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Dade's Conjecture for Steinberg Triality Groups ${}^3D_4(q)$ in Non-defining Characteristics

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0 Introduction

Let G be a finite group, r a prime and B an r-block. Dade in [8] introduced several conjectures on counting the numbers of ordinary irreducible characters in B of a given defect and announced that his final conjecture can be confirmed by verifying it for all non-abelian finite simple groups. In this paper, we prove the final conjecture for the Steinberg triality groups $G = {}^3D_4(q)$ when r is distinct from the characteristic of the group.

We fix some notation and state the invariant conjecture in Section 1, which is equivalent to the final conjecture for G. In Section 2, we choose (and then fix) an embedding of $G_2(q)$ as a maximal subgroup of G and using the subgroup structures of $G_2(q)$ given in [3] and [4], we reduce the family of radical chains to a G-invariant subfamily $\mathcal{R}^0(G)$ and then determine the stabilisers in the automorphism group $\mathrm{Aut}(G)$ of chains in $\mathcal{R}^0(G)$. Dade's invariant conjecture for G is proved when r is odd and even in Sections 3 and 4, respectively using the block structure of G given by Deriziotis and Michler [10].

1 Dade's invariant conjecture

We shall follow the notation of [8]. Given an r-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{1.1}$$

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of a finite group G, define $|C| = n$ and $N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n)$, where $N(P_i)$ is the normaliser $N_G(P_i)$ of P_i in G. In addition, let

$$C_k : P_0 < P_1 < \cdots < P_k$$
$$C^k : 1 < P_{k+1} < \cdots < P_n, \quad 0 \le k \le n-1.$$

The chain C is said to be *radical* if

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$

for $1 \le k \le n$. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical r-chains of G.

Let $\mathrm{Irr}(G)$ be the set of irreducible ordinary characters of G, $\mathrm{Blk}(G)$ the set of r-blocks of G, $D(B)$ a defect group of $B \in \mathrm{Blk}(G)$, $\mathrm{Irr}(B)$ the set of irreducible ordinary characters of B and $\mathrm{Irr}^i(B)$ the characters of $\mathrm{Irr}(B)$ with height i.

If $Z(G) = 1$, we can identify G with its inner automorphism group, so that $G \trianglelefteq A = \mathrm{Aut}(G)$ and $O = \mathrm{Out}(G) = A/G$. Let $N_A(C)$ and $N_A(C, \psi)$ be the stabilizers of C and (C, ψ) in A, respectively, where $\psi \in \mathrm{Irr}(N(C))$. Given $B \in \mathrm{Blk}(G)$, $C \in \mathcal{R}(G)$, an integer $d \ge 0$ and $U \le O$, let $\mathrm{Blk}(N(C)|B) = \{b \in \mathrm{Blk}(N(C)) : b^G = B\}$, and let $\mathrm{k}(N(C), B, d, U)$ be the number of characters in the set

$$\mathrm{Irr}(N(C), B, d, U) = \big\{\psi \in \mathrm{Irr}(N(C)) : B(\psi)^G = B,\ N_O(C, \psi) = U,\ \mathrm{d}(\psi) = d\big\}$$

where $N_O(C, \psi) = N_A(C, \psi)/N(C)$, $B(\psi)$ is the block of $N(C)$ containing ψ and $\mathrm{d}(\psi)$ is the r-defect of ψ.

Dade's Invariant Conjecture ([8] 2.5). If $O_p(G) = 1$ and B is an r-block of G with defect $\mathrm{d}(B) > 0$, then for any integer $d \ge 0$ and any subgroup $U \le \mathrm{Out}(G)$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, U) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

If $O = \mathrm{Out}(G)$ is cyclic, then U is determined uniquely by its order $|U|$. We set $\mathrm{k}(N(C), B, d, |U|) = \mathrm{k}(N(C), B, d, U)$. Since Steinberg triality groups ${}^3D_4(q)$ have a trivial Schur multiplier, the invariant conjecture is equivalent to the final one.

2 The radical chains

The notation and terminology of Section 1 are continued. Let $\mathbb{F}_q$ be the field of $q = p^e$ elements of characteristic p, G the Steinberg triality group ${}^3D_4(q)$, and r a prime distinct from p. Throughout this paper we shall also follow the notation of [10], [17] and [1]. In particular, if L and K are groups, then $L \cdot K$ denotes an extension of L by K, and $L \circ K$

or simply LK denotes a central product of L and K. Given a non-negative integer m, we denote by D_{2m} a dihedral group of order $2m$ and write $\mathbb{Z}_m$ or simply m for the cyclic group of order m. Given a sign δ, $\mathrm{SL}_m^\delta(q) = \mathrm{SL}_m(q)$ or $\mathrm{SU}_m(q)$ according as $\delta = +$ or $-$.

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and α the Frobenius automorphism of $\overline{\mathbb{F}}_q$, so that $t^\alpha = t^p$ for all $t \in \overline{\mathbb{F}}_q$ and $\mathbb{F}_{q^3}$ is the fixed-point set $C_{\overline{\mathbb{F}}_q}(\alpha^{3e})$ of α^{3e}. Let V be an 8-dimensional orthogonal space of type $+$ over $\mathbb{F}_{q^3}$, and $Q : V \to \mathbb{F}_{q^3}$ a non-degenerate quadratic form of Witt defect 0. Denote by $\Delta(V)$ and $\mathrm{O}(V)$ the conformal and orthogonal groups, and by $\Omega(V) = [\,\mathrm{O}(V), \mathrm{O}(V)\,]$ the commutator group of $\mathrm{O}(V)$. In addition, let $P\Omega = \Omega(V)/(Z(\mathrm{GL}(V)) \cap \Omega(V))$ and θ a graph-field triality automorphism such that

$$G = {}^3D_4(q) = C_{P\Omega}(\theta).$$

Let $\mathcal{B} = \{w_0, w_0', v_i, v_i' : 1 \le i \le 3\}$ be a standard basis of V, so that $Q(w_0) = Q(w_0') = Q(v_i) = Q(v_i') = 0$ and $Q(w_0, w_0') = Q(v_i, v_i') = 1$ for $1 \le i \le 3$, where for simplicity, we also denote by Q the symmetric bilinear form of the quadratic form Q. If p is odd, set $v_0 = w_0 - w_0'$ and $v_0' = w_0 + w_0'$, while if $p = 2$, set $v_0 = w_0$ and $v_0' = w_0 + w_0'$. Let

$$W = \mathbb{F}_q v_0 \oplus \mathbb{F}_q v_1 \oplus \mathbb{F}_q v_1' \oplus \mathbb{F}_q v_2 \oplus \mathbb{F}_q v_2' \oplus \mathbb{F}_q v_3 \oplus \mathbb{F}_q v_3',$$

$V_0 = \mathbb{F}_q v_0' \oplus W$ and f the alternating trilinear form on W whose monomials in the basis $\mathcal{B}_W = \{v_0, v_i, v_i' : 1 \le i \le 3\}$ is

$$f = v_0 v_1 v_1' + v_0 v_2 v_2' + v_0 v_3 v_3' + v_1 v_2 v_3 + v_1' v_2' v_3'.$$

Then $V = \mathbb{F}_{q^3} \otimes W \oplus \mathbb{F}_{q^3} v_0' = \mathbb{F}_{q^3} \otimes V_0$ and the restriction Q_W in the basis $\mathcal{B}_W$ of the bilinear form Q is $Q_W = -2v_0 + v_1 v_1' + v_2 v_2' + v_3 v_3'$. By Aschbacher [4], p.205,

$$G_2(q) = \mathrm{O}(W, f, Q_W).$$

The restriction, denoted again by α, of α to $\mathbb{F}_{q^3}$ has order $3e$ and $\langle \alpha \rangle = \mathrm{Aut}(\mathbb{F}_{q^3})$. Define the map

$$(\sum_{i=0}^{3} a_i v_i + b_i v_i')^\alpha = \sum_{i=0}^{3} (a_i)^\alpha v_i + (b_i)^\alpha v_i',$$

where $a_i, b_i \in \mathbb{F}_{q^3}$. Then α is a field automorphism of $P\Omega$ and we may suppose

$$A = \mathrm{Aut}({}^3D_4(q)) = {}^3D_4(q) \rtimes \langle \alpha \rangle. \tag{2.1}$$

Define $U_4 = \mathbb{F}_q v_1 \oplus \mathbb{F}_q v_2 \oplus \mathbb{F}_q v_1' \oplus \mathbb{F}_q v_2'$ and $V_3 = \mathbb{F}_q v_1 \oplus \mathbb{F}_q v_2 \oplus \mathbb{F}_q v_3$. Then $(U_4)^\alpha = U_4$ and $(V_3)^\alpha = V_3$. If $\beta = \alpha^e$, then by [13], p.104, we may suppose $\theta = \beta\gamma$ for some graph

automorphism $\gamma \in \mathrm{Aut}(P\Omega)$ of order 3 with $[\,\beta,\gamma\,]=1$ (see also [17], p.189), moreover, as shown in the proof of [13], (9.1.3) $C_{P\Omega(V_0)}(\gamma) \simeq G_2(q)$. Now $C_{P\Omega}(\beta) = P\Omega(V_0)$ and

$$C_G(\beta) = C_{P\Omega}(\langle\beta,\gamma\rangle) = C_{C_{P\Omega}(\beta)}(\gamma).$$

Thus $C_G(\beta) \simeq G_2(q)$ and $C_G(\beta) \le P\Omega(V_0)$. By Proposition 3.1.1 (i) of [16], $C_G(\beta)$ fixes a non-singular 1-space X in V_0. Since $P\Delta(V_0)$ is transitive on non-singular 1-spaces, we may suppose $X = \mathbb{F}_q v_0'$, so $C_G(\beta) \le \mathrm{O}(W)$ as $W = (\mathbb{F}_q v_0')^\perp$. Since $\mathrm{O}(W)$ has a unique class of subgroups isomorphic to $G_2(q)$ (cf. [16], p.219), we may suppose $G_2(q) = \mathrm{O}(W, f, Q_W) \le G$.

By Proposition 1.2 of [10] or Table I of [17], G contains 7 classes of maximal tori, denoted the representatives by T_δ, $T_{1,\delta}$, $T_{2,\delta}$, and T_3, where $\delta = \pm$, $T_\delta \simeq \mathbb{Z}_{q^3-\delta} \times \mathbb{Z}_{q-\delta}$, $T_{1,\delta} \simeq \mathbb{Z}_{(q^3-\delta)(q+\delta)}$, $T_{2,\delta} \simeq \mathbb{Z}_{q^2+\delta q+1} \times \mathbb{Z}_{q^2+\delta q+1}$, and $T_3 \simeq \mathbb{Z}_{q^4-q^2+1}$. Here, for simplicity, we always identify $q-\delta$ with $q-\delta 1$. Moreover, $N_G(T_\delta)/T_\delta \simeq D_{12}$, $N_G(T_{1,\delta})/T_{1,\delta} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, $N_G(T_{2,\delta})/T_{2,\delta} \simeq \mathrm{SL}_2(3)$, and $N_G(T_3)/T_3 \simeq \mathbb{Z}_4$.

Let $K(q^2) = \mathrm{SL}(\mathbb{F}_{q^2} \otimes V_3)$ and $L_+ = \mathrm{SL}(V_3) \simeq \mathrm{SL}_3(q)$. By (2.3) of [4], L_+ and $K(q^2)$ can be viewed as subgroups of $G_2(q)$ and $G_2(q^2) = \mathrm{O}(\mathbb{F}_{q^2} \otimes W, f, Q_W)$, respectively. Let $r(1) = (v_1, v_2', -v_1, -v_2')(v_2, -v_1', -v_2, v_1')(v_3, v_3')(v_0, -v_0') \in \mathrm{GL}(W)$ (which is denoted by r in [4], p.204). Then $r(1) \in G_2(q)$ and $L_- = C_{r(1)^{-1}K(q^2)r(1)}(\alpha^e) = \mathrm{SU}_3(q) \le G_2(q)$. Moreover, $N_{G_2(q)}(L_\delta) = \langle L_\delta, r(1)\rangle$ and α induces a field automorphism of order e on L_δ. Let

$$M_\delta \simeq (\mathrm{SL}_3^\delta(q) \circ \mathbb{Z}_{q^2+\delta q+1}) \cdot f_\delta \cdot 2,$$

be a maximal subgroup of G, where $f_\delta = (3, q^2+\delta q+1)$.

If $3|q-\delta$, then $Z(L_\delta) = \langle z\rangle$ for some element z of order 3, $L_\delta = C_{G_2(q)}(z) \le C_G(z)$, and by Table 1 of [1] (or Table II of [17]), $C_G(z) = H_\delta$ and $\mathbb{Z}_{q^2+\delta q+1} = O(C_G(z))$, where $H_\delta = (\mathrm{SL}_3^\delta(q) \circ \mathbb{Z}_{q^2+\delta q+1}) \cdot f_\delta$ and the radical $O(K)$ of a finite group K is its largest normal soluble subgroup. Thus α stabilizes L_δ, $\mathbb{Z}_{q^2+\delta q+1}$ and $M_\delta = N_G(L_\delta)$.

(2A) *In the notation above, if $3|q-\delta$, then we may suppose α stabilizes both M_δ and H_δ. In addition,*

$$N_A(M_\delta) = M_\delta \rtimes \langle\alpha\rangle, \qquad M_\delta = \langle H_\delta, \rho\rangle$$

and α induces a field automorphism of order e on $H_\delta/O(H_\delta) = \mathrm{PGL}_3^\delta(q)$, where $[\,\rho,\alpha\,]=1$ and ρ induces the inverse-transpose automorphism on $L_\delta = \mathrm{SL}_3^\delta(q)$.

Proof. Set $\rho = r(1)$. Then ρ induces the inverse-transpose automorphism on $L_\delta = \mathrm{SL}_3^\delta(q)$ and $N_{G_2(q)}(L_\delta) = \langle L_\delta, \rho\rangle$. By Tables 2.2a and 2.2b of [10], $H_\delta/O(H_\delta) \simeq \mathrm{PGL}_3^\delta(q)$ and α induces an automorphism of $H_\delta/O(H_\delta)$. But α is a field automorphism of order e on $PL_\delta = L_\delta/Z(L_\delta) = \mathrm{PSL}_3^\delta(q)$, so α induces a field automorphism on $H_\delta/O(H_\delta)$.

By Theorem 3 (2) of [4], $N_{G_2(q)}(U_4) = \mathrm{SO}(U_4)$, so that α induces a field automorphism of order e on $N_{G_2(q)}(U_4)$. If q is odd, the $Z(\mathrm{SO}(U_4))$ is generated by an involution s_2. Since $\mathrm{SO}(U_4)^\alpha = \mathrm{SO}(U_4)$, it follows that $(s_2)^\alpha = s_2$ and α stabilizes $M_2 = C_G(s_2) \simeq (\mathrm{SL}_2(q^3) \circ \mathrm{SL}_2(q)) \cdot 2$. The proof of the following Lemma is similar to that of [3] (2C) (a).

(2B) *In the notation above, suppose p is odd. Then α stabilizes M_2 and*

$$N_A(M_2) = \langle L_1 \circ L_2, g \rangle \rtimes \langle \alpha \rangle, \tag{2.2}$$

where $L_1 = \mathrm{SL}_2(q^3)$, $L_2 = \mathrm{SL}_2(q)$ and $g \in \mathrm{SO}(U_4)$ is an involution with $[\, g, \alpha \,] = 1$. In addition, $(L_i)^\alpha = L_i$ and α induces a field automorphism of order $3e$ or e on L_i according as $i = 1$ or 2.

Proof. Suppose $H = C_{M_2}(\alpha^e) = \mathrm{SO}(U_4)$. Then $[\, H, H \,] = \Omega(U_4)$, so that $L_1 \circ L_2 = [\, M_2, M_2 \,]$ and α stabilizes $L_1 \circ L_2$. If $q > 3$ and N is a normal subgroup of $L_1 \circ L_2$ such that $|N| > 2$, then $|N \cap L_i| = |L_i|$ or 2. In the former case, $N = L_i$ and in the latter case we may suppose $i = 1$. So $\eta(N)$ is a non-trivial normal subgroup of $L_2/Z(L_2)$, where η is the natural group homomorphism $\eta : L_1 \circ L_2 \to L_2/Z(L_2) = (L_1 \circ L_2)/L_1$. Thus $\eta(N) = L_2/Z(L_2)$ and $N = L_2$, so that $\alpha(L_i) = L_i$ for $i = 1, 2$. If $q = p = 3$, then $L_2 = O(L_1 \circ L_2)$ and $L_1 = C_{L_1 \circ L_2}(L_2)$, so that $(L_i)^\alpha = L_i$ for $i = 1, 2$.

Now $C_{L_1 \circ L_2}(\alpha^e) = [\, C_{G_2(q)}(s_2), C_{G_2(q)}(s_2) \,] = \Omega(U_4) = H_1 \circ H_2$, where $H_i \simeq \mathrm{SL}_2(q)$. The same proof to above shows that if $q > 3$, then H_i are the only normal subgroups of $H_1 \circ H_2$ with order > 2, so $C_{L_1}(\alpha^e) = H_i$ for some i. We may suppose $i = 1$, so that $C_{L_2}(\alpha^e) = H_2 = L_2$. If $q = p = 3$, then $|\alpha| = 3$ on G, so that the induced automorphism on L_i has order 1 or 3. Since $C_{L_1}(\alpha) \circ C_{L_2}(\alpha) = \Omega(U_4)$ is soluble, it follows that $C_{L_1}(\alpha) \neq L_1$ and the order of the reduced automorphism of α on L_1 is 3. By Proposition 7.2 of [13], α induces a field automorphism of order 3 on L_1, and so $C_{L_1}(\alpha) = \mathrm{SL}_2(p)$.

Suppose $e \geq 2$ and e' is a factor of e with $e' \neq e$. Then $C_{L_i}(\alpha^{e'}) \neq L_i$, since $C_{\Omega(U_4)}(\alpha^{e'}) = \mathrm{SL}_2(q^{e'}) \circ \mathrm{SL}_2(q^{e'})$. Thus the order of the induced automorphism of α on L_i is $3e$ or e according as $i = 1$ or 2, Again by Proposition 7.2 of [13], α induces a field automorphism of order $3e$ and e on L_i according as $i = 1$ or 2. By [3] (2.5), $\mathrm{SO}(U_4) = \langle H_1 \circ H_2, g \rangle$ for some involution g with $[\, g, \alpha \,] = 1$. Then $M_2 = C_G(s_2) = \langle L_1 \circ L_2, g \rangle$. This completes the proof.

Let Y be a subgroup of $\mathrm{SO}(\mathbb{F}_{q^2} \otimes U_4)$ such that

$$Y = \left\{ \begin{pmatrix} X & \\ & X^{-t} \end{pmatrix} : X \in \mathrm{GL}(\mathbb{F}_{q^2} v_1 \oplus \mathbb{F}_{q^2} v_2) \right\}.$$

So α induces a field automorphism of order $2e$ on Y and $N_{\mathrm{SO}(\mathbb{F}_{q^2} \otimes U_4)}(Y) = \langle Y, v \rangle$, where $v \in C_{\mathrm{SO}(U_4)}(\alpha) = \mathrm{SO}_4^+(p)$ induces the inverse-transpose automorphism on Y. Thus $J_+ =$

$C_Y(\alpha^e) \simeq \mathrm{GL}_2(q)$ and $J_- = C_{v^{-1}Yv}(\alpha^e) \simeq \mathrm{U}_2(q)$. Let $h \in \mathrm{O}(U_4)$ be the reflection on $v_1 + v_1'$ and $J_\delta^* = h^{-1}J_\delta h$, where $\delta = +$ or $-$. Then J_δ and J_δ^* are not $G_2(q)$-conjugate, $J_\delta^* \leq \mathrm{SO}(U_4)$, $J_\delta^* \simeq \mathrm{GL}_2^\delta(q)$ and α induces a field automorphism of order e on J_δ^*, since $h^\alpha = h$. Let $I_\delta = C_G(Z(J_\delta))$ and $I_\delta^* = C_G(Z(J_\delta^*))$. Then

$$I_\delta = (\mathrm{SL}_2(q) \circ \mathbb{Z}_{q^3-\delta}) \cdot d', \qquad I_\delta^* = (\mathrm{SL}_2(q^3) \circ \mathbb{Z}_{q-\delta}) \cdot d' \tag{2.3}$$

and α stabilizes both I_δ and I_δ^*, where $d' = (2, q-1)$. Since $C_{I_\delta^*}(\alpha^e) = J_\delta^*$, it follows by Proposition 7.2 of [13] that α induces the filed automorphism of order $3e$ on $\mathrm{SL}_2^\delta(q^3)$. Similarly, since $C_{O(I_\delta)}(\alpha^e) = Z(J_\delta) \simeq \mathbb{F}_{q-\delta}$, α induces an automorphism of order $3e$ on $\mathbb{F}_{q^3-\delta}$. It follows that

$$N_A(Z(I_\delta)) = N_G(I_\delta) \rtimes \langle\alpha\rangle, \qquad N_A(Z(I_\delta^*)) = N_G(I_\delta^*) \rtimes \langle\alpha\rangle. \tag{2.4}$$

Let $\overline{T}$ be the largest diagonal subgroup of $\overline{P\Omega} = P\Omega(\overline{\mathbb{F}}_q \otimes V)$. Choose elements $x, y, z \in C_{\overline{P\Omega}}(\alpha)$ so that $N_{\overline{P\Omega}}(\overline{T}) = \langle \overline{T}, x, y, z\rangle$ and $N_{\overline{P\Omega}}(\overline{T})/\overline{T}$ is the Wyle group of $\overline{P\Omega}$. If T is a maximal torus of G, then $T = C_{\overline{T}}(w\theta)$ for some $w \in \langle x, y, z\rangle$. Since $[\, w, \alpha\,] = 1$, it follows that α stabilizes T and so

$$N_A(T) = N_G(T) \rtimes \langle\alpha\rangle, \tag{2.5}$$

in addition, $N_G(T) = \langle T, x_1, x_2, x_3\rangle$ for some $x_i \in C_G(\alpha)$.

Suppose $B \in \mathrm{Blk}(G)$ with $D = D(B)$. If D is cyclic, then Dade's conjecture for B follows by [7] and we may always suppose D is non-cyclic. Thus either $r|q^2-1$ or $r|q^2+\delta q+1$, where $\delta = \pm$. In the former case let r^a or r^{a+1} be the exact power of r dividing q^2-1 according as r is odd or even, and let the sign ϵ be chosen such that r^a divides $q-\epsilon$. If r divides both $q-\epsilon$ and $q^2+\delta q+1$, then $r=3$ and $\delta = \epsilon$. If $r > 3$, then let the sign ϵ be chosen such that $r|q^2+\epsilon q+1$, and let r^a be the exact power of r dividing $q^2+\epsilon q+1$.

If $K \leq G$, then we denote by K^* a non-conjugate subgroup of G which is isomorphic to K. Let $\Phi(G, r)$ be the set of representatives for radical r-subgroups of G. Suppose $R \in \Phi(G, r)$ with $|R| \geq r$. If $r \in \{2, 3\}$, then the local structures of R are given in Sections 1 and 2 of [1]. If $r \geq 5$ and R is non-cyclic, then we may suppose $R = O_r(T)$. If R is cyclic, then we may suppose $R \leq O_r(T)$, so that $T \leq C_G(R)$. By Table 1 of [1], we may suppose $C_G(R) \in \{I_\epsilon, {I_\epsilon}^*\}$ or $C_G(R) = H_\epsilon$ according as $r|q-\epsilon$ or $r|q^2+\epsilon q+1$. In the former case set $\mathbb{Z}_{r^a} = O_r(I_\epsilon)$ and $\mathbb{Z}_{r^a}^* = O_r({I_\epsilon}^*)$. In the latter case set $\mathbb{Z}_{r^a} = O_r(H_\epsilon)$. Thus we may suppose

$$\Phi(G, r) = \begin{cases} \{1, \mathbb{Z}_{r^a}, \mathbb{Z}_{r^a}^*, \mathbb{Z}_{r^a} \times \mathbb{Z}_{r^a}\} & \text{if } r|q-\epsilon \text{ and } r \geq 5, \\ \{1, \mathbb{Z}_{r^a}, \mathbb{Z}_{r^a} \times \mathbb{Z}_{r^a}\} & \text{if } r|q^2+\epsilon q+1 \text{ and } r \geq 5. \end{cases}$$

Suppose $r = 3$, then let $\mathbb{Z}_3 = O_3(M_\epsilon)$ and $\mathbb{Z}_{3^{a+1}} = O_3(I_\epsilon)$. If $a \neq 1$, then set $\mathbb{Z}_{3^a} = O_3({I_\epsilon}^*)$ and if $a = 1$, then set $\mathbb{Z}_3^* = O_3({I_\epsilon}^*)$. By [1] (1A) and (1B), we may suppose

$$\Phi(G,3) = \begin{cases} \{1, \mathbb{Z}_3, \mathbb{Z}_3^*, \mathbb{Z}_9, O_3(T_\epsilon), O_3(T_{2,\epsilon}), 3_+^{1+2}, S\} & \text{if } a = 1 \text{ and } 3 \neq q - \epsilon, \\ \{1, \mathbb{Z}_3, \mathbb{Z}_3^*, \mathbb{Z}_9, O_3(T_{2,\epsilon}), 3_+^{1+2}, S\} & \text{if } a = 1 \text{ and } 3 = q - \epsilon, \\ \{1, \mathbb{Z}_3, \mathbb{Z}_{3^a}, \mathbb{Z}_{3^{a+1}}, O_3(T_\epsilon), O_3(T_{2,\epsilon}), 3_+^{1+2}, S\} & \text{if } a \geq 2 \text{ and } 3^a \neq q - \epsilon, \\ \{1, \mathbb{Z}_3, \mathbb{Z}_{3^a}, \mathbb{Z}_{3^{a+1}}, O_3(T_{2,\epsilon}), 3_+^{1+2}, S\} & \text{if } a \geq 2 \text{ and } 3^a = q - \epsilon, \end{cases}$$

where 3_+^{1+2} is an extraspecial subgroup of L_ϵ with exponent 3, and $S \in \mathrm{Syl}_3(G)$ is a Sylow 3-subgroup of order 3^{2a+2}.

Suppose $T'_\delta = \mathbb{Z}_{q-\delta} \times \mathbb{Z}_{q-\delta}$ is a maximal torus of L_δ. We may suppose $T'_\delta \leq T_\delta \leq H_\delta$, so $T'_\delta = T_\delta \cap G_2(q) = T_\delta \cap L_\delta$. Let

$$N_{G_2(q)}(T_\delta \cap L_\delta) = \langle T_\delta \cap L_\delta, \sigma, \tau, \rho \rangle,$$

where σ, τ and ρ are given by [3] (2.2). Then $N_{G_2(q)}(T_\delta \cap L_\delta)/(T_\delta \cap L_\delta) \simeq D_{12}$, $|\sigma| = 3$, $|\rho| = 2$, $\sigma, \tau \in L_\delta$ and $x^\alpha = x$ for each $x \in \langle \sigma, \tau, \rho \rangle$. Since $C_G(T_\delta \cap L_\delta) = T_\delta$, it follows that $\sigma, \tau, \rho \in N_G(T_\delta)$. But $N_G(T_\delta)/T_\delta \simeq D_{12}$, so

$$N_G(T_\delta) = \langle T_\delta, \rho, \sigma, \tau \rangle. \tag{2.6}$$

Thus σ and τ both centralize each element of $Z(H_\delta)$, and $N_G(T_\delta) = N_{M_\delta}(T_\delta)$.

Suppose $r = 2$, so that q is odd. A 2-group R is called of symplectic type if R is a central product $E \circ P$ over $Z(E) = \Omega_1(Z(P))$ of an extraspecial 2-group E and a 2-group P such that P is either cyclic or isomorphic to a semidihedral group SD_{2^β}, a dihedral group D_{2^β} or a generalized quaternion group Q_{2^β} of order 2^β, where $\beta \geq 4$.

Let $\mathbb{Z}_2 = O_2(M_2)$, $\mathbb{Z}_{2^a} = O_2(I_\epsilon)$ and $\mathbb{Z}_{2^a}^*$. By Section 2 of [1] (with a few corrections), we may suppose

$$\Phi(G,2) = \{1, \mathbb{Z}_2, \mathbb{Z}_{2^a}, \mathbb{Z}_{2^a}^*, O_2(T_\delta), O_2(T_{1,\delta}), E_8, \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, \langle O_2(T_\epsilon), \rho \rangle, EP, S\},$$

where $\delta = \pm$, E_8 is elementary abelian of order 8, ρ is given by (2.6), EP is a non-abelian 2-group of symplectic type and $S \in \mathrm{Syl}_2(G)$. In addition, G has exactly one or two classes of subgroups $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ according as $q - 1 = 2^a$ or $q - 1 \neq 2^a$.

Suppose $r \geq 3$. Define

$$\begin{aligned} C(1) &: \begin{cases} 1 < \mathbb{Z}_{r^a} < \mathbb{Z}_{r^a} \times \mathbb{Z}_{r^a} & \text{if } r \geq 5, \\ 1 < O_3(M_\epsilon) < O_3(T_{2,\epsilon}) & \text{if } r = 3, \end{cases} \\ C(2) &: 1 < \mathbb{Z}_{r^a}^* < O_r(T_\epsilon) \quad \text{if } r \geq 3 \text{ and } r | q - \epsilon, \end{aligned} \tag{2.7}$$

where for simplicity, we set $\mathbb{Z}_{3^a}^* = \mathbb{Z}_{3^a}$ when $a \geq 2$. If $r \geq 5$, then

$$\mathcal{R}(G)/G = \begin{cases} \{C(1)_0, C(1)_1, C(2)_1, C(1)^1, C(1), C(2)\} & \text{if } r|q-\epsilon, \\ \{C(1)_0, C(1)_1, C(1)^1, C(1)\} & \text{if } r|q^2+\epsilon q+1, \end{cases}$$

and we set $\mathcal{R}^0(G) = \mathcal{R}(G)$. If $r = 3$, then let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that

$$\mathcal{R}^0(G)/G = \{C(1)_0, C(1)_1, C(1)^1, C(1), C(2)_1, C(2)\}.$$

(2C) *Let $G = {}^3D_4(q)$. In the notation above, suppose $r \geq 3$. If $B \in \mathrm{Blk}(G)$ with $\mathrm{d}(B) > 0$ and $d, u \in \mathbb{Z}$, then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, u) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, u).$$

Proof. Let C' be a radical r-chain

$$C' : 1 < P_1' < \ldots < P_{n'}'. \tag{2.8}$$

Suppose $r = 3$ and $C \in \mathcal{R}(G)$ given by (1.1). We may suppose $P_1 \in \Phi(G, r)$.

Suppose $P \in \Phi(M_\epsilon, 3)$. If $Z(P)$ is cyclic, then $\mathbb{Z}_3 = \Omega_1(Z(P))$, $N_G(P) \leq N_G(\mathbb{Z}_3) = M_\epsilon$ and P is a radical subgroup of G. If $Z(P)$ is non-cyclic, then we may suppose $T = C_G(\Omega_1(Z(P))) = T_\epsilon$ or $T_{2,\epsilon}$ as shown in the proof of [17], p.187. Thus $P \leq T$, $C_G(P) = C_{M_\epsilon}(P) = T$ and by Lemma 2.1 of [19], $O_3(T) \leq P$, so that $P = O_3(T)$. We may suppose

$$\Phi(M_\epsilon, 3) = \{\mathbb{Z}_3, \mathbb{Z}_{3^{a+1}}, O_3(T_\epsilon), O_3(T_{2,\epsilon}), 3_+^{1+2}, S\} \subseteq \Phi(G, 3).$$

Let $R \in \Phi(M_\epsilon, 3) \backslash \{\mathbb{Z}_3, O_3(T_\epsilon), O_3(T_{2,\epsilon})\}$. Then $O_3(M_\epsilon) = \Omega_1(Z(R)) = \mathbb{Z}_3$, so $N_G(R) \leq N_G(\mathbb{Z}_3)$ and $N_A(R) \leq N_A(\mathbb{Z}_3)$. If $R = O_3(T_\epsilon)$, then $N_A(R) = N_A(T_\epsilon) = N_G(T_\epsilon) \rtimes \langle \alpha \rangle \leq N_A(\mathbb{Z}_3)$. Define G-invariant subfamilies $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ such that

$$\begin{aligned} \mathcal{M}^+(R)/G &= \{C' \in \mathcal{R}(G)/G : P_1' = R\}, \\ \mathcal{M}^0(R)/G &= \left\{C' \in \mathcal{R}(G)/G : P_1' = \mathbb{Z}_3,\ P_2' = R\right\}. \end{aligned} \tag{2.9}$$

For each $C' \in \mathcal{M}^+(R)$ given by (2.8), the chain

$$g(C') : 1 < \mathbb{Z}_3 < R < P_2' < \ldots < P_{n'}'$$

is a chain of $\mathcal{M}^0(R)$ and $|C'| = |g(C')| - 1$. Moreover, $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C'))$. Thus

$$\mathrm{k}(N(C'), B, d, u) = \mathrm{k}(N(g(C')), B, d, u), \tag{2.10}$$

for each $C' \in \mathcal{M}^+(R)$, $B \in \mathrm{Blk}(G)$ with $\mathrm{d}(B) > 0$ and $d, u \in \mathbb{Z}$. We may suppose

$$C \notin \bigcup_{R \in \Phi(M_\epsilon, 3) \setminus \{\mathbb{Z}_3, O_3(T_{2,\epsilon})\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

In particular, P_1 and R are not G-conjugate and if $P_1 = \mathbb{Z}_3$, then P_2 and R are not G-conjugate, so that $C = C(1)_1$ or P_2 is conjugate to $O_3(T_{2,\epsilon})$.

Let $Z(H_\epsilon) = \mathbb{Z}_{q^2+\epsilon q+1} = \langle x \rangle$, so that $\langle x \rangle$ contains an element x_3 of order 3 and $O_3(M_\epsilon) = \langle x_3 \rangle = O_3(L_\epsilon)$. By Table 1.1 and Table 2.1 of [10], we may suppose $T = T_{2,\epsilon} \leq C_G(x) = H_\epsilon = U \times \mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)}$, where $U = L_\epsilon \cdot \mathbb{Z}_3$. Thus $T = \langle x \rangle \times \langle y \rangle$ for some element y of order $q^2+\epsilon q+1$ and $\langle y \rangle$ contains a unique subgroup $\langle y_3 \rangle$ of order 3. Thus $y_3 \in U$ and $U = \langle L_\epsilon, y_3 \rangle$. If $T' = T \cap G_2(q)$, then $T' = \mathbb{Z}_{q^2+\epsilon q+1}$ and by [4], p.254 or the remark before [3] (2.2), $\sigma, \rho \in N_{G_2(q)}(T')$ and $N_{G_2(q)}(T') = \langle T', \sigma, \rho \rangle$. Thus $\rho, \sigma \in N_G(T)$ as $M_\epsilon = \langle H_\epsilon, \rho \rangle$ and $T = C_{M_\epsilon}(T')$. For any $g \in N_{M_\epsilon}(T)$, we may suppose $g \in L_\epsilon$, since $M_\epsilon = \langle H_\epsilon, \rho \rangle$ and $H_\epsilon = \langle L_\epsilon, y_3 \rangle \times \mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)}$ Thus $g \in N_{G_2(q)}(T')$ and

$$N_{M_\epsilon}(T_{2,\epsilon}) = \langle T_{2,\epsilon}, \sigma, \rho \rangle. \tag{2.11}$$

The actions of σ, ρ on T' are given by [4], p.254. Thus ρ inverses each $t \in T'$ and σ centralizes each $t \in \mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)}$ as $\sigma \in L_\epsilon$. Since $\rho \notin H_\epsilon$, $\langle Z(H_\epsilon), \rho \rangle$ is nonabelian. But $Z(H_\epsilon) \leq C_{P\Omega}(L_\epsilon) \simeq \mathrm{O}(\mathbb{F}_{q^3} w_0 \oplus \mathbb{F}_{q^3} w_0')$ and $\mathrm{O}(\mathbb{F}_{q^3} w_0 \oplus \mathbb{F}_{q^3} w_0')$ is dihedral, so $\langle Z(H_\epsilon), \rho_1 \rangle$ is dihedral and ρ inverses each element of $Z(H_\epsilon)$, where $\rho_1 \in C_{P\Omega}(L_\epsilon)$ is the restriction of ρ to the subspace $\mathbb{F}_{q^3} w_0 \oplus \mathbb{F}_{q^3} w_0'$. So ρ inverses each element of $T = T_{2,\epsilon}$.

We may suppose $P_1 \in \{\mathbb{Z}_3^*, \mathbb{Z}_{3^a}, O_3(T_{2,\epsilon})\}$. If $P_1 = \mathbb{Z}_3^*$ or $\mathbb{Z}_{3^a}$, then $N_G(P_1) = N_G(I_\epsilon{}^*)$ and we may suppose $T_\epsilon \leq C_G(P_1) = I_\epsilon{}^*$. Since $\Phi(I_\epsilon{}^*, 3) = \{P_1, O_3(T_\epsilon)\}$, it follows that C is conjugate to either $C(2)_1$ or $C(2)$. If $P_1 = O_3(T_{2,\epsilon})$, then $N_G(P_1) = N_G(T_{2,\epsilon}) = T_{2,\epsilon} \cdot \mathrm{SL}_2(3)$, and $\Phi(N_G(T_{2,\epsilon}), 3) = \{P_1, S'\}$ for some $S' \in \mathrm{Syl}_3(N_G(P_1))$. Let

$$C' : 1 < O_3(T_{2,\epsilon}) < S' \quad \text{and} \quad g(C') : 1 < \mathbb{Z}_3 < O_3(T_{2,\epsilon}) < S'.$$

By (2.11) we may suppose $N_G(T_{2,\epsilon})/T_{2,\epsilon} = \mathrm{SL}_2(3) = Q_8 \rtimes \langle \sigma \rangle$ and $S' = \langle O_3(T_{2,\epsilon}), \sigma \rangle$, where Q_8 is the quaternion group. Since $N_{\mathrm{SL}_2(3)}(\langle \sigma \rangle) = \langle \rho\sigma \rangle$, $N_{N_G(T_{2,\epsilon})}(S') \leq \langle T_{2,\epsilon}, \sigma, \rho \rangle = N_{M_\epsilon}(T_{2,\epsilon})$, $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C')) = \langle N(C'), \alpha \rangle$, so that (2.10) holds. Thus if $P_1 = O_3(T_{2,\epsilon})$ or $P_1 = \mathbb{Z}_3$ and P_2 is conjugate to $O_3(T_{2,\epsilon})$, then C is conjugate to either $C(1)^1$ or $C(1)$. This completes the proof.

Define 2-chains $C(i)$ for $1 \leq i \leq 4$ as follows:

$$\begin{aligned} &C(1) : 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon}), \\ &C(2) : 1 < \mathbb{Z}_2 < O_2(T_\epsilon), \\ &C(3) : 1 < E_8 < 2_+^{1+4}, \\ &C(4) : 1 < E_8 < \langle \Omega_2(O_2(T_\epsilon)), \rho \rangle < S'', \end{aligned} \tag{2.12}$$

where $E_8 = \langle \Omega_1(O_2(T_\epsilon)), \rho\rangle$, 2_+^{1+4} is an extraspecial subgroup of order 2^5 with plus type in $N_G(E_8)$ and $S'' = \langle \Omega_2((O_2(T_\epsilon)), \rho, \tau\rangle$. Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that

$$\mathcal{R}^0(G)/G = \{C(1)_0, C(1)_1, C(1)^1, C(1), C(2)^1, C(2), C(3)_1, C(3), C(4)_2, C(4)\}.$$

(2D) *Let* $\mathrm{Blk}^0(G, r)$ *be the set of blocks of* $G = {}^3D_4(q)$ *with non-cyclic defect groups, and* $B \in \mathrm{Blk}^0(G, 2)$. *Then for non-negative integers* d, u

$$\sum_{C\in\mathcal{R}(G)/G} (-1)^{|C|}\,\mathrm{k}(N(C), B, d, u) = \sum_{C\in\mathcal{R}^0(G)/G} (-1)^{|C|}\,\mathrm{k}(N(C), B, d, u).$$

Proof. Suppose $C \in \mathcal{R}(G)$ given by (1.1) with $P_1 \in \Phi(G, 2)$.

(1) Let $M = M_2$ and $\mathcal{X} = \{1, \mathbb{Z}_2, O_2(T_\delta), E_8, \langle O_2(T_\epsilon), \rho\rangle\} \subseteq \Phi(G, 2)$. As shown in the proofs of [1] (2B) and (2C) $\Phi(M, 2)\backslash\{\mathbb{Z}_2, O_2(T_\delta)\} = \Phi(G, 2)\backslash\mathcal{X}$. Take $R \in \Phi(G, 2)\backslash\mathcal{X}$. Then $\mathbb{Z}_2 = \Omega_1(Z(R))$, $N_G(R) = N_M(R)$ and $N_A(R) = N_{N_A(\mathbb{Z}_2)}(R)$. Replace $\mathbb{Z}_3$ by $\mathbb{Z}_2$ in the definition (2.9). Then (2.10) holds for each $C' \in \mathcal{M}^+(R)$ and we may suppose

$$C \notin \bigcup_{R\in\Phi(G,2)\backslash\mathcal{X}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)),$$

so in particular, $P_1 \in \mathcal{X}\backslash\{1\}$ and if $P_1 = \mathbb{Z}_2$, then $C = C(1)_1$ or $P_2 = O_2(T_\delta)$.

(2) Suppose $P_1 = O_2(T_\epsilon) \le M = M_2$. Then $N_M(T_\epsilon) = \langle T_\epsilon, \rho, \tau\sigma\rangle$ and we may take

$$\Phi(N_G(T_\epsilon), 2) = \{O_2(T_\epsilon), \langle O_2(T_\epsilon), \rho\rangle, \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S_1\},$$

where $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2 = \langle O_2(T_\epsilon), \tau\sigma\rangle$, $(\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^* = \langle O_2(T_\epsilon), \rho\tau\sigma\rangle$ and $S_1 = \langle O_2(T_\epsilon), \rho, \tau\sigma\rangle$. Replace R by $\langle O_2(T_\epsilon), \rho\rangle$ and $\mathbb{Z}_3$ by $O_2(T_\epsilon)$ in (2.9). Since $N_G(\langle O_2(T_\epsilon), \rho\rangle) \le N_G(O_2(T_\epsilon))$ and $N_A(\langle O_2(T_\epsilon), \rho\rangle) \le N_A(O_2(T_\epsilon))$, it follows that (2.10) holds.

Suppose $R \in \{\mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S_1\}$. Then

$$N_G(R) = N_G(\mathbb{Z}_2) \cap N_G(R) \quad \text{and} \quad N_A(R) = N_A(\mathbb{Z}_2) \cap N_A(R).$$

If $C' : 1 < O_2(T_\epsilon) < R < P_3' < \ldots < P_{n'}'$ is in $\mathcal{R}(G)$ and $g(C') : 1 < \mathbb{Z}_2 < O_2(T_\epsilon) < R < P_3' < \ldots < P_{n'}'$, then $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C'))$, so (2.10) holds. Since $\Phi(N_M(O_2(T_\epsilon)), 2) = \{O_2(T_\epsilon), \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S_1\}$, it follows that the remaining chains such that $P_1 \in \{\mathbb{Z}_2, O_2(T_\epsilon)\}$ and P_2 is non-conjugate to $O_2(T_{-\epsilon})$ are G-conjugate to $C(1)_1$, $C(2)$ and $C(2)^1$.

(3) Suppose $P_1 = E_8 = \langle \Omega_1(O_2(T_\epsilon)), \rho\rangle$. Then $N_G(E_8) = N_{G_2(q)}(E_8)$ and

$$N_A(E_8) = N_{G_2(q)\rtimes\langle\alpha\rangle}(E_8) = N_{G_2(q)}(E_8) \rtimes \langle\alpha\rangle.$$

We can apply the proof (2) of [3] (2E) to the chains C with $P_1 = E_8$. After cancellations, the remaining chains are G-conjugate to $C(3)$, $C(3)_1$, $C(4)$ and $C(4)_2$.

(4) Suppose $P_1 = O_2(T_{-\epsilon})$ and $T_{-\epsilon} \leq M$. We may take

$$\Phi(N_G(T_{-\epsilon}), 2) = \{O_2(T_{-\epsilon}), E_8, D_8, D_8^*, S_2\},$$

where $E_8 = \langle O_2(T_{-\epsilon}), \rho\rangle$, $D_8 = \langle O_2(T_{-\epsilon}), \tau\sigma\rangle$, $D_8^* = \langle O_2(T_{-\epsilon}), \rho\tau\sigma\rangle$ and S_2 the Sylow 2-subgroup $\langle O_2(T_{-\epsilon}), \rho, \tau\sigma\rangle$ of $N_G(T_{-\epsilon})$. In addition, $N_A(R) = N_G(R) \rtimes \langle\alpha\rangle$ for $R \in \Phi(N_G(T_{-\epsilon}), 2)$. Let

$$C' : 1 < O_2(T_{-\epsilon}) < E_8 \quad \text{and} \quad g(C') : 1 < O_2(T_{-\epsilon}) < E_8 < S_2.$$

Then both $N(C')$ and $N(g(C'))$ are subgroups of $N_G(E_8) = N_{G_2(q)}(E_8)$, and both $N_A(C')$ and $N_A(g(C'))$ are subgroups of $N_A(E_8) = N_{G_2(q)}(E_8) \rtimes \langle\alpha\rangle$. As shown in the proof (3) of [3] (2E) (2.10) holds and we may suppose C is not G-conjugate to C' and $g(C')$.

We may take $\Phi(N_M(T_{-\epsilon}), 2) = \{O_2(T_{-\epsilon}), D_8, D_8^*, S_2\} \subseteq \Phi(N_G(T_{-\epsilon}), 2)$. Let $R \in \Phi(N_M(T_{-\epsilon}), 2)\backslash\{O_2(T_{-\epsilon})\}$. Then

$$N_G(R) = N_G(R) \cap N_G(\mathbb{Z}_2) \quad \text{and} \quad N_A(R) = N_A(R) \cap N_A(\mathbb{Z}_2).$$

Define radical 2-chains

$$C' : 1 < O_2(T_{-\epsilon}) < R < P_3' < \ldots < P_{n'}',$$
$$g(C') : 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon}) < R < P_3' < \ldots < P_{n'}'.$$

Then $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C'))$, so (2.10) holds. The remaining chains $C \in \mathcal{R}(G)$ such that $P_1 = \mathbb{Z}_2$ or $O_2(T_{-\epsilon})$ and P_2 is not conjugate to $O_2(T_\epsilon)$ have representatives, $C(1)_1$, $C(1)$ and $C(1)^1$. Thus (2D) follows.

(2E) *Let $C \in \mathcal{R}^0(G)$ with $|C| \geq 1$. If $r \geq 3$, then*

C	$N_A(C)$	Conditions
$C(1)_1 : 1 < \mathbb{Z}_{r^a}$	$\langle I_\epsilon, \rho, \alpha\rangle$	$r \geq 5, r \mid q - \epsilon$
$C(1)^1 : 1 < O_r(T_\epsilon)$	$N_A(T_\epsilon)$	$r \geq 5, r \mid q - \epsilon$
$C(1)_1 : 1 < \mathbb{Z}_{r^a}$	$\langle H_\epsilon, \rho, \alpha\rangle$	$r \geq 5, r \mid q^2 + \epsilon q + 1$
$C(1)^1 : 1 < O_r(T_{2,\epsilon})$	$N_A(T_{2,\epsilon})$	$r \geq 3, r \mid q^2 + \epsilon q + 1$
$C(2)_1 : 1 < \mathbb{Z}_{r^a}^*$	$\langle {I_\epsilon}^*, \rho, \alpha\rangle$	$r \geq 3, r \mid q - \epsilon$
$C(1)_1 : 1 < \mathbb{Z}_3$	$\langle H_\epsilon, \rho, \alpha\rangle$	$r = 3$
$C(1) : 1 < \mathbb{Z}_{r^a} < O_r(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau, \alpha\rangle$	$r \geq 5, r \mid q - \epsilon$
$C(1) : 1 < O_r(M_\epsilon) < O_r(T_{2,\epsilon})$	$\langle T_{2,\epsilon}, \rho, \sigma, \alpha\rangle$	$r \geq 3, r \mid q^2 + \epsilon q + 1$
$C(2) : 1 < \mathbb{Z}_{r^a}^* < O_r(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau, \alpha\rangle$	$r \geq 3, r \mid q - \epsilon$

where $\rho, \sigma, \tau, \alpha$ are given by (2.6) *and* (2.1).

Proof. The proof is given by that of (2C) and Section 1 of [1].

(2F) *Let $C \in \mathcal{R}^0(G)$ with $|C| \geq 1$. If $r = 2$, then*

C	$N_A(C)$
$C(1)_1 : 1 < \mathbb{Z}_2$	$\langle C_G(\mathbb{Z}_2), \alpha \rangle$
$C(1)^1 : 1 < O_2(T_{-\epsilon})$	$N_A(T_{-\epsilon})$
$C(2)^1 : 1 < O_2(T_{\epsilon})$	$N_A(T_{\epsilon})$
$C(3)_1 : 1 < E_8$	$\langle E_8 \cdot L_3(2), \alpha \rangle$
$C(1) : 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon})$	$\langle T_{-\epsilon}, \rho, \tau\sigma, \alpha \rangle$
$C(2) : 1 < \mathbb{Z}_2 < O_2(T_{\epsilon})$	$\langle T_{\epsilon}, \rho, \tau\sigma, \alpha \rangle$
$C(3) : 1 < E_8 < 2_+^{1+4}$	$\langle 2_+^{1+4} \cdot S_3, \alpha \rangle$
$C(4)_2 : 1 < E_8 < \langle \Omega_2(O_2(T_{\epsilon})), \rho \rangle$	$\langle \Omega_2(O_2(T_{\epsilon})), \rho, \sigma, \tau, \alpha \rangle$
$C(4) : 1 < E_8 < \langle \Omega_2(O_2(T_{\epsilon})), \rho \rangle < S''$	$\langle S'', \alpha \rangle$

where ρ, σ, τ are given by (2.6)*, and E_8, S'' are given by* (2.12).

Proof. The proof is given by that of (2D) (cf. [2] (2D)).

3 The conjecture for odd primes

The notation and terminology of Sections 1 and 2 are continued in this section. We shall identify $G = {}^3D_4(q)$ with its dual group. Let $\mathcal{E}\big(G, (s)\big)$ be the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with the conjugacy class $(s) = (s)_G$ of a semisimple element $s \in G$, and let

$$\mathcal{E}_r(G, (s)) = \bigcup_w \mathcal{E}(G, (sw)),$$

where $s \in G$ is a semisimple r'-element and w runs over all the r-elements of $C_G(s)$. Then $\mathcal{E}_r\big(G, (s)\big)$ is a union of r-blocks.

Let T be torus of G, and

$$\phi : T \longrightarrow \mathrm{Irr}(T) \tag{3.1}$$

the bijection from T to $\mathrm{Irr}(T)$ such that (T, w) corresponding to $(T, \phi(w))$ for $w \in T$ under the correspondence of [9] (5.21.5). Let $Q \leq G$ and $B \in \mathrm{Blk}(G)$. We denote

$$\mathcal{Y}(B, Q) = \mathrm{Irr}(B) \bigcap \left(\bigcup_t \mathcal{E}(G, (t)) \right), \tag{3.2}$$

where t runs over G such that $C_G(t) = Q$. Denote by $\chi_{t,\mu}$ the character of $\mathcal{E}(G,(t))$ labelled by (t,μ), where μ is a irreducible unipotent character of $C_G(t)$.

Since $t^\alpha = t^p$ and ϕ is a group isomorphism, it follows that $\phi(t)^\alpha = \phi(t^\alpha)$, $\chi_{t,\mu}^\alpha \in \mathcal{E}_r(G,(t^\alpha))$ and moreover, by the degrees of characters in $\mathrm{Irr}(G)$ given by Table 4.4 of [10]

$$\chi_{t,\mu}^\alpha = \chi_{t^\alpha,\mu}. \tag{3.3}$$

(3A) *If $r = 3$, let $R \in \{\mathbb{Z}_{3^a}^*, \mathbb{Z}_3\}$. If $r \geq 5$, let $R \in \{\mathbb{Z}_{r^a}, \mathbb{Z}_{r^a}^*\}$ or $R = \mathbb{Z}_{r^a}$ according as $r|q-\epsilon$ or $r|q^2+\epsilon q+1$. Set $H = N_G(R)$ and $T = T_{2,\epsilon}$ or T_ϵ according as $H = M_\epsilon$ or $H \in \{N_G(I_\epsilon), N_G(I_\epsilon{}^*)\}$. Let B be a block of $\mathrm{Blk}(H)$ with defect group $D = O_r(T)$ and b a block of $K = N_H(D)$ with $b^H = B$. Then there is a defect preserving bijection Ψ between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(B)$ such that $\Psi(\varphi^\alpha) = \Psi(\varphi)^\alpha$ for all $\varphi \in \mathrm{Irr}(b)$.*

Proof. If $L = C_G(R)$, then $H = \langle L, \rho\rangle$, $N_L(D) = \langle T, x\rangle$ and $K = \langle T, x, \rho\rangle$, where $x = \sigma$ or τ according as $T = T_{2,\epsilon}$ or T_ϵ.

Let $(1,b) \leq (D, b_D)$, $B' = b_D^L$, $b' = b_D^{N_L(D)}$ and $N_L(b_D)$ the stabilizes of b_D in $N_L(D)$. Then $|N_L(b_D)/T| \leq 3$, and by [22], there are perfect isometries between b_D, b', and b', B'. Since $\mathrm{k}(b_D) = \mathrm{k}(b_D, 0)$, it follows $\mathrm{k}(b') = \mathrm{k}(b', 0) = \mathrm{k}(B') = \mathrm{k}(B', 0)$. But $|H\colon L| = |K\colon N_L(D)| = 2$, so $\mathrm{k}(B) = \mathrm{k}(B, 0)$ and $\mathrm{k}(b) = \mathrm{k}(b, 0)$. It suffices to show that there is a bijection Ψ between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(B)$ such that $\Psi(\varphi)^\alpha = \Psi(\varphi)^\alpha$.

If $[T, \phi(w)]$ is the L-class containing $(T, \phi(w))$, then

$$[T, \phi(w)]^\rho = [T, \phi(w^{-1})]$$

since $\phi(w)^\rho = \phi(w^\rho) = \phi(w^{-1})$ for $w \in T$. By Broué and Michel [6], Theorems 2.2 and 3.2, $B' \subseteq \mathcal{E}_r(L,(s))$ for some semisimple r'-element $s \in L$ and we may suppose $b_D = \mathcal{E}_r(T,(s))$.

If $C_L(sy) = T$, then $\chi = \pm R_T^L(\phi(sy)) \in \mathrm{Irr}(B')$ (for some sign $\pm$), $\chi^\rho = \chi$ if and only if $[T, \phi(sy)] = [T, \phi(sy)^{-1}]$ if and only if $(s^{-1}) = (s)$ and $(y^{-1}) = (y)$, where (w) denotes the L-class of $w \in T$. Similarly, if $\psi \in \mathrm{Irr}(b')$ covers $\phi(sy)$, then $\psi^\rho = \psi$ if and only if $(s^{-1}) = (s)$ and $(y^{-1}) = (y)$, since $N_L(D) = \langle T, x\rangle$ and $\rho x = x\rho$. If $\chi^\rho \neq \chi$, define

$$\Psi(\mathrm{Ind}_{N_L(D)}^K(\psi)) = \mathrm{Ind}_L^H(\chi).$$

If $\chi^\rho = \chi$, then χ has two extensions φ_1 and φ_2 to H, and ψ also has two extensions ζ_1 and ζ_2 to K. Define $\Psi(\zeta_i) = \varphi_i$ for $i = 1, 2$.

Suppose $C_L(sy) = L$ and let $(\lambda_1, \lambda_2, \lambda_3) = (1, \mathrm{St}, \chi_{q(q+\epsilon)})$ or $(\lambda_1, \lambda_2) = (1, \mathrm{St})$ according as $L = H_\epsilon$ or $L \in \{I_\epsilon, I_\epsilon{}^*\}$, where 1 is the trivial character, St is the Steinberg character, and $\chi_{q(q+\epsilon)}$ is the irreducible unipotent character of H_ϵ with degree $q(q+\epsilon)$. By the degrees, ρ and α both stabilizes each λ_i. Let $\chi_i \in \mathrm{Irr}(B')$ labelled by (sy, λ_i). If

$L = H_\epsilon$, then $N_L(D) = \langle T, \sigma\rangle$ and $\phi(sy)$ has 3 extensions μ_1, μ_2 and μ_3. If $L \in \{I_\epsilon, {I_\epsilon}^*\}$, then $N_L(D) = \langle T, \tau\rangle$ and $\phi(sy)$ has two extensions μ_1 and μ_2. Define $\Psi(\mu_i) = \chi_i$ for all i. Then Ψ is a bijection between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(B)$. Since $[\,\alpha, g\,] = 1$ for $g \in \langle\sigma, \rho, \tau\rangle$, it follows that $\Phi(\varphi)^\alpha = \Phi(\varphi^\alpha)$ for all $\varphi \in \mathrm{Irr}(b)$. This completes the proof.

Suppose $T = T_\epsilon$ or $T_{2,\epsilon}$, $N = N_G(T)$ and $N(\phi(t))$ the stabiliser of $\phi(t) \in \mathrm{Irr}(T)$ in N. A proof similar to that of [2] (4.2) shows that we may suppose

$$N(\phi(t)) \in \{N, \langle T, \rho\sigma\rangle, \langle T, \rho, \tau\sigma\rangle, \langle T, \tau, \sigma\rangle, \langle T, \rho^j\tau\sigma^i\rangle,\ j, i = 0, 1\}.$$

Let $\mathcal{S}(t)$ be the subset of $\mathrm{Irr}(N(\phi(t)))$ consisting of characters covering $\phi(t)$.

(3B) *Let $G = {}^3D_4(q)$ and $t \in T = T_\epsilon$ or $T_{2,\epsilon}$.*

(a) *Suppose $t = 1$ and $T = T_\epsilon$. Then $\mathcal{S}(t) = \{\xi_{1,i} : 1 \le i \le 6\}$, where $\xi_{1,i}(1) = 1$ for $1 \le i \le 4$ and $\xi_{1,5}(1) = \xi_{1,6}(1) = 2$.*

(b) *Suppose $t = 1$ and $T = T_{2,\epsilon}$. Then $\mathcal{S}(t) = \{\xi_{1,i} : 1 \le i \le 7\}$, where $\xi_{1,i}(1) = 1$ for $1 \le i \le 3$, $\xi_{1,j}(1) = 2$ for $4 \le j \le 6$ and $\xi_{1,7}(1) = 3$.*

(c) *Suppose $|t| = 2$ and $N(\phi(t)) = \langle T_\epsilon, \rho, \tau\sigma\rangle$. Then $\mathcal{S}(t) = \{\xi_{t,i} : 1 \le i \le 4\}$ with $\xi_{t,i}(1) = 1$ for all i.*

(d) *Suppose $C_G(t) \simeq H_\epsilon$. Then $\mathcal{S}(t) = \{\xi_{t,i} : 1 \le i \le 3\}$ with $\xi_{t,1}(1) = \xi_{t,2}(1) = 1$ and $\xi_{t,3}(1) = 2$.*

(e) *Suppose $C_G(t) \in \{I_\epsilon, {I_\epsilon}^*\}$. Then $\mathcal{S}(t) = \{\xi_{t,1}, \xi_{t,2}\}$ with $\xi_{t,1}(1) = \xi_{t,2}(1) = 1$.*

(f) *Suppose $C_G(t) = T$. Then $\mathcal{S}(t) = \{\xi_{t,1}\}$ with $\xi_{t,1}(1) = 1$.*

In all cases, $\xi_{t,i}^\alpha = \xi_{t^\alpha,i}$ for all i.

Proof. Since $N(\phi(1))/T = N/T \simeq \mathbb{Z}_2 \times S_3$ or $\mathrm{SL}_2(3) = Q_8 \rtimes \mathbb{Z}_3$ according as $T = T_\epsilon$ or $T_{2,\epsilon}$, the trivial character $\phi(1)$ has 6 or 7 extensions to N. Now $N = \langle T_\epsilon, \sigma, \tau, \rho\rangle$ or $N = \langle T_{2,\epsilon}, x_1, x_2, x_3\rangle$, and $w^\alpha = w$ for each $w \in \langle\sigma, \tau, \rho\rangle$ or $w \in \langle x_1, x_2, x_3\rangle$. It follows that α stabilizes each extension of $\phi(1)$, which implies (a) and (b).

The proofs of (c)-(f) are similar.

(3C) *Let $G = {}^3D_4(q)$ and let r be a prime such that $\gcd(r, q) = 1$. If $r \ge 3$, then Dade's invariant conjecture holds for G.*

Proof. We may suppose $B \in \mathrm{Blk}^0(G, r)$. Let $B \subseteq \mathcal{E}_r(G, (s))$ with defect group D, and (D, b_D) a Brauer B-pair. Let $\mathcal{R}(B)$ be the chains $C \in \mathcal{R}(G)$ such that $\mathrm{Blk}(N(C)|B) \ne \emptyset$ and $\mathcal{R}^0(B) = \mathcal{R}^0(G) \cap \mathcal{R}(B)$. Then

$$\mathcal{R}^0(B)/G = \begin{cases} \mathcal{R}^0(G)/G & \text{if } D \in \mathrm{Syl}_r(G), \\ \{C(1)_0, C(1)_1, C(1)^1, C(1)\} & \text{if } D = O_3(T_{2,\epsilon}), \\ \{C(1)_0, C(1)_1, C(2)_1, C(2)\} & \text{if } D = O_3(T_\epsilon). \end{cases}$$

(1) Let

$$(C, C') = \begin{cases} (C(2)_1, C(2)) & \text{if } r|q-\epsilon \text{ and } D = O_r(T_\epsilon), \\ (C(2)_1, C(2)) & \text{if } r = 3 \text{ and } D \in \mathrm{Syl}_3(H_\epsilon), \\ (C(1)_1, C(1)) & \text{if } r|q^2+\epsilon q+1 \text{ and } D = O_r(T_{2,\epsilon}). \end{cases}$$

In the notation of (3A) $N(C) = H$ and $N(C') = K$.

If D is abelian, then $N(C') = N_{N(C)}(D)$ and the Brauer correspondence $b(C') \mapsto b(C) = b(C')^{N(C')}$ is a bijection between $\mathrm{Blk}(N(C')|B)$ and $\mathrm{Blk}(N(C)|B)$. If $r = 3$ and $D \in \mathrm{Syl}_3(G)$, then $N(C) = \langle {I_\epsilon}^*, \rho \rangle$. If $b(C) \in \mathrm{Blk}(N(C)|B)$, then by Lemma V.6.1 of [11], $\mathbb{Z}_3 = Z(D) \leq Q \leq D$ for some $Q = D(b(C))$. If $O_3({I_\epsilon}^*) = Q$, then $\mathbb{Z}_3 = \Omega_1(O_3({I_\epsilon}^*))$, ${I_\epsilon}^* \leq H_\epsilon$ and ${J_\epsilon}^* \leq L_\epsilon$, which is impossible. So $O_3({I_\epsilon}^*) < Q$ and Q is non-cyclic. Thus $C_{N(C)}(Q) = T_\epsilon$, $Q = O_3(T_\epsilon)$ and $N(C') = N_{N(C)}(Q)$. The Brauer correspondence $b(C') \mapsto b(C) = b(C')^{N(C')}$ is a bijection between $\mathrm{Blk}(N(C')|B)$ and $\mathrm{Blk}(N(C')|B)$. It follows by (3A) that

$$\mathrm{k}(N(C), B, d, u) = \mathrm{k}(N(C'), B, d, u). \tag{3.4}$$

(2) If $r \geq 5$ and $r|q-\epsilon$, then the same proofs above can be applied to $C = C(1)_1$ and $C' = C(1)$. Thus (3.4) still holds.

(3) If $r \geq 5$ or $r = 3$ but $D = O_3(T_{2,\epsilon})$, let $T = T_\epsilon$ or $T_{2,\epsilon}$ according as $D = O_r(T_\epsilon)$ or $O_r(T_{2,\epsilon})$ and set $C = C(1)^1 : 1 < O_r(T)$. Then $N(C) = N(T) = N(D)$ and $\mathrm{Blk}(N(C)|B) = \{b(C)\}$. By Corollaries 5.10 and 5.12 of [10], $\mathrm{k}(B) = \mathrm{k}(B, 0) = \mathrm{k}(b(C)) = \mathrm{k}(b(C), 0)$. We shall define a defect preserving bijection

$$\Psi : \mathrm{Irr}(b(C)) \longrightarrow \mathrm{Irr}(B), \tag{3.5}$$

such that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ for any $\varphi \in \mathrm{Irr}(b(C))$.

Let $Q \leq G$, and $\mathcal{Y}_G(b(C), Q) = \{I(\xi) : \xi \in \mathcal{S}(t)\}$, where $I(\xi) = \mathrm{Ind}_{N(\phi(t))}^{N(C)}(\xi)$, $\phi(t) \in b_D = \mathcal{E}_r(T, (s))$ and $C_G(t) \simeq Q$. Define the map

$$\Psi : \mathcal{Y}_G(b(C), Q) \longrightarrow \mathcal{Y}(B, Q) \tag{3.6}$$

as follows: In the notation of (3B),

if $C_G(t) = T$, then $\mathcal{E}(G, (t)) = \{\chi_{t,1}\}$ and define $\Psi(I(\xi_{t,1})) = \chi_{t,1}$.

If $C_G(t) = I_\epsilon$ or ${I_\epsilon}^*$, then $\mathcal{E}(G, (t)) = \{\chi_{t,1}, \chi_{t,\mathrm{St}}\}$ and define $\Psi(I(\xi_{t,1})) = \chi_{t,1}$ and $\Psi(I(\xi_{t,2})) = \chi_{t,\mathrm{St}}$, where St is the Steinberg character of $C_G(t)$.

If $C_G(t) = H_\epsilon$, then $\mathcal{E}(G, (t)) = \{\chi_{t,1}, \chi_{t,\mathrm{St}}, \chi_{t,q(q+\epsilon)}\}$ (see [10] Table 4.4), and we define $\Psi(I(\xi_{t,1})) = \chi_{t,1}$, $\Psi(I(\xi_{t,2})) = \chi_{t,\mathrm{St}}$, and $\Psi(I(\xi_{t,3})) = \chi_{t,q(q+\epsilon)}$ for $\xi_{t,i} \in \mathcal{S}(t)$

If $|t| = 2$, then $\mathcal{E}(G,(t)) = \{\chi_{t,1}, \chi_{t,\mathrm{St}}, \chi_{t,\mathrm{St}'}, \chi_{t,\mathrm{StSt}'}\}$, and we define $\Psi(I(\xi_{t,1})) = \chi_{t,1}$, $\Psi(I(\xi_{t,2})) = \chi_{t,\mathrm{St}}$, $\Psi(I(\xi_{t,3})) = \chi_{t,\mathrm{St}'}$, and $\Psi(I(\xi_{t,4})) = \chi_{t,\mathrm{StSt}'}$ for $\xi_{t,i} \in \mathcal{S}(t)$.

Suppose $U(B_0) = \mathcal{E}(G,(1)) \cap \mathrm{Irr}(B_0)$. In the notation of [10], Table 4.4

$$U(B_0) = \begin{cases} \{1, [\epsilon_1], [\epsilon_2], \mathrm{St}, \rho_1, \rho_2\} & \text{if } r|q-1, \\ \{1, [\epsilon_1], [\epsilon_2], \mathrm{St}, {}^3D_4[-1], {}^3D_4[1]\} & \text{if } r|q+1, \\ \{1, [\epsilon_1], [\epsilon_2], \mathrm{St}, \rho_1, \rho_2, {}^3D_4[1]\} & \text{if } r|q^2+q+1, \\ \{1, [\epsilon_1], [\epsilon_2], \mathrm{St}, \rho_2, {}^3D_4[-1], {}^3D_4[1]\} & \text{if } r|q^2-q+1, \end{cases}$$

It follows by (3B) (a), (b) that there is a bijection Ψ between $U(B_0)$ and the characters of $\mathrm{Irr}(b(C))$ covering $\phi(1) = 1$.

By [10], Theorem 5.9, $\chi_{sy,\mu} \in \mathrm{Irr}(B)$ for $y \in D$ if and only if the block $b(sy)$ of $L = C_G(sy)$ containing $\phi(sy)\mu$ induces B, where μ is an irreducible unipotent character of $C_G(sy)$. It follows that the map Ψ defined above is a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$. Since

$$(\mathrm{Ind}_{N(\phi(t))}^{N(C)}(\xi))^\alpha = \mathrm{Ind}_{N(\phi(t^\alpha))}^{N(C)}(\xi^\alpha),$$

it follows that $\Psi(I(\xi))^\alpha = \Psi(I(\xi)^\alpha)$ for $\xi \in \mathcal{S}(t)$. Thus (3.4) holds for $C = C(1)^1$ and $C' = C(1)_0$.

(4) Suppose $r = 3$ and $D = O_3(T_\epsilon)$, so that $C_G(D) = T_\epsilon$. By the proof (1), we may suppose $C = C(1)_1$, so that $N(C) = M_\epsilon$ and $\mathrm{Blk}(N(C)|B) = \{b(C)\}$. If $K = N_{N(C)}(D) = N_G(T_\epsilon)$, then $b^{N(C)} = b(C)$ and $b^G = B$ for some $b \in \mathrm{Blk}(K|B)$. If $(1,b) \le (D, b_D)$, then $|N_{H_\epsilon}(b_D)/T_\epsilon| \le 2$, since $N_{H_\epsilon}(T_\epsilon)/T_\epsilon \simeq S_3$. The proof of (3A) can be applied to $b(C)$ and b. Thus there is a defect preserving bijection Ψ_1 between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(b(C))$ such that $\Psi_1(\varphi)^\alpha = \Psi_1(\varphi^\alpha)$.

Let Ψ_2 be the bijection defined by (3.6) between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(B)$, and let $\Psi = \Psi_2 \circ \Psi_1^{-1}$. Then Ψ a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$ such that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ for all $\varphi \in \mathrm{Irr}(b(C))$. Thus (3.4) holds for $C = C(1)_1$ and $C' = C(1)_0$.

Finally, suppose $r = 3$ and $D \in \mathrm{Syl}_3(G)$, so we may suppose $D \le M_\epsilon$. By the proof (1) above, we may suppose

$$C \in \{C(1)_0, C(1)_1, C(1)^1, C(1)\}.$$

(5). If $C = C(1)^1$ and $C' = C(1)$, then $N(C)/T_{2,\epsilon} \simeq \mathrm{SL}_2(3) = Q_8 \rtimes 3$ and $N(C')/T_{2,\epsilon} \simeq \mathbb{Z}_6$. Moreover, we may suppose $N(C) = \langle T_{2,\epsilon}, x_1, x_2, x_3\rangle$ such that $x_i^\alpha = x_i$ for all i.

Suppose B is not the principal block $B_0 = B_0(G)$. If $b \in \mathrm{Blk}(T_{2,\epsilon}|B)$, then by Theorem 3.2 of [6], we may suppose $b = \mathcal{E}_3(T_{2,\epsilon}, (s))$. Thus $b^{N(C)} = b(C) \in \mathrm{Blk}(N(C)|B)$,

$b^{N(C')} = b(C') \in \mathrm{Blk}(N(C')|B)$ and the induction $b(C')^{N(C)}$ induces a bijection between $\mathrm{Blk}(N(C')|B)$ and $\mathrm{Blk}(N(C)|B)$. Let $N(b)$ and $N'(b)$ be the stabilizers of b in $N(C)$ and $N(C')$, respectively. By Theorem 5.9 (c) of [10], $N(b) = \langle T_{2,\epsilon}, \sigma\rangle = N'(b)$. If b^* is a block of $N(b)$ covering b, then by a theorem of Fong and Reynolds, [11], Theorem V.2.5, the induction $\mathrm{Ind}_{N(b)}^{N(C)}$ is a bijection between $\mathrm{Irr}(b^*)$ and $\mathrm{Irr}(b(C))$. Similarly, $\mathrm{Ind}_{N(b)}^{N(C')}$ is a bijection between $\mathrm{Irr}(b^*)$ and $\mathrm{Irr}(b(C'))$. Thus there is a defect preserving bijection Ψ between $\mathrm{Irr}(b(C'))$ and $\mathrm{Irr}(b(C))$. Since $N(C) = \langle T_{2,\epsilon}, x_1, x_2, x_3\rangle$ with $x_i^\alpha = x_i$ for all i, it follows that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ and (3.4) holds.

Suppose $B = B_0$, $b_0(C) = B_0(N(C))$, $b_0(C') = B_0(N(C'))$, and let $\mathrm{Irr}(b_0) = \{\phi(y) : y \in O_3(T_{2,\epsilon})\}$. Then $N_{N(C)}(\phi(1)) = N(C)$, $N_{N(C')}(\phi(1)) = N(C')$ and by (3B),

$$\mathcal{Y}_G(b_0(C'), G) = \{\xi_{1,i} : 1 \le i \le 6\},$$

and $\mathcal{Y}_G(b_0(C), G) = \{\zeta_{1,j} : 1 \le j \le 7\}$, where $\xi_{1,i}(1) = 1$ for all i, $\zeta_{1,j}(1) = 1$ for $1 \le i \le 3$, $\zeta_{1,j}(1) = 2$ for $4 \le i \le 6$ and $\zeta_{1,7}(1) = 3$. Define

$$\Psi : \mathcal{Y}_G(b_0(C'), G) \longrightarrow \mathcal{Y}_G(b_0(C), G)\backslash\{\zeta_{1,7}\}$$

such that $\Psi(\xi_{1,i}) = \zeta_{1,i}$. Then Ψ is a defect preserving bijection and $\Psi(\xi_{1,i})^\alpha = \Psi(\xi_{1,i}^\alpha)$, since α stabilizes each character of $\mathcal{Y}_G(b_0(C), G) \cup \mathcal{Y}_G(b_0(C'), G)$.

If $y \ne 1$, then by Table 1 of [1], $C_G(y) \simeq H_\epsilon$ and $N_{N(C)}(\phi(y))/T_{2,\epsilon} \simeq \mathbb{Z}_3$. Thus $\phi(y)$ has three extensions $\eta_{y,i}$ in $N_{N(C)}(\phi(y))$. So the nontrivial characters of $\mathrm{Irr}(b_0)$ have 24 extensions and $\mathrm{Irr}(b_0(C))$ has 3 characters $\zeta_{y,i}$ covering these characters, each contains $8 = |\mathrm{SL}_2(3)\colon \mathbb{Z}_3|$ such extensions, where $\zeta_{y,i} = \mathrm{Ind}_{N(\phi(y))}^{N(C)}(\eta_{y,i})$.

If $\langle c\rangle = Z(\langle O_3(T_{2,\epsilon}), \sigma\rangle) \le O_3(T_{2,\epsilon})$, then $N_{N(C')}(\phi(w)) = \langle T_{2,\epsilon}, \sigma\rangle$ for $w = c$ or c^{-1}, so that $b_0(C')$ has 3 irreducible characters $\xi_{c,i}$ covering both $\phi(c)$ and $\phi(c^{-1})$. If $y \in O_3(T_{2,\epsilon})\backslash\langle c\rangle$, then $N_{N(C')}(\phi(y)) = T_{2,\epsilon}$ and b_0 has 6 such characters $\phi(y)$. Thus $\mathrm{Irr}(b_0(C'))$ contains exactly one character ψ_y covering these 6 characters and $(\psi_y)^\alpha = \psi_y$.

Define $\Psi(\psi_y) = \zeta_{1,7}$, $\Psi(\xi_{c,i}) = \zeta_{c,i}$ for $1 \le i \le 3$. Then $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ for all $\varphi \in \mathrm{Irr}(b_0(C'))$ and (3.4) holds.

Thus we may suppose $C = C(1)_1$ and so $N(C) = M_\epsilon = \langle H_\epsilon, \rho\rangle$.

(6). Let $B = B_0$ and $b_0 = B_0(L_\epsilon)$. In addition, let $L_\epsilon \le H = \mathrm{GL}_3^\epsilon(q)$, and $\zeta_{w,\mu}$ the irreducible character of H labelled by (w, μ), where μ is an irreducible unipotent character of $C_H(w)$. We may suppose ρ induces the inverse-transpose on H. Now

$$\mathrm{Irr}(b_0) = \mathrm{Irr}^0(b_0) \cup \mathrm{Irr}^1(b_0) \cup \mathrm{Irr}^{2a-1}(b_0)$$

(see [12], Theorem 4.5). In the notation of [20] and [12], set $\eta_i = \chi^{(i)}_{(q+\epsilon)(q^2+\epsilon q+1)/3}$, then

$$\mathrm{Irr}^0(b_0) = \{\chi_1, \chi_{q^2+\epsilon q}, \chi_{q^3}, \eta_i : 0 \le i \le 2\}.$$

Thus $\mathcal{E}(L_\epsilon,(1))=\{\chi_1,\chi_{q^2+\epsilon q},\chi_{q^3}\}$ and $\mathcal{E}(L_\epsilon,(t))=\{\eta_1,\eta_2,\eta_3\}$ for some 3-element $t=\mathrm{diag}\,\{t_1,t_1^{-1},1\}\in T_\epsilon$ with $C_H(t)=\mathbb{Z}_{q-\epsilon}\times\mathbb{Z}_{q-\epsilon}\times\mathbb{Z}_{q-\epsilon}$ (cf. the proof (2) of [3] (3C)). So ρ permutes $\mathcal{E}(L_\epsilon,(t))$ and fixes exactly one η_i, say η_0, since $\langle H,\rho\rangle/Z(H)L_\epsilon\simeq S_3$.

Fix a $x\in O_3(T_\epsilon)\backslash O_3(T_\epsilon\cap L_\epsilon)$, so that $H_\epsilon=\langle L_\epsilon\circ\mathbb{Z}_{q^2+\epsilon q+1},x\rangle$. Note that $b_0=B_0(L_\epsilon)=B_0(L_\epsilon\circ\mathbb{Z}_{q^2+\epsilon q+1})$. Since x and ρ stabilizes each character of $\mathcal{E}(L_\epsilon,(1))$, it follows by Clifford theory that $\mathcal{E}(L_\epsilon,(1))$ has 9 extensions to M_ϵ and α stabilizes each of them.

It follows from the remarks of [12], p.571 that x acts fixed-point freely on $\mathcal{E}(L_\epsilon,(t))$, so that M_ϵ has exactly two irreducible characters $\varphi_{t,1}$ and $\varphi_{t,2}$ covering characters of $\mathcal{E}(L_\epsilon,(t))$ and $\varphi_{t,i}(1)=3\eta_1(1)$ for $i=1,2$. Thus $\mathrm{Irr}^0(b(C))$ contains exactly 9 characters and each is stabilized by α. By [10], p.69

$$\text{(3.7)}\quad \mathrm{Irr}^0(B_0)=\begin{cases}\{1,[\epsilon_1],[\epsilon_2],\mathrm{St},\rho_1,\rho_2,\chi_{s_4,1},\chi_{s_4,\mathrm{St}},\chi_{s_4,q(q+1)}\} & \text{if } r|q-1,\\ \{1,[\epsilon_1],[\epsilon_2],\mathrm{St},,{}^3D_4[-1],{}^3D_4[1],\chi_{s_9,1},\chi_{s_9,\mathrm{St}},\chi_{s_9,q(q-1)}\} & \text{if } r|q+1\end{cases}$$

and α stabilizes each character of $\mathrm{Irr}^0(B_0)$. Thus there is a bijection Ψ between $\mathrm{Irr}^0(b(C))$ and $\mathrm{Irr}^0(B_0)$ such that $\Psi(\varphi)^\alpha=\Psi(\varphi^\alpha)$ for $\varphi\in\mathrm{Irr}^0(b(C))$.

(7). In the notation of [20], suppose $\xi_i=\chi^{(i)}_{(q-\epsilon)^2(q+\epsilon)/3}$ for $i=0,1,2$. (In the notation of [12], $\xi_i=\chi^{(i,1)}_{(q-\epsilon)^2(q+\epsilon)/3}$.) Then the ξ_i are the three irreducible constituents of $\zeta_{y,1}|_{L_\epsilon}$ for some 3-element $y\in L_\epsilon$ such that $C_H(y)\simeq\mathbb{Z}_{q^3-\epsilon}$ (see [20], Corollary 4.8), so that $C_{L_\epsilon}(y)=\mathbb{Z}_{q^2+\epsilon q+1}$. Since y and y^ρ are not H-conjugate, it follows that $\zeta_{y,1}\neq\zeta_{y^\rho,1}$ and $\xi_i^\rho\neq\xi_i$ for all i. Thus $\mathrm{Irr}^{2a-1}(b_0)=\{\xi_i,\xi_i^\rho:i=0,1,2\}$ (see [12]), $\mathcal{E}(L_\epsilon,(y))=\{\xi_i:1\le i\le 3\}$ and $\mathcal{E}(L_\epsilon,(y^{-1}))=\{\xi_i^\rho:1\le i\le 3\}$. It follows by the remarks of [12], p.571 that x acts fixed-point freely on both $\mathcal{E}(L_\epsilon,(y))$ and $\mathcal{E}(L_\epsilon,(y^{-1}))$. Thus M_ϵ has exactly one irreducible character φ_0 covering all characters of $\mathrm{Irr}^{2a-1}(b_0)$, and in addition, $\varphi_0(1)=2(q-\epsilon)^2(q+\epsilon)$, so that $\mathrm{Irr}^{2a}(b(C))=\{\varphi_0\}$.

By [10], pp.65 and 66, $\mathcal{Y}(B_0,G)\backslash(\mathrm{Irr}^0(B_0)\cap\mathcal{E}(G,(1)))=\{\chi_0\}$, where $\chi_0={}^3D_4[1]$ or ρ_2 according as $\epsilon=+$ or $-$. Define $\Psi(\varphi_0)=\chi_0$, so that $\Psi(\varphi_0)^\alpha=\Psi(\varphi_0^\alpha)$, since α stabilizes both φ_0 and χ_0 by the degrees. Then Ψ is a bijection between $\mathrm{Irr}^{2a}(b(C))$ and $\mathrm{Irr}^{2a}(B_0)$.

(8). Now we consider $\mathrm{Irr}^1(b_0(C))$ and $\mathrm{Irr}^1(B_0)$. As shown above L_ϵ has exactly one class of non-trivial 3-elements $t=\mathrm{diag}\,\{t_1,t_1^{-1},1\}$ such that $C_{L_\epsilon}(t)=T_\epsilon{}'=T_\epsilon\cap L_\epsilon$ and $\zeta_{t,1}|_{L_\epsilon}$ is reducible with constituents η_i. Let

$$\mathcal{X}(b_0)=\mathcal{Y}(b_0,T_\epsilon{}')\backslash\{\eta_0,\eta_1,\eta_2\}.$$

By Table 1b of [20] and Theorem 4.5 of [12], $|\mathcal{X}(b_0)|=\frac{1}{6}3^a(3^a-3)$. Let

$$\mathcal{X}_0(b_0)=\{\xi\in\mathcal{X}(b_0):\xi^\rho=\xi\},\qquad \mathcal{X}_1(b_0)=\mathcal{X}(b_0)\backslash\mathcal{X}_0(b_0).$$

If $\xi_y = \zeta_{y,1}|_{L_\epsilon} \in \mathcal{X}_0(b_0)$, then we may suppose $y = \text{diag}\ \{y_1, y_1^{-1}, 1\}$ for some $y_1 \in O_3(\mathbb{Z}_{q-\epsilon})\backslash\{t_1, t_1^{-1}, 1\}$, so that $C_{G_2(q)}(y) = {J_\epsilon}^*$. Thus $|\mathcal{X}_0(b_0)| = \frac{1}{2}(3^a - 3)$ and $|\mathcal{X}_1(b_0)| = \frac{1}{6}3^a(3^a - 3) - \frac{1}{2}(3^a - 3) = \frac{1}{6}(3^a - 3)^2$.

If $\xi_y \in \mathcal{X}_0(b_0)$, then ξ_y has exactly 3 extensions $\varphi_{y,i}$ to M_ϵ, where $\varphi_{y,1}|_{H_\epsilon} = \varphi_{y,2}|_{H_\epsilon} = \xi_{y,1}$ and $\varphi_{y,3}(1) = 2\xi_y(1)$. Since $[\,\rho, \alpha\,] = 1$, it follows that $\varphi_{y,i}^{\alpha} = \varphi_{y^\alpha,i}$ for all i. Let

$$\mathcal{X}_0(C) = \{\varphi_{t,i}, \varphi_{y,i} \in \text{Irr}(M_\epsilon) : \varphi_{y,i} \ \text{ covers } \ \xi_y \in \mathcal{X}_0(b_0), \varphi_{y,i}(1) = \xi_y(1), i = 1, 2\}$$

and $\mathcal{X}_0^*(C) = \{\varphi_{y,3} \in \text{Irr}(M_\epsilon) : \varphi_{y,3}$ covers $\xi_y \in \mathcal{X}_0(b_0), \varphi_{y,3}(1) = 2\xi_y(1)\}$, where $\varphi_{t,i}$ is defined in the proof (6) above. In addition, $|\mathcal{X}_0(C)| = (3^a - 3) + 2 = 3^a - 1$ and $|\mathcal{X}_0^*(C)| = \frac{1}{2}(3^a - 3)$. By Table 4.4 of [10], $\mathcal{Y}(B_0, {I_\epsilon}^*)$ contains exactly $\frac{1}{2}(3^a - 1)$ characters of degrees $(q+\epsilon)(q^8+q^4+1)$ and $q^3(q+\epsilon)(q^8+q^4+1)$, respectively labelled by $(y, 1)$ and (y, St) for some 3-elements $y \in G$ such that $C_G(y) = {I_\epsilon}^*$. Since $C_{G_2(q)}(y) = {J_\epsilon}^*$, we may suppose $y = \text{diag}\ \{y_1, y_1^{-1}, 1\} \in T_\epsilon \cap L_\epsilon$ for $1 \neq y_1 \in O_3(\mathbb{Z}_{q-\epsilon})$ (c.f. [14], p.358). Define

$$\Psi : \mathcal{X}_0(C) \longrightarrow \mathcal{Y}(B_0, {I_\epsilon}^*)$$

such that $\Psi(\varphi_{y,1}) = \chi_{y,1}$ and $\Psi(\varphi_{y,2}) = \chi_{y,\text{St}}$. Then Ψ is a defect preserving bijection and $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$.

If $\xi_y \in \mathcal{X}_1(b_0)$, then $C_{G_2(q)}(y) = {T_\epsilon}'$, $\xi_y(1) = (q+\epsilon)(q^2+\epsilon q+1)$, $\xi_y^\rho \neq \xi_y$ and $\xi_y^x = \xi_y$. Thus ξ_y has 3 extensions $\xi_{y,i}$ to H_ϵ, and M_ϵ has $\frac{3}{2}\frac{1}{6}(3^a - 3)^2 = \frac{1}{4}(3^a - 3)^2$ irreducible characters covering characters of $\mathcal{X}_1(b_0)$. Let

$$\mathcal{X}_1(C) = \{\varphi_{y,i} = \text{Ind}_{H_\epsilon}^{M_\epsilon}(\xi_{y,i}) : \xi_y \in \mathcal{X}_1(b_0), i = 1, 2, 3\} \cup \mathcal{X}_0^*(C),$$

so that $|\mathcal{X}_1(C)| = \frac{1}{4}(3^a - 3)(3^a - 1)$. We can relabel the index i such that $\xi_{y,1}^\alpha = \xi_{y^\alpha,1}$, $\xi_{y,2} = \xi_{y,1} \otimes \phi({x{T_\epsilon}'})$ and $\xi_{y,3} = \xi_{y,1} \otimes \phi(x^2{T_\epsilon}')$. By Table 4.4 of [10], $|\mathcal{Y}(B_0, T_\epsilon)| = \frac{1}{12}(3^{a+1} - 9)(3^a - 1) = \frac{1}{4}(3^a - 3)(3^a - 1)$,

$$\mathcal{Y}(B_0, T_\epsilon) = \{\chi_{w,1} : w \in O_3(T_\epsilon), C_G(w) = T_\epsilon\}.$$

In particular, G has $\frac{1}{4}(3^a - 3)(3^a - 1)$ classes of 3-elements w such that $C_G(w) = T_\epsilon$.

By Tables 1 and II of [15], $G_2(q)$ has $\frac{1}{12}(3^a - 3)^2$ classes of 3-elements y such that $C_{G_2(q)}(y) = {T_\epsilon}'$. Thus $C_G(y) = C_G(yx) = C_G(yx^2) = T_\epsilon$. If y and yx are G-conjugate, then they are $N_G(T_\epsilon)$-conjugate. But $N_G(T_\epsilon) = \langle T_\epsilon, \rho, \tau, \sigma\rangle$ with $\rho, \tau, \sigma \in G_2(q)$, so y and yx are $G_2(q)$-conjugate, which is impossible. Similarly, we can show that $(y)_G$, $(yx)_G$ and $(yx^2)_G$ are three distinct G-classes. This gives $\frac{1}{4}(3^a - 3)^2$ G-classes of 3-elements $w \in T_\epsilon$ such that $C_G(w) = T_\epsilon$.

Again by Tables 1 and II of [15] $G_2(q)$ has $\frac{1}{2}(3^a - 3)$ classes of 3-elements $v \in {T_\epsilon}'$ such that $C_{L_\epsilon}(v) = {T_\epsilon}'$ and $C_{G_2(q)}(v) = {J_\epsilon}^*$. We may suppose $v = \text{diag}\ \{v_1, v_1^{-1}, 1\} \in {T_\epsilon}' \leq L_\epsilon$.

Thus $C_G(vx) = T_\epsilon$ and G has $\frac{1}{2}(3^a-3)$ such classes. Note that $(vx)_G = (vx^2)_G$ and $(vx)_G \notin \{(y)_G, (yx)_G, (yx^2)_G\}$. Since $\frac{1}{4}(3^a-3)^2 + \frac{1}{2}(3^a-3) = \frac{1}{4}(3^a-3)(3^a-1)$, it follows that $\{(vx)_G, (y)_G, (yx)_G, (yx^2)_G\}$ are all the classes of 3-elements w such that $C_G(w) = T_\epsilon$. Define $\Psi(\varphi_{v,3}) = \chi_{vx,1}$ for $\varphi_{v,3} \in \mathcal{X}_0^*(C)$ and $\Psi(\varphi_{y,i}) = \chi_{yx^{i-1},1}$ for $\varphi_{y,i} \in \mathcal{X}_1(C)\backslash\mathcal{X}_0^*(C)$. Then Ψ is a bijection between $\mathcal{X}_1(C)$ and $\mathcal{Y}(B_0, T_\epsilon)$ such that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$.

It follows by Theorem 4.5 of [12] that $\mathcal{Y}(b_0, J_\epsilon)$ contains exactly 3^a-1 characters of degrees $q^2+\epsilon q+1$ and $q(q^2+\epsilon q+1)$, respectively. If $\xi_{y,\mu} \in \mathcal{Y}(b_0, J_\epsilon)$ with $\mu \in \{1, \mathrm{St}\}$, then x stabilizes $\xi_{y,\mu}$ and $\mathrm{Irr}(H_\epsilon)$ contains exactly 3 extensions $\xi_{y,\mu,i}$ covering $\xi_{y,\mu}$. We choose the index i such that $\xi_{y,\mu,1}^\alpha = \xi_{y^\alpha,\mu,1}$, $\xi_{y,\mu,2} = \xi_{y,\mu,1} \otimes \phi(xT_\epsilon{}')$ and $\xi_{y,\mu,3} = \xi_{y,\mu,1} \otimes \phi(x^2T_\epsilon{}')$. Since ρ acts fixed-point freely on $\mathcal{Y}(b_0, J_\epsilon)$, it follows that $\xi_{y,\mu,i}^\rho \neq \xi_{y,\mu,i}$ for all y and i. Let

$$\mathcal{X}_2(C) = \{\varphi_{y,\mu,i} = \mathrm{Ind}_{H_\epsilon}^{M_\epsilon}(\xi_{y,\mu,i}) : \xi_{y,\mu} \in \mathcal{Y}(b_0, J_\epsilon), \mu = 1, \mathrm{St}, i = 1, 2, 3\},$$

so that $|\mathcal{X}_2(C)| = \frac{1}{2}(3^{a+1}-3)$. By Table 4.4 of [10], $|\mathcal{Y}(B_0, I_\epsilon)| = \frac{1}{2}(3^{a+1}-3)$, so that G has exactly $\frac{1}{2}(3^{a+1}-3)$ classes of 3-elements y such that $C_G(y) = I_\epsilon$. By Tables 1 and II of [15], $G_2(q)$ has $\frac{1}{2}(3^a-1)$ classes of 3-elements $y \in T_\epsilon{}'$ such that $C_{G_2(q)}(y) = J_\epsilon$. A similar proof to above shows that $\{(y)_G, (yx)_G, (yx^2)_G\}$ are all the G-classes of 3-elements w such that $C_G(w) = I_\epsilon$, where $y \in O_3(T_\epsilon{}')$ such that $C_{G_2(q)}(y) = J_\epsilon$. Define $\Psi(\varphi_{y,\mu,i}) = \chi_{yx^{i-1},\mu}$, so $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ and Ψ is a bijection between $\mathcal{X}_2(C)$ and $\mathcal{Y}(B_0, I_\epsilon)$. Thus Ψ is a required defect preserving bijection between $\mathrm{Irr}^1(b_0(C))$ and $\mathrm{Irr}^1(B_0)$. So Dade's invariant conjecture holds for B_0.

(9). Suppose $B \neq B_0$ and $C = C(1)_1$. Then $\mathrm{Blk}(N(C)|B) = \{b(C)\}$ and we may suppose $B \subseteq \mathcal{E}_r(G, (s))$ for some $s \in Z(H_\epsilon)$. Let $b = B_0(L_\epsilon \cdot 3) \times b_s$ such that $b^{N(C)} = b(C)$, so that $b^G = B$, where b_s is a block of $\mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1))}$ with a unique character φ_s. Since $N_{M_\epsilon}(b) = H_\epsilon$, $\mathrm{Ind}_{H_\epsilon}^{M_\epsilon}$ induces a defect preserving bijection between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(b(C))$. It suffices to show that there is a defect preserving bijection Ψ between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(B)$ such that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ for $\varphi \in \mathrm{Irr}(b)$. The characters of $\mathrm{Irr}(B_0(L_\epsilon \cdot 3))$ can be obtained from the proofs (6)-(8) above. From the proofs (6),

$$\mathrm{Irr}^0(B_0(L_\epsilon \cdot 3)) = \{\varphi_{i,j} : \varphi_{1,j}(1) = 1, \varphi_{2,j}(1) = q^2+\epsilon q, \varphi_{3,j}(1) = q^3, 1 \leq j \leq 3\},$$

where $\varphi_{i,j}$ covering characters of $\mathcal{E}(L_\epsilon, (1))$, $\varphi_{i,2} = \varphi_{i,1} \otimes \phi(xT_\epsilon{}')$ and $\varphi_{i,3} = \varphi_{i,1} \otimes \phi(x^2T_\epsilon{}')$ for all i. By Table 4.4 of [10],

$$\mathrm{Irr}^0(B) = \{\chi_{sc^i,\mu} : 0 \leq i \leq 2, \mu = 1, \mathrm{St}, q(q+\epsilon)\},$$

where $O_3(Z(H_\epsilon)) = \langle c \rangle$. Define a bijection $\Psi : \mathrm{Irr}^0(b) \to \mathrm{Irr}^0(B)$ by

$$\Psi(\varphi_{i,j} \times \varphi_s) = \begin{cases} \chi_{sc^{j-1},1} & \text{if } i = 1, \\ \chi_{sc^{j-1},\mathrm{St}} & \text{if } i = 2, \\ \chi_{sc^{j-1},q(q+\epsilon)} & \text{if } i = 3 \end{cases}$$

for all i. Then $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$.

¿From the proof (7) above $\mathrm{Irr}^{2a}(B_0(L_\epsilon{\cdot}3)) = \{\varphi_1, \varphi_2\}$ such that $\varphi_i(1) = (q-\epsilon)^2(q+\epsilon)$ for all i. Let $E = \langle O_3(T_{2,\epsilon}), \sigma\rangle \leq H_\epsilon$ and $O_3(T_{2,\epsilon}) = \langle x\rangle \times \langle c\rangle$, where $O_3(L_\epsilon) = \langle c\rangle$. Then $E \simeq 3_+^{1+2}$ and $O_3(T_{2,\epsilon})$ consists of two H_ϵ-orbits, $\{x, cx, c^2x\}$ and $\{x^2, cx^2, c^2x^2\}$ of non-central elements in E. So G has exactly two classes $(xs)_G$ and $(x^2s)_G$ such that $C_G(xs) = C_G(x^2s) = T_{2,\epsilon}$. By Table 4.4 of [10],

$$\mathrm{Irr}^{2a}(B) = \{\chi_{sx,1}, \chi_{sx^2,1}\}$$

and we define $\Psi(\varphi_i \times \varphi_s) = \chi_{sx^i,1}$ for all i. Thus $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ and Ψ is a bijection between $\mathrm{Irr}^{2a}(b)$ and $\mathrm{Irr}^{2a}(B)$.

As shown in the proof (8) above there is a unique character $\varphi_t \in \mathrm{Irr}(B_0(L_\epsilon{\cdot}3))$ covering η_0, η_1, η_2, so $\varphi_t(1) = (q+\epsilon)(q^2+\epsilon q+1)$ and $C_G(ts) = T_\epsilon$. In addition, there are (3^a-1) characters $\xi_{y,\mu}$ in $\mathcal{Y}(b_0, J_\epsilon)$ and each $\xi_{y,\mu}$ has 3 extensions $\varphi_{yx^i,\mu}$ in $\mathrm{Irr}(B_0(L_\epsilon \cdot 3))$, where $i = 0, 1, 2$. Thus there are $(3^{a+1}-3)$ characters $\varphi_{yx^i,\mu}$ covering characters in $\mathcal{Y}(b_0, J_\epsilon)$.

Similarly, there are $\frac{1}{2}3^a(3^a-3)$ characters $\varphi_{yx^i,1}$ covering characters $\xi_{y,1} \in \mathcal{X}(b_0)$, so $\varphi_{yx^i,1}(1) = \xi_{y,1}(1) = (q+\epsilon)(q^2+\epsilon q+1)$ and so there are $\frac{1}{2}3^a(3^a-3)+1$ characters $\varphi \in \mathrm{Irr}(B_0(L_\epsilon \cdot 3))$ with this degree.

The group G has $\frac{1}{6}(3^{a+1}-9)(3^a-1)$ classes $y \in O_3(T_\epsilon)$ such that $C_G(y) = T_\epsilon$, and has $\frac{1}{2}(3^a-1)$ classes $y \in O_3(T_\epsilon)$ such that $C_G(y) = {I_\epsilon}^*$, so that $C_G(ys) = T_\epsilon$. Note that for each $y \in O_3(T_\epsilon)$ with $C_G(ys) = T_\epsilon$, either $C_G(y) = T_\epsilon$ or $C_G(y) = {I_\epsilon}^*$. Therefore, G has $\frac{1}{2}(3^a-3)(3^a-1) + \frac{1}{2}(3^a-1) = \frac{1}{2}3^a(3^a-3)+1$ classes $y \in O_3(T_\epsilon)$ such that $C_G(sy) = T_\epsilon$. Define

$$\Psi(\varphi_{yx^i,1} \times \varphi_s) = \chi_{syx^i,1}, \qquad \Psi(\varphi_t \times \varphi_s) = \chi_{st,1},$$

where $t \neq y \in O_3(T_\epsilon)$ with $C_G(sy) = T_\epsilon$. There are $\frac{1}{2}(3^{a+1}-3)$ G-classes $y \in O_3(T_\epsilon)$ such that $C_G(y) = I_\epsilon$. For such a class $(y)_G$, we define $\Psi(\varphi_{yx^i,\mu} \times \varphi_s) = \chi_{syx^i,\mu}$, where $\mu = 1$ or St. Thus Ψ is a required defect preserving bijection between $\mathrm{Irr}^1(b)$ and $\mathrm{Irr}^1(B)$. This proves (3C).

4 The conjecture for the even prime

The notation and terminology of Sections 1 and 2 are continued in this section. Suppose $r = 2$, B is a 2-block of $G = {}^3D_4(q)$ with non-cyclic defect groups D and $B \subseteq \mathcal{E}_2(G, (s))$ for some semi-simple $2'$-element $s \in G$. Then q is odd and (see [10], Theorem 5.9]),

$$D \in \{O_2(T_\delta), SD_{2^{a+2}}, \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, S\},$$

where $S \in \mathrm{Syl}_2(G)$. If $D \in \{O_2(T_{-\epsilon}), SD_{2^a+2}\}$, then the invariant conjecture for B follows by Uno [21]. Suppose $D = S$, then $B = B_0$, $N(C) = N_{G_2(q)}(C) = N_{G_2(p)}(C)$ and

$N_A(C) = N(C) \times \langle\alpha\rangle$ for $C \in \{C(3)_1, C(3), C(4)_2, C(4)\}$. By the proof (3) and (4) of [2] (4C) or the proof (4) of [3] (4B)

$$\sum_{C\in\{C(3)_1,C(3),C(4)_2,C(4)\}} (-1)^{|C|}\,\mathrm{k}(N(C), B_0, d, u) = 0.$$

Thus we may suppose $C \notin \{C(3)_1, C(3), C(4)_2, C(4)\}$.

(4A) *If $G = {}^3D_4(q)$ with q odd and $r = 2$, then Dade's invariant conjecture holds for G.*

Proof. The proof of (4A) is similar to that of (3C) and [3] (4A) and (4B).

(1) Suppose $D = O_2(T_\epsilon)$, so that $C_G(s) = T_\epsilon$. Let $L_1 = \mathrm{SL}_2(q^3)$ and $L_2 = \mathrm{SL}_2(q)$ such that $M_2 = (L_1 \circ L_2).2$. Then $s = t_1t_2 \in L_1 \circ L_2$ for some $t_i \in L_i$, $S_1 = C_{L_1}(t_1) = \mathbb{Z}_{q^3-\epsilon}$, $S_2 = C_{L_2}(t_2) = \mathbb{Z}_{q-\epsilon}$, and

$$O_2(T_\epsilon) = \langle O_2(\mathbb{Z}_{q^3-\epsilon}) \circ O_2(\mathbb{Z}_{q-\epsilon}), g\rangle,$$

where $g = \mathrm{diag}\,\{1,-1\} \otimes \mathrm{diag}\,\{1,-1\} \in M_2$.

Let $C = C(1)_1$ and $C' = C(2)$. Then $N(C) = M_2$ and $N(C') = N_{N(C)}(D)$. If $b_D = \mathcal{E}_2(T_\epsilon, (s))$, then $b(C) = b_D^{N(C)} \in \mathrm{Blk}(N(C)|B)$, $b(C') = b_D^{N(C')} \in \mathrm{Blk}(N(C')|B)$ and $N_{N(C')}(b_D) = T_\epsilon$. Thus $\mathrm{Irr}(b(C)) = \{\varphi_{sy}, \varphi'_{sy} : y = y_1y_2, y_i \in O_2(S_i)\}$ and

$$\mathrm{Irr}(b(C')) = \{\mathrm{Ind}_{T_\epsilon}^{N(C')}(\phi(sy)), \mathrm{Ind}_{T_\epsilon}^{N(C')}(\phi(syg)) : y \in O_2(S_1) \circ O_2(S_2)\},$$

where φ_{sy} and φ'_{sy} are the two extensions of $(\pm R_{S_1}^{L_1}(y_1)) \circ (\pm R_{S_2}^{L_2}(y_2))$ for suitable sign $\pm$. Define a defect preserving bijection $\Psi : \mathrm{Irr}(b(C')) \longrightarrow \mathrm{Irr}(b(C))$ by $\Psi(\mathrm{Ind}_{T_\epsilon}^{N(C')}(\phi(sy))) = \varphi_{sy}$ and $\Psi(\mathrm{Ind}_{T_\epsilon}^{N(C')}(\phi(syg)) = \varphi'_{sy}$. Since $g^\alpha = g$, it follows that $\Psi(\varphi^\alpha) = \Psi(\varphi)^\alpha$, so that (3.4) holds.

If $C = C(2)^1$, then $N(C) = N(D)$ and $\mathrm{Blk}(N(C)|B) = \{b(C)\}$. The map defined by (3.6) is a required bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$.

(2) Suppose $D = \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ and $L = C_G(s)$ is regular in G, so that $L = I_\epsilon$ or ${I_\epsilon}^*$. Let $C = C(2)^1$ and $C' = C(2)$. Then $N(C) = N(T_\epsilon)$ and $N(C') = \langle T_\epsilon, \rho, \tau\sigma\rangle$. Since $O_2(T_\epsilon)$ is the unique maximal normal abelian subgroup of D, it follows that $N(D) \le N(O_2(T_\epsilon)) = N(C)$, and so $N(D) \le N(C')$ as $\Omega_1(Z(D)) = \mathbb{Z}_2$. If $b = \mathcal{E}_2(T_\epsilon, (s))$, then $b^{N(C)} = b(C) \in \mathrm{Blk}(N(C)|B)$, $b^{N(C')} = b(C') \in \mathrm{Blk}(N(C')|B)$, and $b(C') \mapsto b(C)$ is a bijection between $\mathrm{Blk}(N(C)|B)$ and $\mathrm{Blk}(N(C')|B)$. Since $\sigma \notin N(b)$, it follows that $N(\xi) = N'(\xi)$ for each $\xi \in \mathrm{Irr}(b)$, where $N(\xi) = N_{N(C)}(\xi)$ and $N'(\xi) = N_{N(C')}(\xi)$. Since α centralizes both τ and ρ, $\mathrm{Ind}_{N(\xi)}^{N(C)}$ and $\mathrm{Ind}_{N'(\xi)}^{N(C')}$ induce a required bijection Ψ between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(b(C'))$.

If $C = C(1)_1$, then $N(C) = M_2$ and $N(D) \le N(C)$. Since $L = C_G(s)$ is regular in G and $L^\alpha = L$, it follows that by Theorem 2.3 of [5] that $R_L^G : \mathrm{Irr}(b_L) \to \mathrm{Irr}(B)$ is

a perfect isometry between B and b_L, where $b_L = \mathcal{E}_2(L,(s))$ is a block of L. Note that M_2 is the fixed-point set of a Frobenius map on some algebraic group, L is regular in M_2 and $R_L^{M_2} : \mathrm{Irr}(b_L) \to \mathrm{Irr}(b(C))$ is also a perfect isometry, where $b(C)$ is the unique block in $\mathrm{Blk}(N(C)|B)$. If $\Psi = R_L^G \circ (R_L^{M_2})^{-1}$, then Ψ is a defect preserving bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$, and $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$.

(3) Suppose $D = \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ and $C_G(s) = H_\epsilon$. Then $N(C(1)_1) = M_2$ and $\mathrm{Blk}(M_2|B) = \{b(M_2)\}$. By the proof (2) above, there exists a defect preserving bijection Ψ between $\mathrm{Irr}(b(M_2))$ and $\mathrm{Irr}(B_\epsilon)$, where B_ϵ is a block of G such that $B_\epsilon \subseteq \mathcal{E}_2(G,(t))$ with $C_G(t) = I_\epsilon$ and $D(B_\epsilon) = \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$. We may identify $b(M_2)$ with B_ϵ, and so s with t. Thus

$$\mathrm{Irr}^j(B_\epsilon) = \begin{cases} \{\varphi_{sy,\mu} : C_G(sy) = I_\epsilon, \mu = 1, \mathrm{St}\} & \text{if } j = 0, \\ \{\varphi_{sy,1} : C_G(sy) = T_\epsilon\} & \text{if } j = 1, \\ \{\varphi_{sy,1} : C_G(sy) = \mathbb{Z}_{(q^3-\epsilon)(q+\epsilon)}\} & \text{if } j = a, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $(sy)_G \neq (s^{-1}y^{-1})_G$, it follows that $|\mathrm{Irr}^0(B_\epsilon)| = 2^{a+1}$, $|\mathrm{Irr}^1(B_\epsilon)| = \frac{1}{2}2^a(2^a-1)$ and $|\mathrm{Irr}^a(B_\epsilon)| = \frac{1}{2}(2^{a+1}-2^a) = 2^{a-1}$. Now

$$\mathrm{Irr}^j(B) = \begin{cases} \{\chi_{sy,\mu}, \chi_{s,\lambda_i} : C_G(sy) = I_\epsilon, \mu = 1, \mathrm{St}, i = 1,2\} & \text{if } j = 0, \\ \{\chi_{sy,1}, \chi_{s,\lambda_3} : C_G(sy) = T_\epsilon\} & \text{if } j = 1, \\ \{\chi_{sy,1} : C_G(sy) = \mathbb{Z}_{(q^3-\epsilon)(q+\epsilon)}\} & \text{if } j = a, \\ \emptyset & \text{otherwise,} \end{cases}$$

where λ_i for $1 \le i \le 3$ are irreducible unipotent characters of H_ϵ such that $\lambda_1(1) = 1$, $\lambda_2(1) = q^3$ and $\lambda_3(1) = q(q+\epsilon)$. Thus $|\mathrm{Irr}^0(B)| = 2^{a+1}$, $|\mathrm{Irr}^1(B)| = 1 + \frac{1}{6}(2^a-1)(2^a-2)$ and $|\mathrm{Irr}^a(B)| = 2^{a-1}$. So $\Psi(\varphi_{sy,1}) = \chi_{sy,1}$ is a required defect preserving bijection between $\mathrm{Irr}^a(B_\epsilon)$ and $\mathrm{Irr}^a(B)$.

Let $C = C(2)^1$ and $C' = C(2)$. Since $N(D) \le N(C)$ and $N(D) \le N(C')$, it follows that $\mathrm{Blk}(N(C)|B) = \{b(C)\}$ and $\mathrm{Blk}(N(C')|B) = \{b(C')\}$. If $b = \mathcal{E}_2(T_\epsilon,(s))$, then $b^{N(C)} = b(C)$ and $b^{N(C')} = b(C')$. Since $N_{N(C)}(\phi(s)) = \langle T_\epsilon, \sigma, \tau\rangle$, $\phi(s)$ has three extensions, $\eta_{s,i}$ such that $\eta_{s,3}(1) = 2$ and $\eta_{s,1}(1) = \eta_{s,2}(1) = 1$. Note that

$$N(\phi(sy)) = N_{N(C)}(\phi(sy)) = \langle T_\epsilon, N_{N_{G_2(q)}(T_{\epsilon'})}(\phi(sy))\rangle$$

for $y \in O_2(T_\epsilon)$. As shown in the proof (3) of [3] (4A)

$$\mathrm{Irr}^j(b(C)) = \begin{cases} \{\eta_{sy,i}, \eta_{s,i} : C_G(sy) = I_\epsilon, i = 1,2\} & \text{if } j = 0, \\ \{\eta_{sy,1}, \eta_{s,3} : C_G(sy) = T_\epsilon\} & \text{if } j = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

so that $|\mathrm{Irr}^0(b(C))| = 2(2^a-1)+2 = 2^{a+1}$ and $|\mathrm{Irr}^1(b(C))| = 1 + \frac{1}{6}(2^a-1)(2^a-2)$. Thus $\Psi(\eta_{s,i}) = \chi_{s,\lambda_i}$ for all i, $\Psi(\eta_{sy,1}) = \chi_{sy,1}$ and $\Psi(\eta_{sy,2}) = \chi_{sy,\mathrm{St}}$ defines a required defect preserving bijection between $\mathrm{Irr}^j(b(C))$ and $\mathrm{Irr}^j(B)$, where $j = 0, 1$.

Let $N'(\phi(sy)) = N(\phi(sy)) \cap N(C')$. Then $N'(\phi(sy)) = \langle T_\epsilon, N_{N_{G_2(q)}(T_\epsilon{}')}(\phi(sy))\rangle$, so that (cf. the proof (3) of [3] (4A))

$$\mathrm{Irr}^j(b(C')) = \begin{cases} \{\psi_{sy,i} : C_{M_2}(sy) = I_\epsilon, i = 1, 2\} & \text{if } j = 0, \\ \{\psi_{sy,1} : C_{M_2}(sy) = T_\epsilon\} & \text{if } j = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $|\mathrm{Irr}^0(b(C'))| = 2^{a+1}$ and $|\mathrm{Irr}^1(b(C'))| = 2^{a-1}(2^a - 1)$. Let Ψ be the defect preserving bijection between $\mathrm{Irr}^j(B_\epsilon)$ and $\mathrm{Irr}^j(b(C'))$ for $j = 0, 1$ such that $\Psi(\varphi_{sy,1}) = \psi_{sy,1}$ and $\Psi(\varphi_{sy,\mathrm{St}}) = \psi_{sy,2}$. Then Ψ is a required defect preserving bijection between $\mathrm{Irr}(b(C)) \cup \mathrm{Irr}(B_\epsilon)$ and $\mathrm{Irr}(B) \cup \mathrm{Irr}(b(C'))$.

(4) Finally, suppose $B = B_0$, so that $s = 1$ and $\mathrm{Blk}(N(C)|B_0) = \{b_0(C)\}$ for all C. Let $C = C(1)^1$ and $C' = C(1)$. Then $N(C) = N(T_{-\epsilon})$ and $N(C') = \langle T_{-\epsilon}, \rho, \tau\sigma\rangle$. If $Q = \langle O_2(T_{-\epsilon}), \rho, \tau\sigma\rangle$ and $K = \langle O_2(T_{-\epsilon}), \rho, \tau, \sigma\rangle$, then $Q = \mathbb{Z}_2 \times D_8 \leq G_2(q)$, $K = \mathbb{Z}_2 \times A_4 \rtimes 2 \leq G_2(q)$. We may suppose $b_0(C) = B_0(K)$ and $b_0(C') = B_0(Q)$. It follows by the proof (3) of [3] (2E) that (3.4) holds.

Let $C = C(2)^1$ and $C' = C(2)$. As shown in the proof (2) of [3] (4B) there is a required defect preserving bijection Ψ between $\mathrm{Irr}^j(b_0(C'))$ and $\mathrm{Irr}^j(b_0(C))$ for $j = 0, 1$. Let $H = N(C(1)_1) = M_2 = \langle L_1 \circ L_2, g\rangle$ and $b(H) = B_0(H)$. Suppose St_i is the Steinberg character of L_i. It follows from the characters of $\mathrm{Irr}(L_i)$ that

$$\mathrm{Irr}^0(b(H)) = \{\varphi_1, \varphi_2 : \varphi \in \{1 \otimes 1, 1 \otimes \mathrm{St}_2, \mathrm{St}_1 \otimes 1, \mathrm{St}_1 \otimes \mathrm{St}_2\}\},$$

where φ_1 and φ_2 are the two extensions to H of $\varphi \in \mathrm{Irr}(L_1 \circ L_2)$. But

$$\mathrm{Irr}^0(B_0) = \{1, [\epsilon_1], [\epsilon_2], \mathrm{St}, \chi_{s_2,1}, \chi_{s_2,\mathrm{St}_1}, \chi_{s_2,\mathrm{St}_2}, \chi_{s_2,\mathrm{St}_1\mathrm{St}_2}\}$$

and α atabilizes each characters of both $\mathrm{Irr}^0(B_0)$ and $\mathrm{Irr}^0(b(H))$, so there is a required defect preserving bijection between $\mathrm{Irr}^0(b(H))$ and $\mathrm{Irr}^0(B_0)$.

Suppose $\chi_i' \in \mathrm{Irr}(L_i)$ such that $\chi_i'(1) = q^3 - \epsilon$ or $q - \epsilon$ according as $i = 1$ or 2 and suppose $\chi_i'' = (\chi_i')^{g_i}$, where $g_i = \mathrm{diag}\ \{1, -1\} \in L_i$. Then $\mathrm{Irr}^{2a-1}(b(H)) = \{\chi_1' \otimes \chi_2' + \chi_1'' \otimes \chi_2'', \chi_1' \otimes \chi_2'' + \chi_1'' \otimes \chi_2'\}$. But $\mathrm{Irr}^{2a-1}(B_0) = \{\rho_1, \rho_2\}$ or $\{{}^3D_4[-1], {}^3D_4[1]\}$ according as $\epsilon = -$ or $+$ and α stabilizes each character of $\mathrm{Irr}^{2a-1}(B_0)$ and $\mathrm{Irr}^{2a-1}(b(H))$, so there is a required defect preserving bijection Ψ between $\mathrm{Irr}^{2a-1}(b(H))$ and $\mathrm{Irr}^{2a-1}(B_0)$.

Suppose $\varphi_i' \in \mathrm{Irr}(L_i)$ such that $\varphi_i'(1) = q^3 + \epsilon$ or $q + \epsilon$ according as $i = 1$ or 2 and suppose $\varphi_i'' = (\varphi_i')^{g_i}$. Then $\mathrm{Irr}^1(b(H)) = \{\varphi_1' \otimes \varphi_2' + \varphi_1'' \otimes \varphi_2'', \varphi_1' \otimes \varphi_2'' + \varphi_1'' \otimes \varphi_2'\} \cup \mathcal{Y}(b(H), I_\epsilon) \cup \mathcal{Y}(b(H), I_\epsilon{}^*)$ and

$$\mathrm{Irr}^1(B_0) = \begin{cases} \{{}^3D_4[-1], {}^3D_4[1]\} \cup \mathcal{Y}(B_0, I_\epsilon) \cup \mathcal{Y}(B_0, I_\epsilon{}^*) & \text{if } \epsilon = -, \\ \{\rho_1, \rho_2\} \cup \mathcal{Y}(B_0, I_\epsilon) \cup \mathcal{Y}(B_0, I_\epsilon{}^*) & \text{if } \epsilon = +. \end{cases}$$

In addition, $|\mathcal{Y}(b(H), I_\epsilon)| = |\mathcal{Y}(B_0, I_\epsilon)|$ and $|\mathcal{Y}(b(H), I_\epsilon{}^*)| = |\mathcal{Y}(B_0, I_\epsilon{}^*)|$. It follows that there is a defect preserving bijection Ψ between $\mathrm{Irr}^1(b(H))$ and $\mathrm{Irr}^1(B_0)$ such that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$. Moreover, $\mathrm{Irr}^{a+1}(b(H)) = \mathcal{Y}(b(H), T_{1,\delta})$, $\mathrm{Irr}^{a+1}(B_0) = \mathcal{Y}(B_0, T_{1,\delta})$ for $\delta = +$ and $-$, and $|\mathrm{Irr}^{a+1}(b(H))| = |\mathrm{Irr}^{a+1}(B_0)|$. So there is a required defect preserving bijection Ψ between $\mathrm{Irr}^{a+1}(b(H))$ and $\mathrm{Irr}^{a+1}(B_0)$.

Similarly, there is a defect preserving bijection $\Psi^{-1} : \mathrm{Irr}^2(b_0(C')) \longrightarrow \mathrm{Irr}^2(b(H)) = \mathcal{Y}(b(H), T_\epsilon)$ and $\Psi : \mathrm{Irr}^2(b_0(C)) \longrightarrow \mathrm{Irr}^2(B_0) = \mathcal{Y}(B_0, T_\epsilon)$ such that $\Psi(\varphi)^\alpha = \Psi(\varphi^\alpha)$ (cf. the proofs (2) and (3) of [3] (4B)). Note that $|\mathrm{Irr}^2(b_0(C'))| = |\mathrm{Irr}^2(b(H))| = \frac{1}{4}(2^{2a} - 2^{a+1})$ and $|\mathrm{Irr}^2(b_0(C))| = |\mathrm{Irr}^2(B_0)| = \frac{1}{12}(2^a - 4)(2^a - 2)$. Thus there exists a required defect preserving bijection Ψ between $\mathrm{Irr}(b_0(C)) \cup \mathrm{Irr}(b(H))$ and $\mathrm{Irr}(b_0(C)) \cup \mathrm{Irr}(B_0)$. This completes the proof.

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