THE INFINITUDE OF 7-ARC-TRANSITIVE GRAPHS

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Abstract

By theorems of Tutte and Weiss and others, it is known that there are no finite symmetric graphs of degree greater than 2 with automorphism group transitive on 8-arcs, and that 7-arc-transitivity can occur only in the case of quartic graphs (degree 4). In this paper it is shown that there are infinitely many 7-arc-transitive finite quartic graphs; indeed for all but finitely many positive integers n, there is a finite connected 7-arc-transitive quartic graph with the alternating group A_n acting transitively on its 7-arcs, and another with the symmetric group S_n acting transitively on its 7-arcs. The proof uses a construction involving permutation representations of a generic infinite group to produce an infinite family of finite graphs with the required properties.

1 Introduction

Let Γ be an undirected simple graph. An *automorphism* of Γ is any permutation of the vertices of Γ preserving adjacency. Under composition the set of all such permutations of $V\Gamma$ forms a group known as the (full) automorphism group of Γ and denoted by $\mathrm{Aut}\,\Gamma$.

For any positive integer s, an *s-arc* in Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\}$ is an edge of Γ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$, that is, such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct. We say the graph Γ is *s-arc-transitive* if its automorphism group $\mathrm{Aut}\,\Gamma$ acts transitively on the s-arcs of Γ, and *s-arc-regular* if this action is regular. In particular, a 1-arc-transitive graph is said to be *arc-transitive*, or *symmetric*. For example, simple cycles are s-arc-transitive for all s, the cube graph and the complete graphs K_n are 2-arc-transitive (but not 3-arc-transitive), and Petersen's graph and the complete bipartite graphs $K_{n,n}$ are 3-arc-transitive (but not 4-arc-transitive). Note that connnected arc-transitive graphs are necessarily vertex-transitive and therefore regular (in the sense that every vertex has the same degree).

By a theorem of Tutte [6], a finite symmetric graph of degree 3 is s-arc-regular for some s, where s is at most 5. This theorem was generalized by Richard Weiss in [8], using the classification of doubly-transitive finite permutation groups to prove that if Γ is any s-arc-transitive finite graph of degree $k \geq 3$, then $s \leq 7$, and moreover that $s = 7$ only when $k = 4$. In both cases of maximum possible symmetry (5-arc-transitive for finite 3-valent graphs and 7-arc-transitive for finite 4-valent graphs), and in several other cases besides, properties of the automorphism group can be determined to a considerable extent by local analysis in the graph. Indeed in both cases there exists a generic infinite but finitely-presented group $R_{k,s}$ (see [8]), with generators prescribed in terms of specific types of symmetries, such that if Γ is any finite s-arc-transitive k-valent graph (where $(k, s) = (3, 5)$ or $(4, 7)$), then its automorphism group $\mathrm{Aut}\,\Gamma$ must be a homomorphic image of $R_{k,s}$.

The smallest known example of a 7-arc-transitive 4-valent graph is the incidence graph of the generalised hexagon associated with the simple group $G_2(3)$, on 728 vertices [8]. Examples of 5-arc-transitive 3-valent graphs include Tutte's 8-cage (on 30 vertices) and Wong's graph (on 234 vertices), which are described in [1] along with a method (due to John Conway) for constructing an infinite number of covers of any given example; further examples include many of the sextet graphs constructed by Biggs and Hoare in [2]. For some time these examples (and their covers) were the only examples known, until the first author of this paper showed that they are are not all that rare, proving in [4] that for all but finitely many positive integers n both the alternating group A_n and the symmetric group S_n may be represented as full automorphism groups of 5-arc-transitive cubic graphs.

In this paper we respond to a challenge by Norman Biggs (in a personal communication) to prove a similar result for 7-arc-transitive quartic graphs.

In particular we prove the following

Theorem: *For all but finitely many positive integers n both the alternating group A_n and the symmetric group S_n may be represented as 7-arc-transitve groups of automorphisms of finite connected 4-valent graphs.*

As in [4] our proof is based on a standard method for constructing arc-transitive graphs, using carefully selected permutation representations of a generic infinite group as building blocks. The construction method and the group in question are described in the next Section, and the building blocks are defined in Section 3. The proof of the Theorem is given in Section 4.

2 Preliminaries

Suppose Γ is an undirected simple graph, and G is a group of automorphisms of Γ which acts transitively on the arcs of Γ. Then for any vertex $v \in V\Gamma$, the stabilizer $G_v = \{g \in G : v^g = v\}$ in G of v acts transitively on the set $\Gamma(v)$ of vertices adjacent to v in Γ, or equivalently, on the set of arcs in Γ emanating from the vertex v. Furthermore, if (v, w) is any one such arc, then by arc-transitivity there exists an automorphism $a \in G$ reversing (v, w), and the structure of Γ may be defined completely in terms of this automorphism a and the vertex-stabilizer $H = G_v$: vertices may be labelled with right cosets of H in G, and edges are the images under the action of G (by right multiplication) of the single edge $\{v, w\}$ labelled $\{H, Ha\}$ in the natural order.

Conversely, given any group G containing a subgroup H and an element a such that $a^2 \in H$, we may construct a graph $\Gamma = \Gamma(G, H, a)$ on which G acts as an arc-transitive group of automorphisms, as follows: take as vertices of Γ the right cosets of H in G, and join two cosets Hx and Hy by an edge in Γ whenever $xy^{-1} \in HaH$. Defined in this way, Γ is an undirected graph on which the group G acts as a group of automorphisms under the action $g : Hx \to Hxg$ for each $g \in G$ and each coset Hx in G. The stabilizer in G of the vertex H is the subgroup H itself, and as this acts transitively on the set of neighbours of H (which are all of the form Hah for $h \in H$), it follows that Γ is symmetric. Furthermore, the degree of any vertex of Γ is equal to $|H : H \cap a^{-1}Ha|$, the number of right cosets of H in HaH, and the graph Γ

is connected if and only if the elements of HaH generate G.

An equivalent construction was used by Weiss [8] in developing partial presentations for s-arc-transitive groups of automorphisms of symmetric graphs of degree k, for various values of s and k, and also by Conder [4] in dealing with the special case $(k,s) = (3,5)$. In fact as noted in the introduction, in both this case and the case $(k,s) = (4,7)$, there exists a generic infinite group $R_{k,s}$ such that every s-arc-transitive group of automorphisms of a finite k-valent graph occurs as a homomorphic image of $R_{k,s}$. We will exploit the properties of the group $R_{4,7}$ a great deal below.

In contrast to known presentations for $R_{3,5}$ which are relatively straightforward (see [4] or [8]), based on local analysis by Tutte (see [6] or [1]), the presentation obtained by Weiss for $R_{4,7}$ in [8] is naturally rather complex. Its generators are given in terms of automorphisms with prescribed effects on a particular 7-arc, and its relations are determined in a way which illustrates the fact that the stabiliser of a vertex is an extension of an elementary abelian group of order 3^5 by the finite general linear group $GL_2(3)$. For our purposes, an alternative but equivalent presentation of $R_{4,7}$ is more convenient:

Definition: We take $R_{4,7}$ to be the group generated by elements p, q, r, s, t, u, v, h and b, subject to the following relations:

$$h^4 = p^3 = q^3 = r^3 = s^3 = t^3 = u^3 = v^2 = b^2 = 1,$$
$$(hu)^3 = (uv)^2 = (huv)^2 = [h^2, u] = [h^2, v] = 1,$$
$$[q,r] = [q,s] = [q,t] = [r,s] = [r,t] = 1, \;\; [s,t] = p,$$
$$[p,q] = [p,r] = [p,s] = [p,t] = 1, \;\; h^{-1}ph = p, \;\; h^{-1}qh = q^{-1}r,$$
$$h^{-1}rh = qr, \;\; h^{-1}sh = pq^{-1}r^{-1}s^{-1}t^{-1}, \;\; h^{-1}th = p^{-1}qr^{-1}s^{-1}t,$$
$$u^{-1}pu = p, \;\; u^{-1}qu = q, \;\; u^{-1}ru = q^{-1}r, \;\; u^{-1}su = s, \;\; u^{-1}tu = pqrst,$$
$$vpv = p^{-1}, \;\; vqv = q^{-1}, \;\; vrv = r, \;\; vsv = s, \;\; vtv = t^{-1},$$
$$bpb = q^{-1}, \;\; bqb = p^{-1}, \;\; brb = s^{-1}, \;\; bsb = r^{-1}, \;\; btb = u^{-1}, \;\; bub = t^{-1},$$
$$bvb = v \;\; \text{and} \;\; bh^2b = h^2v.$$

[Note: the equivalence with the presentation given for $R_{4,7}$ by Weiss in [8] is given by $e_0 \mapsto u$, $e_1 \mapsto s$, $e_2 \mapsto q$, $e_3 \mapsto p$, $e_4 \mapsto r$, $e_5 \mapsto t$, $e_6 e_0^{-1} \mapsto h$, $a e_6^2 a^{-1} \mapsto v$, and $e_0 e_6 e_0 (e_6 e_0^{-1})^2 a e_6^2 a^{-2} \mapsto b$.]

In the alternative presentation, consider the four subgroups $L = \langle h, u, v\rangle$, $M = \langle p, q, r, s, t\rangle$, $K = \langle h^2, p, q, r, s, t, u, v\rangle$, and $H = \langle h, p, q, r, s, t, u, v\rangle$ respectively. Now $L = \langle h, u, v\rangle$ is isomorphic to $GL_2(3)$ of order 48 (with h^2 the central involution h^2, and $\langle h, u\rangle \cong SL_2(3)$). Next $M = \langle p, q, r, s, t\rangle$ is a subgroup of order 3^5 which is normalised by each of h, u and v, and thus $H = \langle h, p, q, r, s, t, u, v\rangle$ is a semidirect product of M by L, of order 11664. Finally the involution b normalises the subgroup $K = \langle h^2, p, q, r, s, t, u, v\rangle$ of index 4 in H, indeed $K = H \cap b^{-1}Hb$.

It follows that the (infinite) graph $\Gamma(R_{4,7}, H, b)$ is a connected symmetric graph of degree 4 on which the group $R_{4,7}$ induces an arc-transitive group of automorphisms. Note that the element b takes the role of a in the construction described earlier in this section, while the elements h, p, q, r, s, t, u and v generate the stabiliser of a vertex.

More generally, if G is any non-degenerate homomorphic image of $R_{4,7}$ (in which the orders of the generators and the elements hu, uv and huv are preserved), and if by abuse of notation we retain the same symbols for the images of the generators and the subgroups L, M, K and H in G, then the graph $\Gamma = \Gamma(G, H, b)$ is a connected symmetric graph of degree 4 on which the group G induces an arc-transitive group of automorphisms. Moroever:

- the stabilizer of the vertex H is $\langle p, q, r, s, t, h, u, v\rangle$, of order 11664,
- the stabilizer of the arc (H, Hb) is $\langle p, q, r, s, t, h^2, u, v\rangle$, of order 2916,
- the stabilizer of the 2-arc (Hbh, H, Hb) is $\langle p, q, r, s, t, h^2, uv\rangle$, of order 972,
- the stabilizer of the 3-arc $(Hbh^{-1}, H, Hb, Hbhb)$ is $\langle p, q, r, s, t^{-1}h^2, t^{-1}uv\rangle$, of order 324,
- the stabilizer of the 4-arc $(Hbh^{-1}bh^{-1}, Hbh^{-1}, H, Hb, Hbhb)$ is $\langle p, q, r, t^{-1}h^2, s^{-1}t^{-1}uv\rangle$, of order 108,
- the stabilizer of the 5-arc $(Hbh^{-1}bh^{-1}, Hbh^{-1}, H, Hb, Hbhb, Hbhbhb)$ is $\langle p, q, rt^{-1}h^2, r^{-1}s^{-1}t^{-1}uv\rangle$, of order 36,
- the stabilizer of the 6-arc $(Hbh^{-1}bh^{-1}bh^{-1}, Hbh^{-1}bh^{-1}, Hbh^{-1}, H, Hb, Hbhb, Hbhbhb)$ is $\langle p, rt^{-1}h^2, q^{-1}r^{-1}s^{-1}t^{-1}uv\rangle$, of order 12, and
- the stabilizer of the 7-arc $(Hbh^{-1}bh^{-1}bh^{-1}, Hbh^{-1}bh^{-1}, Hbh^{-1}, H, Hb, Hbhb, Hbhbhb, Hbhbhbhb)$ is $\langle rt^{-1}h^2, p^{-1}q^{-1}r^{-1}s^{-1}t^{-1}uv\rangle$, of order 4.

Finally, as the subgroup $\langle p\rangle$ stabilises the 6-arc $(Hbh^{-1}bh^{-1}bh^{-1}, Hbh^{-1}bh^{-1},$

$Hbh^{-1}, H, Hb, Hbhb, Hbhbhb)$ but permutes the 3 remaining neighbours of the vertex $Hbhbhb$ in a cycle, it follows that the graph Γ is 7-arc-transitive.

3 Building blocks

For our construction we will use two basic building blocks, which are transitive non-degenerate permutation representations of the group $R_{4,7}$ on 2912 and 8825 points respectively, corresponding to the action of $R_{4,7}$ by right multiplication on right cosets of the two subgroups described below. In this context, by the conjugate of x by y we mean the element yxy^{-1}.

The first is a subgroup of index 2912 generated by p, q, r, s, t, u, v, h^2 and b, together with the conjugates of all of these nine elements by $(hb)^5ht^{-1}u^{-1}buth^{-1}(bh^{-1})^5$.

The second is an index 8825 subgroup generated by p, q, r, s, t, u, v, h^2, and b, together with conjugates of each of these nine elements by $(hb)^6vhs^{-1}u^{-1}buth^{-1}(bh^{-1})^5$, plus conjugates of each of the three elements $hvtur^{-1}, phvtur^{-1}$ and $put^{-1}hvt$ by $(hb)^6ht^{-1}bs^{-1}t^{-1}u(h^{-1}b)^3t^{-1}qr^{-1}$, plus the conjugate of b by $(hb)^6ht^{-1}bs^{-1}t^{-1}uh^{-1}(bh^{-1})^2uhu^{-1}$.

The precise representations may be obtained from the above subgroups by coset enumeration with the help of the GAP [5] or Magma [3] systems.

In both representations, the permutations induced by p, q, r, s, t, u, v and h are all even, while the permutation induced by b is odd. To be more precise, p, q, r, s, t and u all induce permutations with cycle structures $3^{936}1^{104}$ and $3^{2880}1^{185}$, and the corresponding cycle structures for v and h are $2^{1408}1^{96}$ and $2^{4328}1^{169}$, and $4^{704}2^{48}$ and $4^{2164}2^{84}1^{1}$ respectively, while those for b are $2^{1455}1^{2}$ and $2^{4407}1^{11}$.

Also in a similar way to the construction used in [4], each of these representations can be decomposed into a number of orbits of the subgroup $H = \langle p, q, r, s, t, u, v, h\rangle$, linked together by the action of the involution b, and each orbit of H in turn can be decomposed into orbits of the subgroup $K = \langle p, q, r, s, t, u, v, h^2\rangle$. The decomposition into orbits of H is illustrated in Figures 1 and 2.

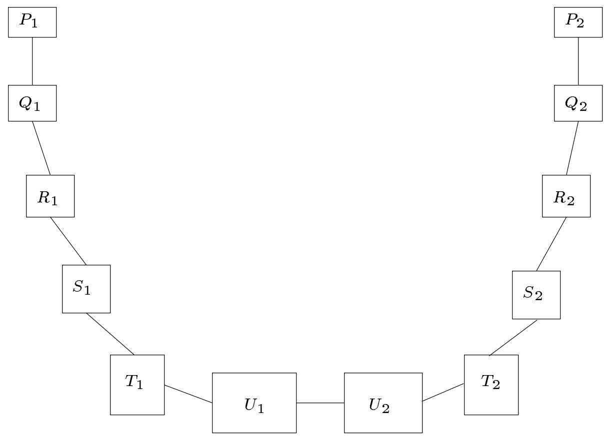

Figure 1: Block A

In each case P_i, Q_i, R_i, S_i, T_i and U_i (for $i \in \{1, 2\}$) are copies of transitive permutation representations of the subgroup $H = \langle p, q, r, s, t, u, v, h \rangle$ with the following properties:

• each P_i is a permutation representation on 4 points, consisting of K-orbits of lengths 1 and 3,
• each Q_i is a permutation representation on 12 points, consisting of K-orbits of lengths 3 and 9,
• each R_i is a permutation representation on 36 points, consisting of K-orbits of lengths 9 and 27,
• each S_i is a permutation representation on 108 points, consisting of K-orbits of lengths 27 and 81,
• each T_i is a permutation representation on 324 points, consisting of K-orbits of lengths 81 and 243, and
• each U_i is a non-degenerate permutation representation on 972 points, consisting of K-orbits of lengths 243 and 729.

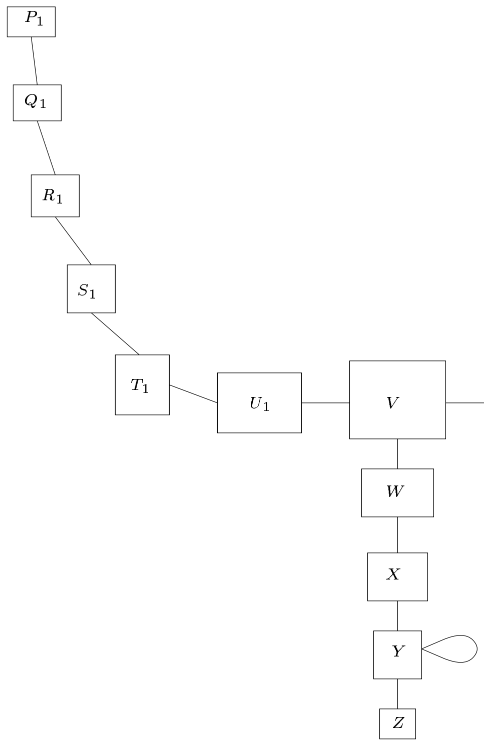

Figure 2: Block B

For the second basic building block B, five further transitive permutation representations of H are required:

- V is a non-degenerate permutation representation on 2916 points, consisting of K-orbits of lengths 729, 729 and 1458,
- W is a permutation representation on 1994 points, consisting of K-orbits of lengths 1458 and 486,
- X is a permutation representation on 648 points, consisting of K-orbits of lengths 486 and 162,
- Y is a permutation representation on 324 points, consisting of K-orbits of lengths 162, 81 and 81, and
- Z is a permutation representation on 81 points, consisting of a single K-orbit of length 81.

In fact the generators given earlier for the two subgroups which produce these building blocks were chosen specifically to produce such decompositions. This was achieved with the help of the Magma system [3] to analyse transitive permutation representations of the group H (of order 11664), of which there are 499 up to conjugacy, and to find ways of linking orbits of the subgroup $K = \langle p, q, r, s, t, u, v, h^2 \rangle$ together using the action of the involution b (consistent with the relations $bpb = q^{-1}$, $brb = s^{-1}$, $btb = u^{-1}$, $bvb = v$ and $bh^2b = h^2v$). For example, in the case of the second subgroup, the inclusion of the conjugate of b by $(hb)^6ht^{-1}bs^{-1}t^{-1}uh^{-1}(bh^{-1})^2uhu^{-1}$ in the generating set ensures that one of the K-orbits of length 81 in the representation Y is linked to itself by the action of b.

For use in what follows, we now make some additional observations. First, in each of the building blocks A and B, there are two K-orbits of length 1, and the single point in each such orbit is fixed by all of p, q, r, s, t, u, v, h^2 and also b. Next, on the 2912 points of the block A the permutation induced by the element bh has cycle structure $36^1 18^{157} 6^8 2^1$, with the two points from K-orbits of length 1 lying together in the single cycle of length 36. Again this can be verified by coset enumeration with the help of GAP or Magma. Similarly, on the 8825 points of the block B the permutation induced by the element bh has cycle structure $364^1 354^1 338^1 334^1 298^1 284^1 282^1 278^1 260^2 252^1 224^1 216^1 208^1 198^1$ $194^1 174^1 160^1 154^2 148^1 147^1 144^1 129^1 128^1 127^1 123^1 118^1 108^1 107^1 103^1 101^1 94^1 90^1$ $88^2 84^2 76^1 69^1 66^2 64^1 58^1 56^4 54^1 52^1 48^1 46^2 42^1 41^1 39^1 38^2 36^1 34^2 32^1 30^5 29^1 28^2 22^2$ $18^{14} 17^2 16^2 12^1 6^2 4^1$, with the two points from K-orbits of length 1 lying in disjoint single cycles of lengths 107 and 127.

Out of these basic building blocks we may construct transitive permutation representations of the generic infinite group $R_{4,7}$ of arbitrarily large degree, in an analogous manner to the approach taken in [4]. For example, several copies of the block A may be strung together, each linked to the next by the introduction of an additional transposition for b which interchanges fixed points of the subgroup K. Note that this process of linkage preserves all the relations for $R_{4,7}$. In particular it produces a transitive non-degenerate permutation representation of $R_{4,7}$ of degree $2912m$ for any positive integer m, and thereby an infinite family of 7-arc-transitive quartic graphs.

But further, since the numbers of points of our basic building blocks (2912 and 8825) are relatively prime, stringing together linear combinations of copies of A and B will produce transitive permutation representations of $R_{4,7}$ of any sufficiently large degree n. We will use this fact in the proof of our Theorem in the next Section, however before then we examine the process of linking blocks together in more detail:

Consider the transitive permutation repesentation of $R_{4,7}$ obtained by stringing together two copies of B with one copy of A in between, linking one of the K-orbits of length 1 from each copy of B with one of the two such orbits from A by the introduction of a new transposition for the involutory generator b. On the 20562 points of the resulting block, which we might call C, again each of the permutations induced by p, q, r, s, t, u, v and h is even, while the permutation induced by b is odd. Also the block C still has two points fixed by all of p, q, r, s, t, u, v, h^2 and b, one from each copy of B.

Finally we observe that the cycle structure of the permutation induced by bh on the points of C may be obtained easily from the cycle structure of bh on the points of each of the three basic blocks used to construct C. Almost all cycles of bh are unaffected, since the effect of b on all but four of the points of the three blocks is unchanged, and the effect of h is completely unchanged. The only points affected by linking together the three blocks are the four points moved by the two new transpositions introduced for b, say α_1 and α_2 from the K-orbits of length 1 in the copy of A, and β and γ from K-orbits of length 1 from the two different copies of B. Of these four, before linkage α_1 and α_2 lie in a single cycle of bh of length 36, while β and γ lie in disjoint cycles of length 107 or 127, and the effect of the linkage is to concatenate these three cycles into a single cycle of length 250, 270 or 290: $(\alpha_1,\beta)(\alpha_2,\gamma)(\alpha_1,\ldots,\alpha_2,\ldots)(\beta,\ldots)(\gamma,\ldots) = (\alpha_1,\ldots,\beta,\ldots,\alpha_2,\ldots,\gamma,\ldots)$.

4 Proof of the Theorem

For any positive integers x and y such that $x < y$, let $n = 2912x + 8825y$. Note that as 2912 and 8825 are relatively prime, and that for large x and y if necessary we may replace the pair (x, y) by $(x-8825, y+2912)$ to produce the same value of n, every sufficiently large positive integer n can be expressed in this form.

Now let us take x copies of the building block A and y copies of B, and link them together into a chain as described in the preceding Section, by introducing new transpositions for b which interchange fixed points of $\langle p, q, r, s, t, u, v, h^2, b\rangle$ in different blocks. Note that the order in which these blocks are linked is not particularly important, except that we will insist that every copy of the block A is linked between two successive copies of B, and also that each link between two copies of B will concatenate a cycle of bh of length 107 from one copy of B with a cycle of length 127 from the other, and that the copies of B at the ends of the chain provide one free cycle of bh of each of the lengths 107 and 127.

The resulting structure will be a transitive non-degenerate permutation representation of the infinite group $R_{4,7}$ on n points. Also the permutations induced by the generators p, q, r, s, t, u, v and h are all even, while that induced by b is odd: each copy of A or B contributes an odd number of transpositions to this permutation, and each link contributes one, and/but the number of copies of A and B exceeds the number of links by 1.

Next consider the cycle structure of the permutation induced by bh. From the analysis made at the end of the preceding Section we see that the length of almost every cycle of this permutation is one of 364, 354, 338, 334, 298, 284, 282, 278, 260, 252, 224, 216, 208, 198, 194, 174, 160, 154, 148, 147, 144, 129, 128, 123, 118, 108, 103, 101, 94, 90, 88, 84, 76, 69, 66, 64, 58, 56, 54, 52, 48, 46, 42, 41, 39, 38, 36, 34, 32, 30, 29, 28, 22, 18, 17, 16, 12, 6, 4 or 2, these being the lengths of cycles unaffected by interblock linkages. Resulting from the linkages we also have several cycles of length $107+36+107 = 250$, or $107+36+127 = 270$ or $127+36+127 = 290$ (all from subsections of the chain of the form B—A—B), and some of length $107+127 = 234$ (from links between successive copies of B). In addition and finally, there are single cycles of lengths 107 and 127, from copies of B at respective ends of the chain.

Note that the lengths of all but one of the cycles of this permutation are relatively prime to 107. Hence if we let m be the least common multiple of all such lengths (except 107), then the permutation induced by $(bh)^m$ is a single cycle of length 107. Note also that the latter cycle contains a point β which is fixed by the subgroup $K = \langle p, q, r, s, t, u, v, h^2 \rangle$ and by b, coming from one of the end copies of the building block B.

This 107-cycle can be used to prove our permutation representation of degree n is primitive. For if not, then all 107 points of the cycle would all lie in the same block of imprimitivity, say J. In particular, J would contain the point β, and therefore be preserved by $K = \langle p, q, r, s, t, u, v, h^2 \rangle$ and by b. But also J would contain $\beta^{bh} = \beta^h$, and therefore be preserved by h. Hence J would be an orbit for the whole group, that is, $|J| = n$, contradicting the assumption of imprimitivity.

Furthermore, the existence of this single cycle of prime length implies that the group generated by our permutations has to be alternating or symmetric of degree n, by Jordan's theorem (*cf* Theorem 13.9 in [9]). In fact we have the symmetric group S_n, as the permutation induced by b is odd, and accordingly S_n acts transitively on the 7-arcs of the corresponding finite quartic graph.

Similarly we may obtain A_{n+1} as a 7-arc-transitive group of automorphisms of a finite quartic graph, by linking a copy of the trivial permutation representation of the subgroup $H = \langle p, q, r, s, t, u, v, h \rangle$ to the end of the chain via the fixed point of b in the single cycle of bh of length 127. This increases the degree by 1, and adds one transposition to the permutation induced by b (making it even), while not affecting the permutations induced by the other generators. Also it lengthens the 127-cycle of bh by one point, and the above argument still applies, so the permutations generate A_{n+1}.

This completes the proof.

Acknowledgements

The first author gratefully acknowledges support from the Marsden Fund (grant number 95-UOA-MIS-0173), and the second author gratefully acknowledges support from a University of Auckland Postgraduate Scholarship.

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