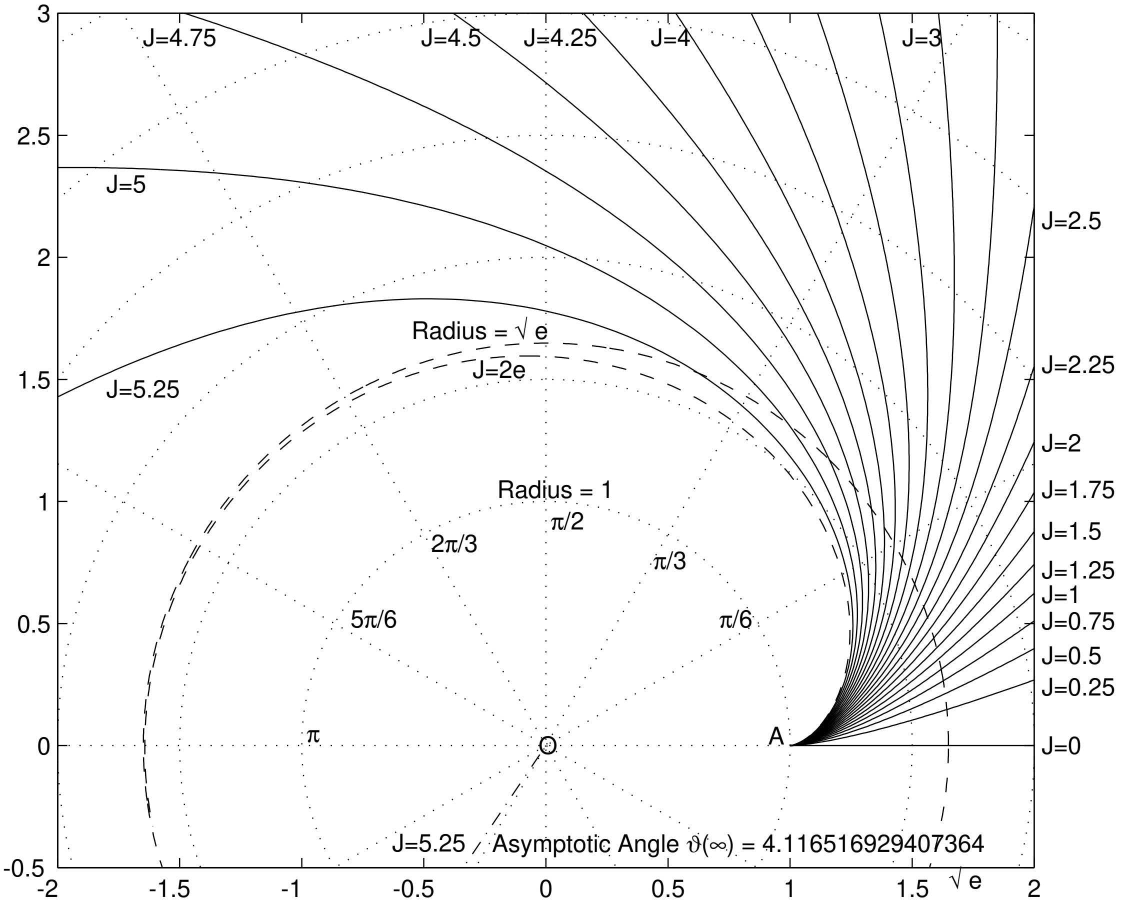
$$\vartheta(\infty) \;=\; \int_0^{\sqrt{2\log\eta}} v^2 \sqrt{\frac{J}{2e^{v^2} - Jv^2}}\;\mathrm{d}v \;+\; \int_0^{1/\log\eta} \frac{\sqrt{J}e^{-1/y}}{y^2\,\sqrt{y - Je^{-2/y}}}\;\mathrm{d}y. \tag{58}$$

Note that $e^{-1/y} < 10^{-43}$ when $0 < y < 0{\cdot}01$; and hence, in computation, $e^{-1/y}$ can usually be replaced by 0 when $0 < y < 0{\cdot}01$. Hence, in (58) the lower limit for integration with respect to y can safely be changed from 0 to $0{\cdot}01$.

4.3.2 Pictures of Unbounded Brachistochrones

Figure 12 shews the unbounded brachistochrones for $J = 0\ (0{\cdot}25)\ 5{\cdot}25$. The unbounded brachistochrone with $J = 0$ is the radius vector $\vartheta = 0$ with $r \geq 1$, as in ([3], Figure 2).

Figure 12. Unbounded Brachistochrones, for J = 0 (0.25) 5.25



The Critical Brachistochrone, with J = 2e, has all r<√ e

If brachistochrones were plotted for equidistant values of K, then they would asymptotically become equally spaced as $r \searrow 1$. But these brachistochrones are plotted for $J = 0,\ 0{\cdot}25, 0{\cdot}5$ *et cetera*, where $J = 2K^2$. Accordingly, in Figure 12 the spacings between successive brachistochrones (e.g. on the circle $r = 1{\cdot}5$) are approximately in the ratio of $1 : (\sqrt{2} - 1) : (\sqrt{3} - \sqrt{2}) : (2 - \sqrt{3}) : \ \ldots$. Also, there is shewn the limit of those unbounded brachistochrones for $r < \sqrt{e}$ as $J \nearrow 2e = 5{\cdot}43656365918090$, which is the (bounded) Critical Brachistochrone with $J = 2e$.

Figure 13 shews the inverses of those unbounded brachistochrones. For $J = 5{\cdot}25$ the tangent at the centre to the inverse curve is drawn, at the asymptotic angle for that brachistochrone of $\vartheta(\infty) = 4{\cdot}116516929407364$.

Figure 13. Inverses of Unbounded Brachistochrones, for J = 0 (0.25) 5.25

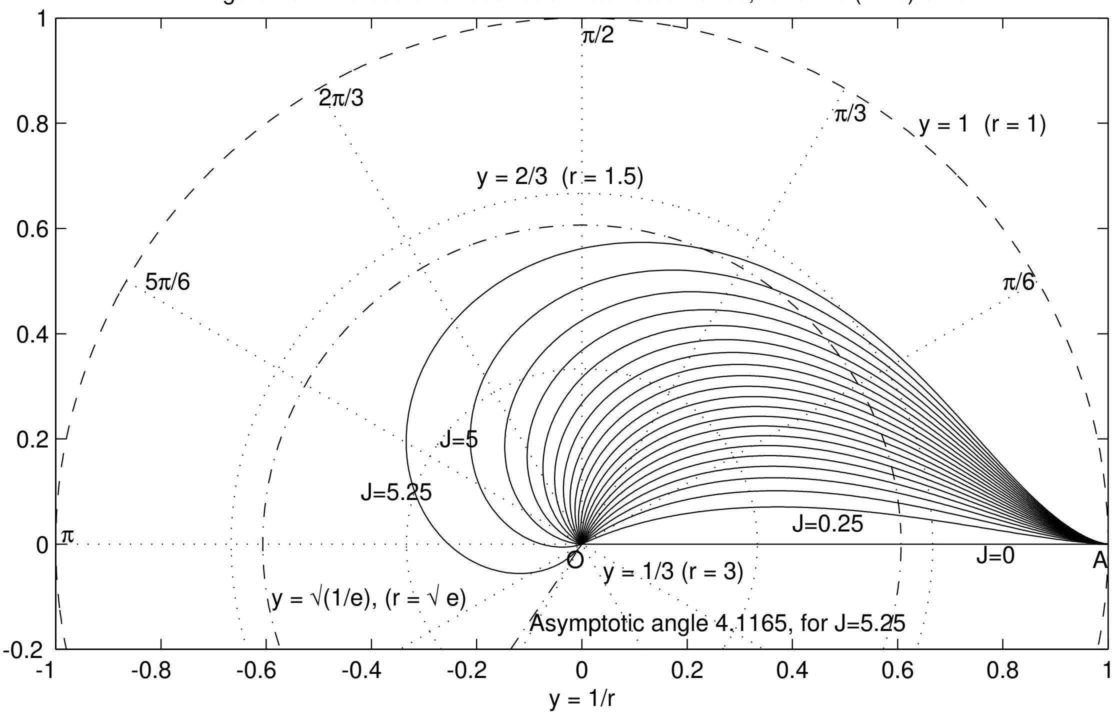


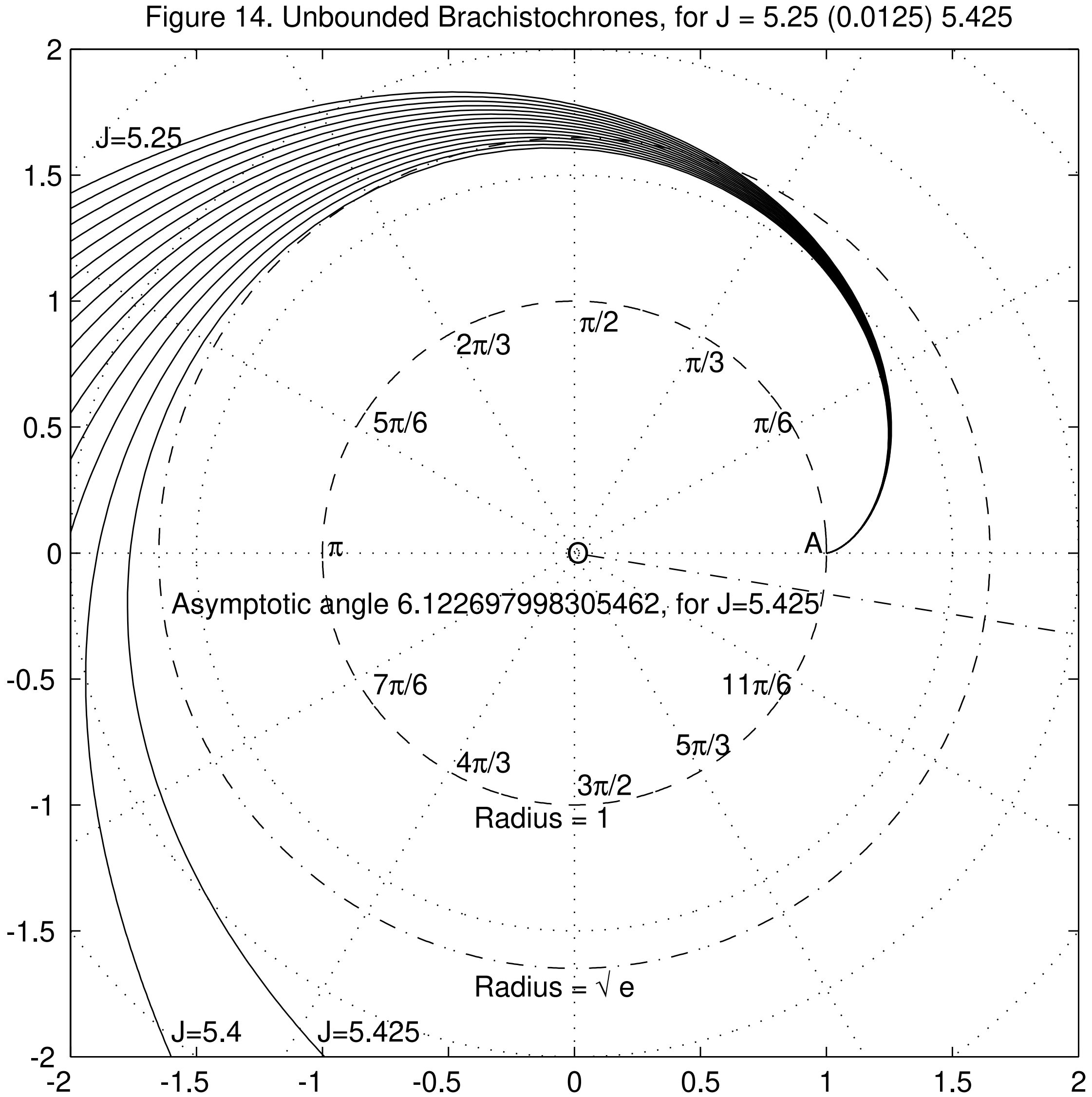
Figure 14 shews the unbounded brachistochrones for $J = 5{\cdot}25\ (0{\cdot}0125)\ 5{\cdot}425$, and Figure 15 shews the inverses of those unbounded brachistochrones. For $J = 5{\cdot}425$ the tangent at the centre to the inverse curve is drawn, at the asymptotic angle for that brachistochrone of $\vartheta(\infty) = 6{\cdot}122697998305462$.

Figure 16 shews the unbounded brachistochrone for $J = 5{\cdot}43655$, and Figure 17 shews the inverse of that unbounded brachistochrone. The tangent at the centre to the inverse curve is drawn, at the asymptotic angle for that brachistochrone of $\vartheta(\infty) = 10{\cdot}893088083061459$.

All graphs in this paper were drawn by the MATLAB 5 procedure "plot", which connects each point to its successor by a straight segment. When the interval $1 \le r \le 2{\cdot}3$ was divided into 5200 equal sub-intervals of $\frac{1}{4000}$, near $r = \sqrt{e}$ the increment in ϑ between successive values of r was large enough for the graph in Figure 16 to appear visibly piecewise–linear. Accordingly, Figure 16 was drawn with the interval $1 \le r \le 2{\cdot}3$ divided into 10400 equal sub-intervals of $\frac{1}{8000}$, and Figure 17 was drawn with the interval $0 \le y \le 1$ divided into 16000 equal sub-intervals of $\frac{1}{16000}$, which did produce smooth curves.

4.4 Arclength as a Function of Radius for Unbounded Brachistochrones

The integrand in (5) is smooth (when the positive square root is used), and hence $s(r)$ can readily be numerically evaluated for $r \ge 1$, by Romberg integration with respect to x.



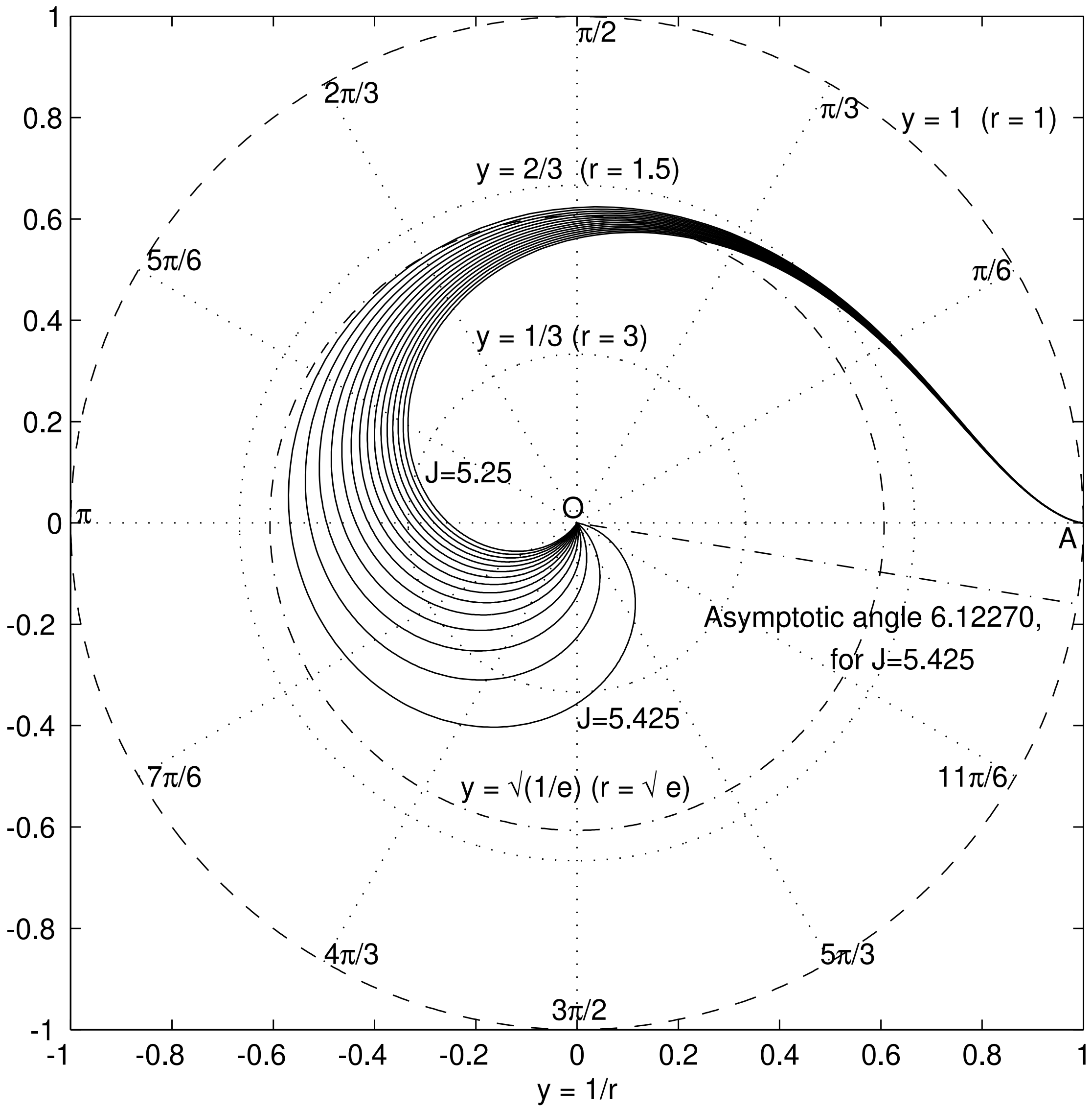
4.4.1 Graphs of Radius versus Arclength

For each unbounded brachistochrone, the angle ξ between the tangent and the radius vector attains its maximum at $r = \sqrt{e}$. Since $\mathrm{d}r/\mathrm{d}s = \cos\xi$, it follows that $\mathrm{d}r/\mathrm{d}s$ attains its minimum value at the critical radius $r = \sqrt{e}$; and hence each graph of r versus s has exactly one point of inflexion, at $r = \sqrt{e}$.

Figure 18 shews graphs of r versus s, for the unbounded brachistochrones with $J = 0$ (0·25) 5·25 . In Figure 18 (and also in Figure 6) the limiting graph for the Critical Brachistochrone, with $J = 2e$, is computed from (8) (with positive square root) by Romberg integration with respect to x.

The graph for $J = 0$, representing radial motion, has the equation $r = s - 1$. The line $r = \sqrt{e}$ is the asymptote for the Critical Brachistochrone graph; and also it is the locus of the points of inflexion on the graphs. The inflexion is seen clearly on the graph for $J = 5{\cdot}25$.

Figure 15. Inverses of Brachistochrones, for J = 5.25 (0.0125) 5.425



As $r \nearrow \infty$ each unbounded brachistochrone with $J < 2e$ converges asymptotically to the radius vector at the asymptotic angle for J. Hence, each graph has slope increasing with s (and r) and converging to 1.

4.5 Time as a Function of Radius for Unbounded Brachistochrones

The integral (9) for t (with positive square root) is suitable for Romberg integration with respect to v.

4.5.1 Graphs of Radius versus Time

Figure 19 shews graphs of r versus t, for the unbounded brachistochrones with $J = 0\ (0{\cdot}25)\ 5{\cdot}25$. In Figure 19 (and also in Figure 7), the limiting graph for the Critical Brachistochrone with $J = 2e$ is constructed by Romberg integration of (9).

Figure 16. Unbounded Brachistochrone, for J = 5.43655

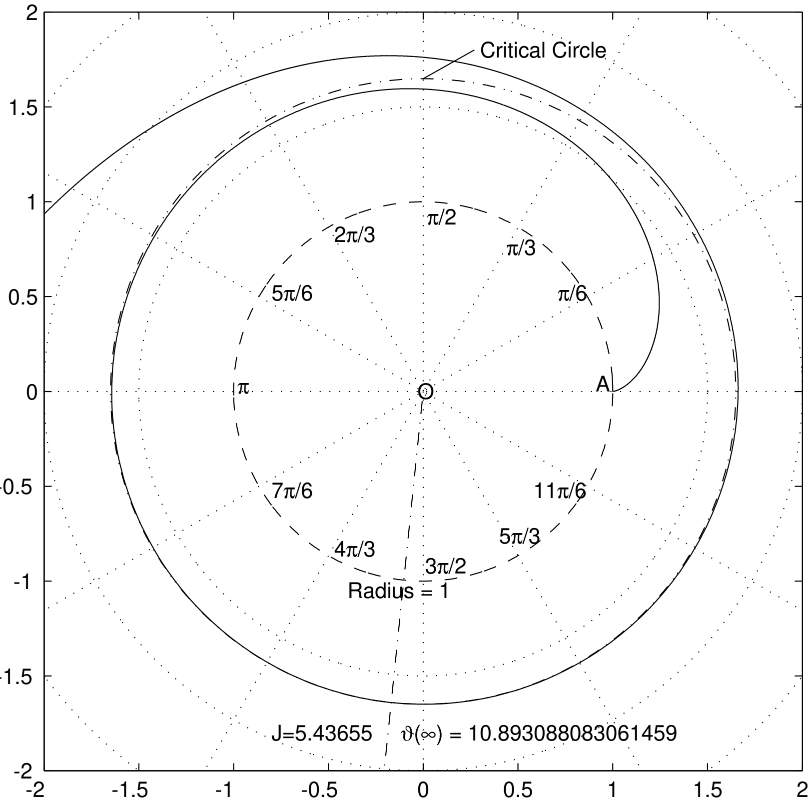


Figure 17. Inverse of Unbounded Brachistochrone, for J = 5.43655

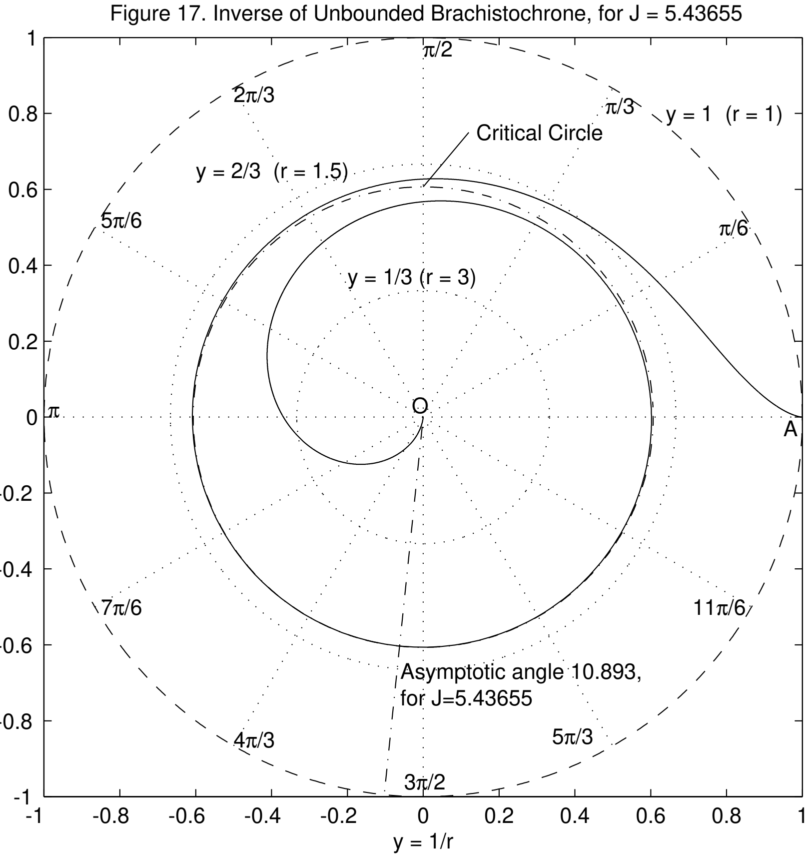


Figure 18: Radius versus Arclength, for J=0 (0.25) 5.25 and 2e

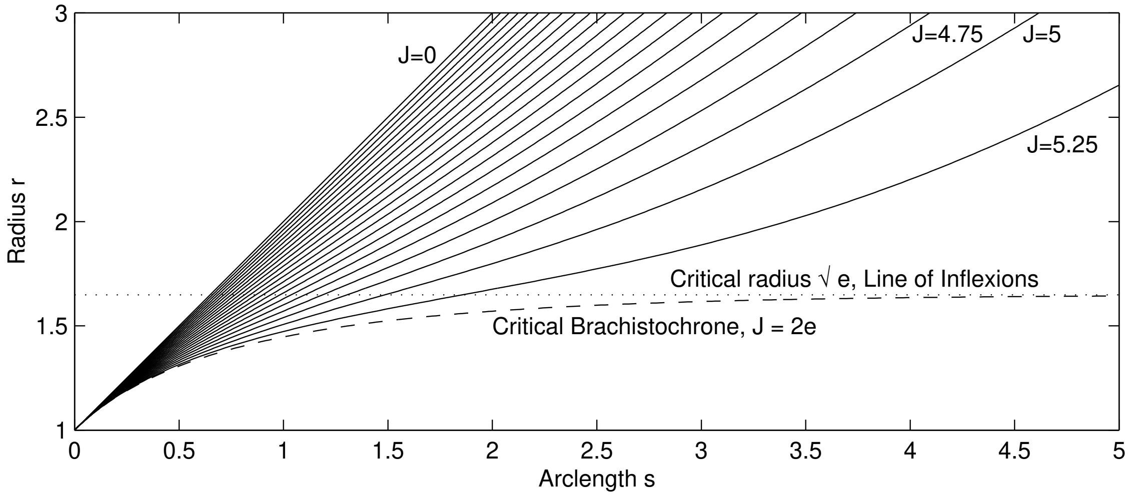
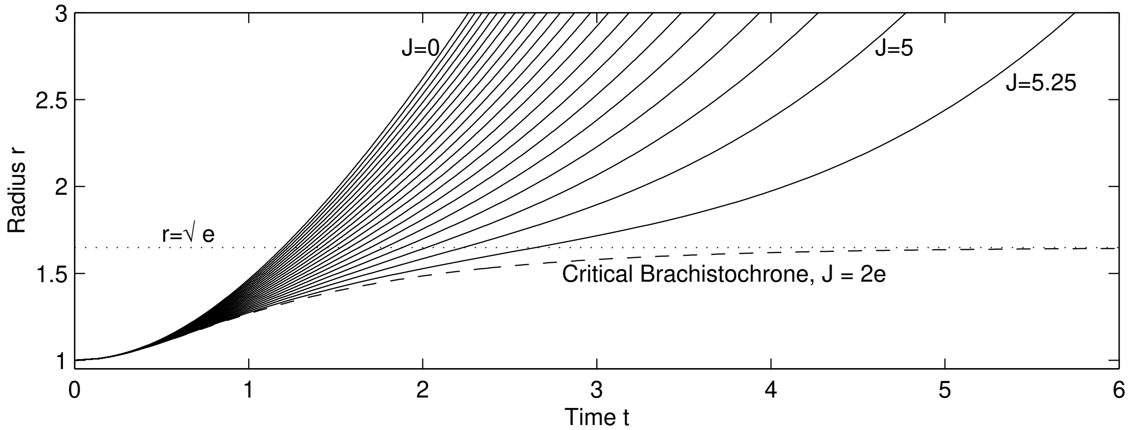


Figure 19: Radius versus Time, for J=0 (0.25) 5.25 and 2e



The graph for $J = 0$, representing radial motion (cf. [3] Figure 2), has the equation

$$t \;=\; \int_0^{\sqrt{2\ \log r}} e^{v^2/2}\ \mathrm{d}v. \tag{59}$$

From (6), the slope of each unbounded brachistochrone $J < 2e$ is

$$\frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \frac{1}{\mathrm{d}t/\mathrm{d}r} \;=\; \sqrt{2\ \log r\left(1 - \frac{2J\ \log r}{r^2}\right)}, \tag{60}$$

which increases unboundedly with r.

The line $r = \sqrt{e}$ is the asymptote for the CriticalBrachistochrone.

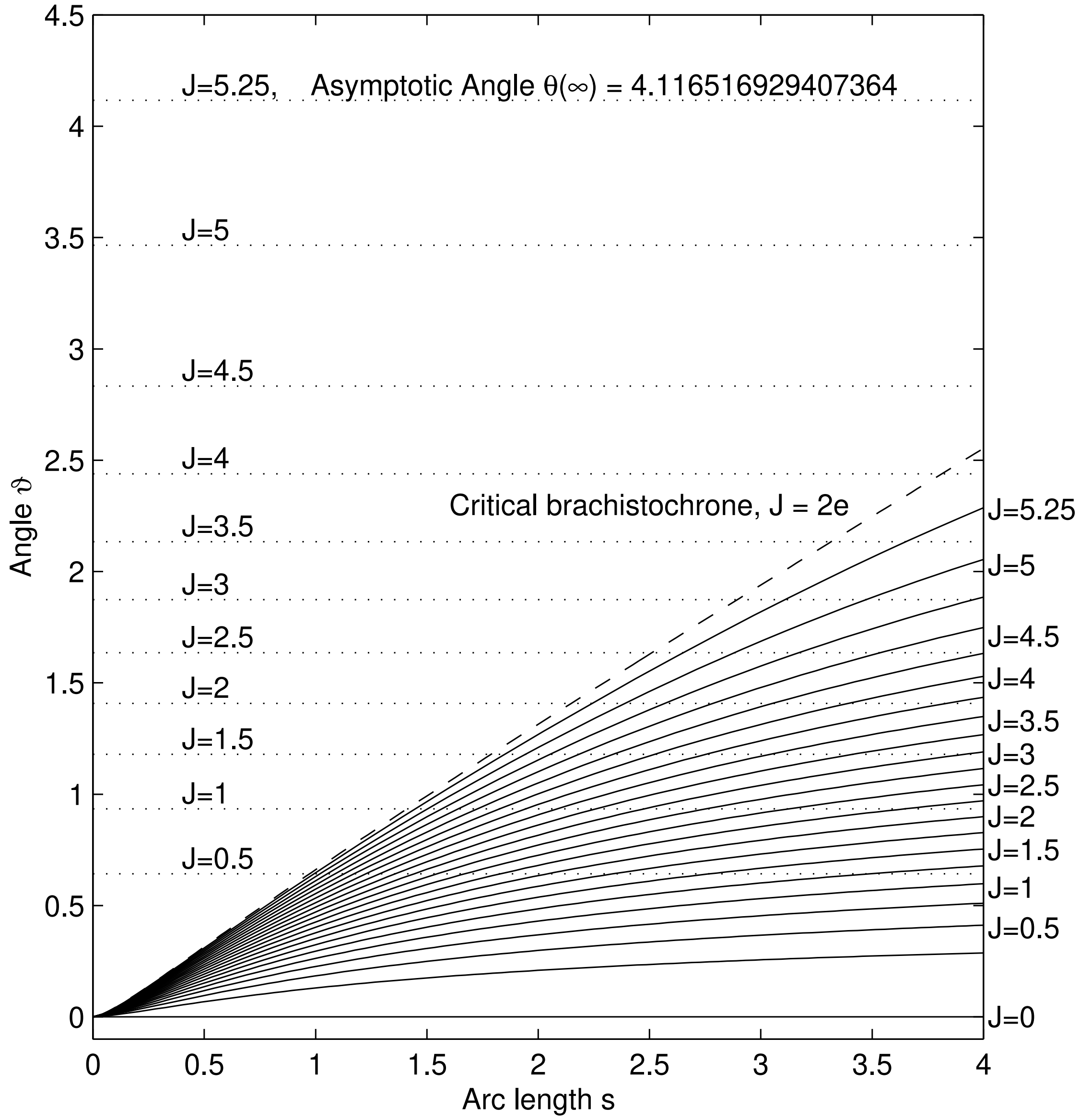
4.6 Angle, Arclength and Time for Unbounded Brachistochrones

Each of $\vartheta,\ s$ and t is a computable function of r. Accordingly, for various values of J, each of the three relations between pairs of these variables can be represented by a graph, with r as the parameter for computing values of the chosen pair of variables.

4.6.1 Relation between Arclength and Angle

Figure 20 shews graphs of ϑ versus s, for the unbounded brachistochrones with $J = 0\ (0{\cdot}25)\ 5{\cdot}25$. For $J < 2e$, as $s \nearrow \infty$ each graph converges towards its horizontal asymptote, with $\vartheta(r) \nearrow \vartheta(\infty)$ for that J.

Figure 20: Angle versus Arclength, for J = 0 (0.25) 5.25 and 2e



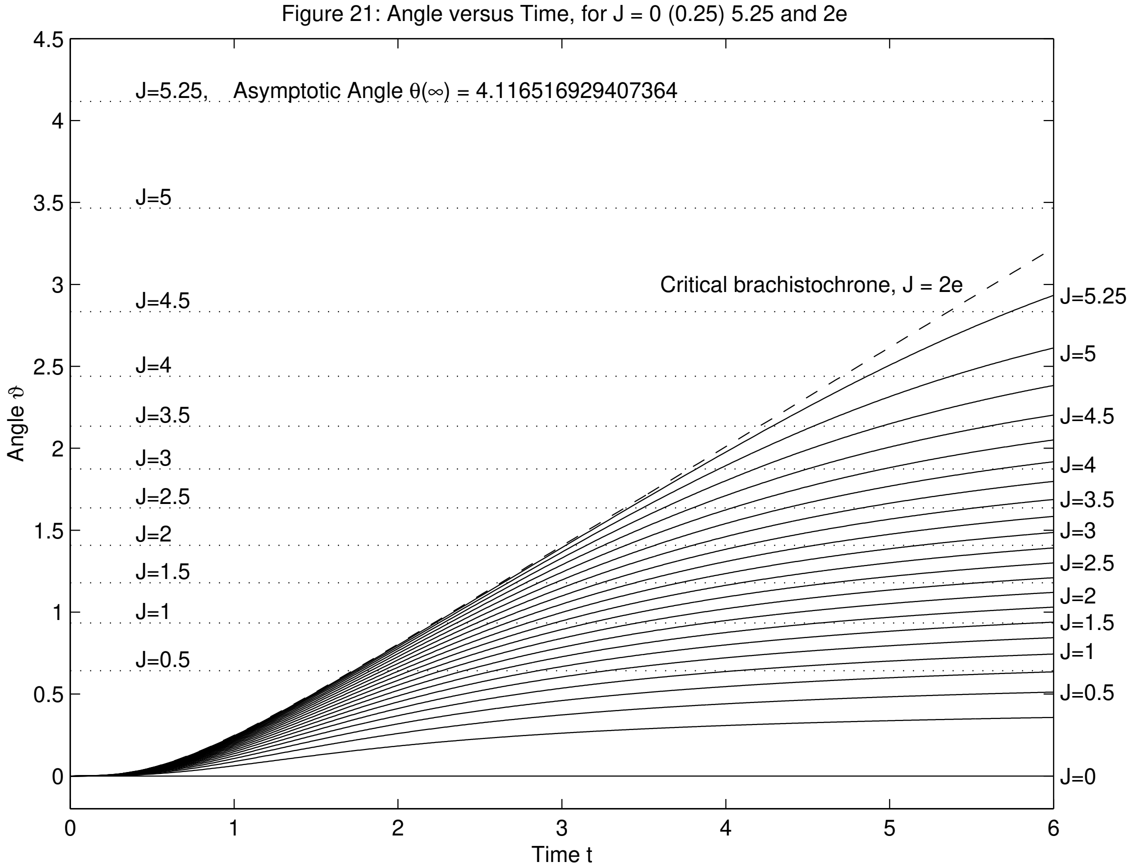
In Figure 20 (and also in Figure 8) the graph for the Critical Brachistochrone with $J = 2e$ is given by Romberg integration of (5), and of (7) with respect to v. As $s \nearrow \infty$, the particle spirals asymptotically towards motion on the circle $r = \sqrt{e}$. Therefore, on the Critical Brachistochrone,

$$\lim_{s\to\infty} \frac{\mathrm{d}\vartheta}{\mathrm{d}s} \;=\; e^{-1/2} \;=\; 0{\cdot}606530659712633424\ . \tag{61}$$

4.6.2 Relation between Time and Angle

Figure 21 shews graphs of ϑ versus t, for the unbounded brachistochrones with $J = 0\ (0{\cdot}25)\ 5{\cdot}25$. For $J < 2e$, as $t \nearrow \infty$ each graph converges towards its horizontal asymptote, with $\vartheta(t) \nearrow \vartheta(\infty)$ for that J.

Figure 21: Angle versus Time, for J = 0 (0.25) 5.25 and 2e



In Figure 21 (and also in Figure 9) the graph for the Critical Brachistochrone with $J = 2e$ is given by Romberg integration of (7) and (9), with respect to v. As $t \nearrow \infty$, the particle spirals asymptotically towards uniform motion on the circle $r = \sqrt{e}$ with speed $v(\sqrt{e}) = 1$. Therefore, on the Critical Brachistochrone,

$$\lim_{t\to\infty} \frac{\mathrm{d}\vartheta}{\mathrm{d}t} \;=\; e^{-1/2}. \tag{62}$$

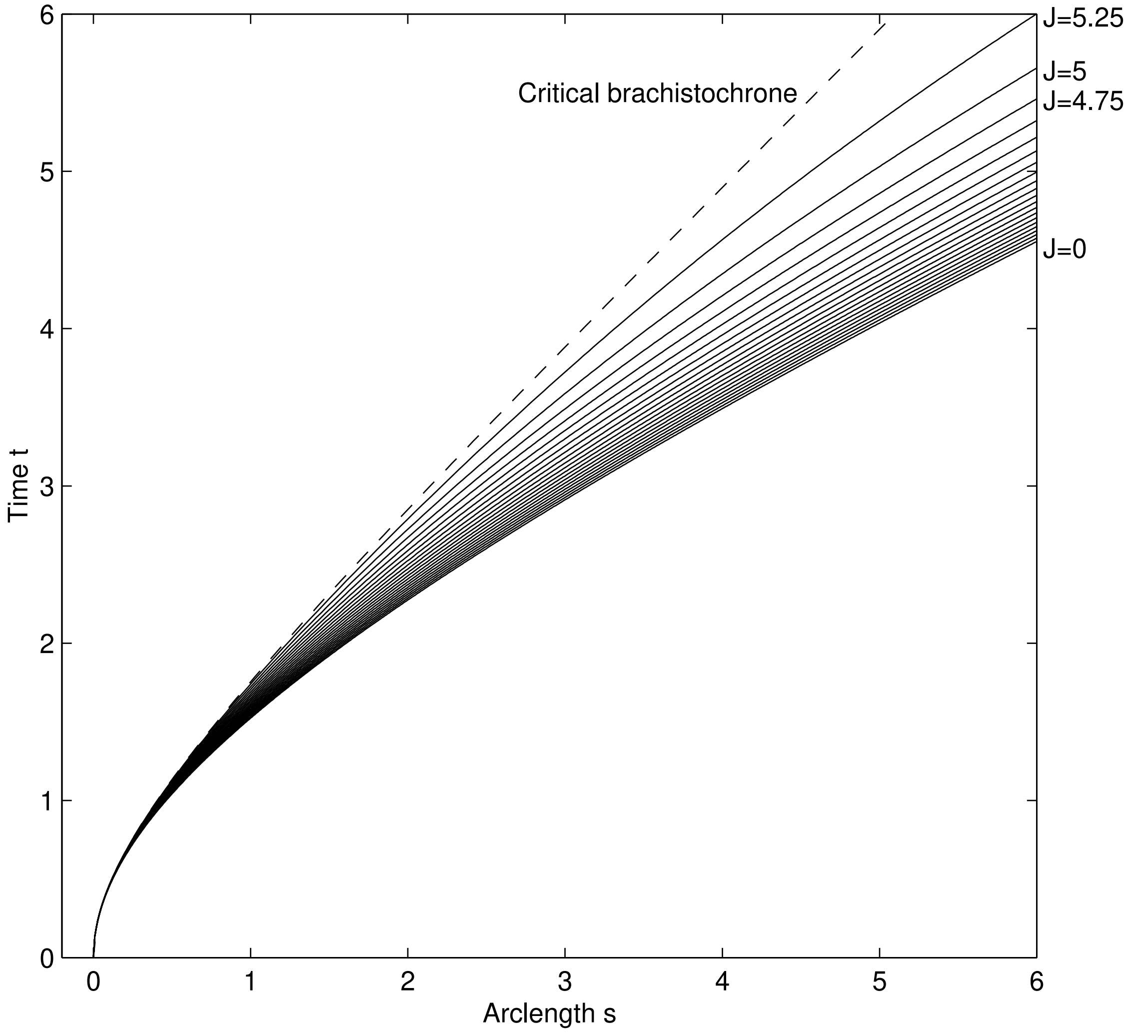
4.6.3 Relation between Arclength and Time

Figure 22 shews graphs of t versus s, for the unbounded brachistochrones with $J = 0\ (0{\cdot}25)\ 5{\cdot}25$. For $J = 0$ with radial motion, $s = r - 1$; and hence the graph is that for $J = 0$ in Figure 19, shifted down by 1 (and with s and t axes interchanged). For $J < 2e$, as $t \nearrow \infty$ the particle's speed $v \nearrow \infty$, and hence on each graph the slope $\mathrm{d}t/\mathrm{d}s$ decreases towards 0.

In Figure 22 (and also in Figure 10) the graph for the Critical Brachistochrone with $J = 2e$ is given by Romberg interation of (5), and of (9) with respect to v. As $t \nearrow \infty$, the particle spirals asymptotically towards uniform motion on the circle $r = \sqrt{e}$ with speed $v(\sqrt{e}) = 1$. Therefore, the limiting slope of the graph for the Critical Brachistochrone is:

$$\lim_{t\to\infty} \frac{\mathrm{d}t}{\mathrm{d}s} \;=\; 1. \tag{63}$$

Figure 22: Time versus Arclength, for J = 0 (0.25) 5.25 and 2e



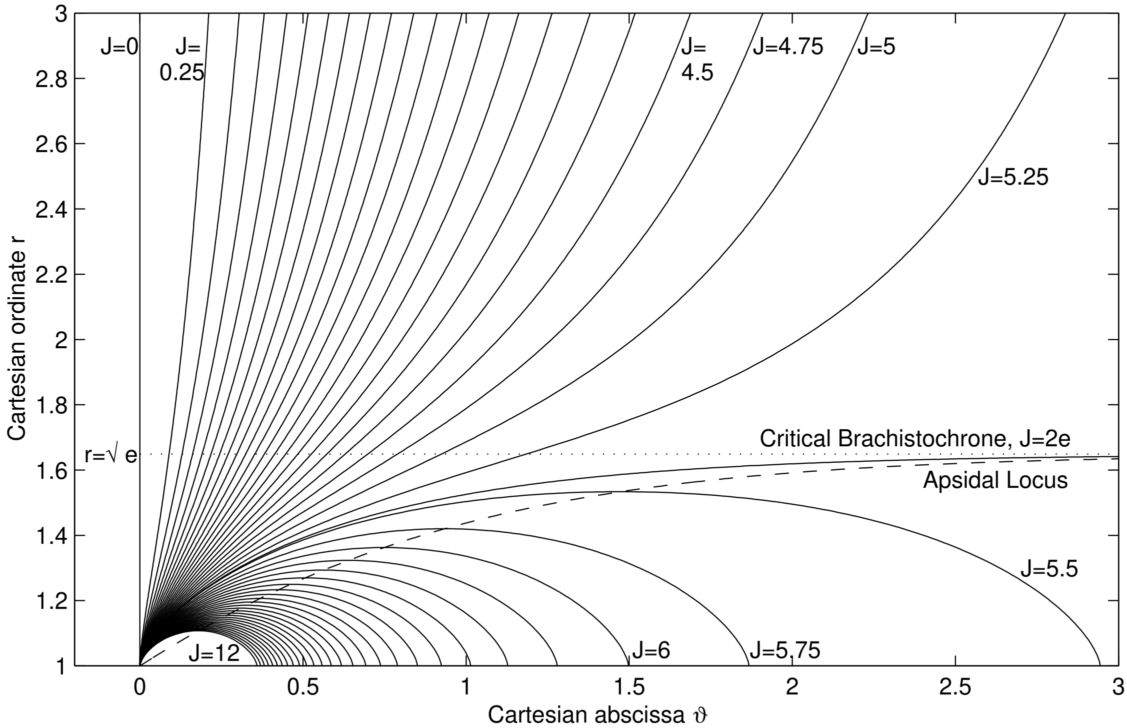
5 Critical Brachistochrone as Separatrix of Bounded and Unbounded Brachistochrones

Figure 23, with rectangular Cartesian axes for ϑ and r, shews graphs for the brachistochrones with $J = 0$ (0·25) 12, the Critical Brachistochrone with $J = 2e$, and the apsidal locus. For each $\vartheta > 0$, the first half–arches of (open) bounded brachistochrones are confined to the range of r between the Apsidal Locus and the Critical Brachistochrone, and the second half–arches are confined between $r = 1$ and the Apsidal Locus. For $J > 2e$, the equation $\mathcal{H}(x) = 0$ (cf. (10)) is solved for R by the bisection method, then for the first half–arch ϑ is computed by (17), and then for the second half–arch ϑ is computed by (20). The graph for each bounded brachistochrone is symmetric about the line $\vartheta = \Theta(R, R)$, the apsidal angle. For each unbounded brachistochrone, as $r \nearrow \infty$, ϑ converges to $\vartheta(\infty)$ for that J.

For brachistochrones at fixed $r \geq 1$, equations (4), (5) and (6) (each taking the positive square root) shew that each of ϑ, s and t is a continuous and strictly increasing function of J. That result holds for all $r \geq 1$ on unbounded brachistochrones with $0 \leq J < 2e$; and for $1 \leq r < \sqrt{e}$ it holds also at $J = 2e$ (on the Critical Brachistochrone), and also on the first half–arches of bounded brachistochrones for $J \geq r^2/\log r > 2e$. These results are illustrated in Figures 1, 3, 4, 5, 11, 12, 14 and

16 for ϑ, in Figures 6 and 18 for s, and in Figures 7 and 19 for t.

Figure 23: Brachistochrones for J = 0 (0.25) 12, Critical Brachistochrone & Apsidal Locus



However, reasoning similar to that in ([1] (83)) shews that, for $J \geq r^2/\log r > 2e$, each of ϑ, s and t is a continuous decreasing function of J on the second half–arches of bounded brachistochrones, where the negative square roots are taken in (4), (5) and (6). Also, the apsidal angle $\Theta(R, R)$ is a decreasing function of J.

For total angle $\vartheta > 0$, r is a decreasing function of J, and J is a decreasing function of r.

Therefore, in the planes with Cartesian coordinates (ϑ, r), (s, r) and (t, r), graphs for distinct brachistochrones never intersect at any point (other than the startpoint), and no graph for any brachistochrone intersects itself. Hence, in each of those 3 Cartesian planes (without the startpoint), the Critical Brachistochrone ($J = 2e$) separates the unbounded brachistochrones ($0 \leq J < 2e$) from the bounded brachistochrones ($J > 2e$). This separation is illustrated in Figure 6 and Figure 18 for (r, s), in Figure 7 and Figure 19 for (r, t), and in Figure 23 for (r, ϑ).

6 Infinite Family of Brachistochrones Through Two Points

Thus, for any point B at (ρ, λ), with $\rho \geq 1$, in the plane OAB there is exactly one brachistochrone ending at B with total angle λ. However, if λ is any nonzero multiple of π, the plane curve can be rotated through any angle about OA, giving an uncountable infinity of brachistochrones with the same passage time.

But, in physical space with polar coordinates (r, ϑ), each physical point (r, ϑ) coincides with the physical points $(r, \vartheta + 2k\pi)$, for all integers k. Hence, each complete bounded brachistochrone with total angle $\geq 2\pi$ intersects itself, as in Figure 3, Figure

4 and Figure 5. For all $r \geq 1$, any point on a brachistochrone with total angle there of ϑ is also on brachistochrones with total angle there of $\vartheta + 2k\pi$, for every integer k.

If $r \geq \sqrt{e}$ then each brachistochrone through C at (r, ϑ) must be unbounded, with $J < 2e$. For each integer k there will be an unbounded brachistochrone through C with total angle $\vartheta + 2k\pi$, which winds positively k times around O from the total angle ϑ, while staying close to $r = \sqrt{e}$ (as in Figure 5 and Figure 16).

For $1 \leq r < \sqrt{e}$, if the Critical Brachistochrone has total angle $> \vartheta$ at radius r, then in Figure 23 C is to the left of the graph for the Critical Brachistochrone; and hence the brachistochrone with total angle ϑ at radius r must be unbounded, with $J < 2e$. If the Critical Brachistochrone has total angle $< \vartheta$ at radius r, then in Figure 23 C is to the right of the graph for the Critical Brachistochrone; and hence the brachistochrone with total angle ϑ at radius r must be bounded, with $J > 2e$. If C is on the Critical Brachistochrone then $J = 2e$ for that total angle ϑ. But, for each integer k the point C in physical space is traversed also by brachistochrones with total angle $\vartheta + 2k\pi$, each of which could be unbounded, critical or bounded.

In each case, as $k \nearrow \infty$ or $k \searrow -\infty$, then $J \to 2e$.

4.7 Global Minimum Passage Time

Reasoning as in ([2], §6.1) shews that, for $\rho \geq 1$, the brachistochrone AB with total angle $0 \leq \lambda \leq \pi$ takes less passage time than does any of the denumerable infinity of brachistochrones ending at B, which wind once or more (in either direction) around the centre. Therefore that curve is the unique curve of quickest descent from A to B — except that if $\lambda = \pi$ (and the force field is radial in 3 dimensions, rather than just a plane field), then the plane curve can be rotated through any angle around OA.

4.8 Computation of Brachistochrones Through Two Points

In order to compute any brachistochrone AB, we need to compute the parameter J for the brachistochrone passing through the endpoint B, with radius ρ and total angle λ (not necessarily $\leq \pi$).

The radius ρ determines some useful bounds for J, without taking the angle λ into account. If $\rho \geq \sqrt{e}$ then $0 \leq J < 2e$. If $1 < \rho < \sqrt{e}$ and $J > \rho^2/\log\rho$, then the maximum radius would be $R < \rho$, which is impossible. Therefore, if $1 < \rho < \sqrt{e}$, then $0 \leq J \leq \rho^2/\log\rho$. If $\rho = 1$ then the brachistochrone is a complete bounded brachistochrone, with $J > 2e$, (unless $\lambda = 0$, in which case J can have any non–negative value).

For fixed radius $\rho \geq 1$, the total angle is given as a function $\Omega(J)$ of the parameter J, where (7) for unbounded brachistochrones (and also for the Critical Brachistochrone, with $\rho < \sqrt{e}$),

$$\Omega(J) \stackrel{\text{def}}{=} \int_0^{\sqrt{2\log\rho}} v^2 \sqrt{\frac{J}{2e^{v^2} - Jv^2}}\, \mathrm{d}v\ . \qquad (J \leq 2e) \tag{64}$$

However, for $\rho \gg 1$, $\Omega(J)$ is better computed as in (58):

$$\Omega(J) \stackrel{\text{def}}{=} \int_0^{\sqrt{2\log\eta}} v^2 \sqrt{\frac{J}{2e^{v^2} - Jv^2}}\,\mathrm{d}v \;+\; \int_{1/\log\rho}^{1/\log\eta} \frac{\sqrt{J}e^{-1/y}}{y^2\sqrt{y - Je^{-2/y}}}\,\mathrm{d}y. \qquad (\rho > \eta,\ J < 2e). \tag{65}$$

For bounded brachistochrones, compute the maximum radius R from J by solving equation (15), and then (25) compute

$$\kappa(-\log\rho) \;=\; \arccos\left(1 - \frac{2\log\rho}{\log R}\right). \tag{66}$$

The endpoint B is on the first half–arch, if and only if λ is less than or equal to the apsidal angle $\Theta(R,R)$ (cf. (34)). In that case (31),

$$\Omega(J) \stackrel{\text{def}}{=} \int_0^{\kappa(-\log\rho)} \frac{\log R\ \sin^2(\kappa/2)}{\sqrt{1 - \log R\,\mathcal{F}(-\log R\ \cos^2(\kappa/2))}}\,\mathrm{d}\kappa\,, \qquad (J > 2e,\ \lambda \leq \Theta(R,R))\,. \tag{67}$$

Otherwise, the endpoint is on the second half–arch, with (20),

$$\Omega(J) \stackrel{\text{def}}{=} 2\Theta(R,R - \int_0^{\kappa(-\log\rho)} \frac{\log R\ \sin^2(\kappa/2)}{\sqrt{1 - \log R\,\mathcal{F}(-\log R\ \cos^2(\kappa/2))}}\,\mathrm{d}\kappa\,, \qquad (J > 2e,\ \lambda > \Theta(R,R))\,. \tag{68}$$

Thus, the construction of the brachistochrone AB has been reduced to computation of the unique non–negative root $J = \tilde{J}$ of the equation

$$\Omega(J) \;=\; \lambda. \tag{69}$$

For any fixed $\rho \geq 1$, the function $\Omega(J)$ can be computed from (64), (65), (67) and (68).

The equation (69) can readily be solved to high accuracy by the secant method, which requires evaluation of the function Ω and which requires two initial estimates J_0 and J_1. When the pair of initial estimates are such that the secant method does converge to a root ($J_n \to \tilde{J}$), then it converges (for a C^2 function) with order $\gamma = (\sqrt{5}+1)/2 = 1{\cdot}6180340$. Since the convergence is of order $\gamma > 1$ (which is faster than linear), then $(J_n - \tilde{J})/(J_n - J_{n-1}) \to 0$ as $n \to \infty$, and so the limit $\tilde{J}$ can reliably be estimated.

Once $\tilde{J}$ has been computed, then the arclength AB can be computed from (5), (8), (38) and (20); and the minimum passage time from A to B can be computed from (9), (40) and (20).

If the computed root $\tilde{J}$ differs from $2e$ by less than some suitable $\epsilon \ll 1$, which allows for roundoff in the arithmetic which has been performed, then the brachistochrone AB is computationally indistinguishable from an arc of the Critical Brachistochrone, with $J = 2e$ exactly.

4.9 Examples of Construction of Brachistochrone Through Two Points

Many examples of brachistochrones through two points have been computed by a program written in Lightspeed PASCAL, using <u>extended</u> variables which have roundoff corresponding to 19 or 20 significant decimal figures. Some examples are presented in Table 1, with initial estimates based on Figure 23. In each case, the secant method ended with two successive estimates of J differing by less than 10^{-16}. (Some other initial estimates J_0 & J_1 gave divergence.)

Here, ρ and λ give the polar coordinates of the endpoint B,
J_0 & J_1 are the initial estimates given for J,
Steps is the number of steps performed of the secant method,
$\tilde{J}$ is the computed value for J,
s is the arclength of the brachistochrone AB, and
t is the time for the particle to reach the endpoint B.

As in ([2], §6.3.1), if the endpoint B is close to the Critical Brachistochrone then, even though the brachistochrone arc AB can be computed to high accuracy, extrapolation of the brachistochrone AB far beyond B may be very ill-conditioned.

5 Higher Powers of Inverse Radius Central Force

Central force proportional to r^{-3} applies to a charge and dipole, when the dipole rotates to keep its axis passing through the charge. Equidistant charges +c -2c +c along an axis converge to give a limiting quadrupole, with force along that axis proportional to r^{-4}; and so a central force proportional to r^{-4} applies to a charge and such a quadrupole, when the quadrupole rotates to keep its axis passing through the charge. And similarly for higher powers of inverse radius.

The details of brachistochrones for such central forces are left as an exercise for the reader.

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Table 1

Brachistochrone Through The Point (ρ, λ)

ρ	λ	J_0	J_1	**Steps**	**Brachistochrone** AB
3	0·3	0·5	0·6	6	$\tilde{J}$ = 0·485084942683883 s = 1·925673720329567 t = 2·570871950631921
1·6	0·5	5	4	7	$\tilde{J}$ = 3·868863878549967 s = 0·406327313616198 t = 0·836541639282948
2·5	1·5	4·8	4·4	7	$\tilde{J}$ = 4·552989324197387 s = 0·785384609449281 t = 3·181666808195187
2·4	2·5	5·25	5·24	6	$\tilde{J}$ = 5·256532082412189 s = 0·529191450779341 t = 4·495886184724548
1·3	0·3	6	5·8	8	$\tilde{J}$ = 5·416026132322073 s = 0·200699678363196 t = 0·406050361583099
1	π	5·5	5·49	7	$\tilde{J}$ = 5·484245087120896 s = 4·758068190271493 t = 6·548129505746249
1·3	2·7	5·5	5·4	9	$\tilde{J}$ = 5·494410501528624 s = 5·208496451752894 t = 1·538392243709977
1	1	8	8·1	10	$\tilde{J}$ = 6·785474122738596 s = 1·386027330503428 t = 2·902651298036730

References

[1] G. J. Tee, "Isochrones and brachistochrones".
University of Auckland Department of Mathematics,
Report Series 405 (1998) 1–23.
(To appear in *Dynamic Systems and Applications*).

[2] G. J. Tee, "Brachistochrones for repulsive inverse square force".
University of Auckland Department of Mathematics,
Report Series 409 (1999) 1–33.

[3] G. J. Tee, "Brachistochrones for attractive logarithmic potential".
University of Auckland Department of Mathematics,
Report Series 410 (1999) 1–16.