ON ONE RECENT RESULT ON THE INTERSECTION OF WEIGHTED HARDY SPACES

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1. INTRODUCTION

The recent paper by K.Kazarjan, I.Spitkovski and F.Soria [1] is devoted to boundary Riemann problem in the weighted L_p spaces with singular weights, but as a consequence contains a number of statements related to the Toeplitz operators in the Hardy space H^2_+ in the unit disk. The corresponding result from [1] (Theorem 2) on Fredholmity of a pair of subspaces gives a nice generalization of the situation considered much earlier in the Helson-Sarason Theorem [2]. But the first statement from [1] (Theorem 1), which actually connects the dimension of the kernel of the corresponding Toeplitz operator with the number of so-called "zeroes" of the weight function on the unit circle is not true and the main purpose of the present paper is to clarify this point and to construct the counterexample to the statement of the above mentioned Theorem 1 from [1]. In what follows we denote as usual by D the unit disk, by $\mathbb{T}$ the unit circle. Let also $\mathcal{P} = \{p(z) = \sum_{k=m}^{n} a_k z^k, m, n \in \mathbb{Z}, m \leq n, z \in \mathbb{T}\}$ be the linear space of all trigonometric polynomials on $\mathbb{T}$ and $\mathcal{P}_- = \{p(z) \in \mathcal{P}, n < 0\}$, $\mathcal{P}_+ = \{p(z) \in \mathcal{P}, m \geq 0\}$. Let us first recall that the famous Helson-Szegö and Hunt-Muckenhoupt- Wheeden theorems [3, 4]describe those positive weights $w \in L_1(\mathbb{T})$ for which the angle between $Clos_{L_2(w,\mathbb{T})}\mathcal{P}_-$ and $Clos_{L_2(w,\mathbb{T})}\mathcal{P}_+$ is nonzero. Namely, such an angle is nonzero if and only if the weight w satisfies the Muckenhoupt condition. We denote in what follows $Clos_{L_2(w,\mathbb{T})}\mathcal{P}_+ = H^2_{w,+}$, $Clos_{L_2(w,\mathbb{T})}\mathcal{P}_- = H^2_{w,-}$ and call these spaces the weighted Hardy spaces of analytic and antianalytic functions respectively.

In the paper "Past and Future" [2] Helson and Sarason have obtained the further result which turned out to be very essential for the Fredholm theory of Toeplitz operators. This theorem asserts that the angle in $L_2(w, \mathbb{T})$ between the closures of $\mathcal{P}_-$ and $z^N\mathcal{P}_+, N \in \mathbb{N}$ is nonzero if and only if $w = |P_n(z)|^2 w_M$, where P_n is a polynomial of degree at most N with all its zeroes lying on $\mathbb{T}$ and w_M is the Muckenhoupt (or Helson-Szegö) weight. We should remind the reader that by Szegö theorem for all such weights $w \log w \in L_1(\mathbb{T})$ and consequently $w = |h|^2$, where h is an outer function from H^2_+.

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In terms of this outer function h the weighted Hardy spaces are related to the ordinary Hardy spaces (with respect to Lebesgue measure $d\varphi$) in a very simple way, namely $H^2_{w,+} = H^2_+/h, H^2_{w,-} = H^2_-/\bar{h}$.

Therefore all the facts on the intersection of $H^2_{w,+}$ and $H^2_{w,-}$ in $L_2(w, \mathbb{T})$ are equivivalent to the facts on the intersection of H^2_- and $\frac{\bar{h}}{h} H^2_+$ in L_2. Going further and recalling the definition of Toeplitz operator T_a with a symbol $a \in L_\infty$:

$$T_a f \stackrel{def}{=} P_+ af, \quad \forall f \in H^2_+,$$

we see that the last intersection is exactly the kernel of the operator $T_{\frac{\bar{h}}{h}}$.

Note that Helson-Szegö theorem is closely related to the following fact on the invertibility of Toeplitz operators with unimodular symbols (see for example [5, Appendix 4]):

Theorem. *Let $a \in L_\infty(\mathbb{T}), |a| = 1$, a.e. on $\mathbb{T}$, the Toeplitz operator $T_a = P_+ a|H^2_+$ is invertible if and only if $a = \frac{\bar{h}}{h}$ for some outer function $h \in H^2_+$, such that the weight $|h|^2$ on $\mathbb{T}$ is the Muckenhoupt one.*

In the same context the Helson-Sarason Theorem is related to the description of those Toeplitz operators $T_a, |a| = 1$ a.e on $\mathbb{T}$ for which the operator $T_{z^n a}$ is invertible for some $n \in \mathbb{N}$ or the same for which $indT_a = dimKerT_a = n$. Here as usual by $indA$ we denote the Fredholm index of the operator A, i.e. in a situation when A-image is closed, $indA = dimKerA - dimKerA^*$.

For such Toeplitz operators $az^n = \frac{\bar{h}}{h}$, where h is outer function from H^2_+ and $|h|^2$ is the Muckenhoupt weight on $\mathbb{T}$, or in terms of Helson-Sarason Theorem, $a = \frac{\bar{\tilde{h}}}{\tilde{h}}$, where $\tilde{h}$ is an outer function from H^2_+, and $|\tilde{h}|^2$ is the weight from the Helson-Sarason Theorem with exactly n zeroes on $\mathbb{T}$.

In terms of weighted Hardy spaces $H^2_{w,+}$ and $H^2_{w,-}$ this result in particular means that

1.the sum $H^2_{w,+} + H^2_{w,-}$ is closed and

2.$dim\{H^2_{w,+} \bigcap H^2_{w,-}\} = n$

if and only if $w = |P_n|^2 w_M$, where degree of P_n is equal to n and all its zeroes are lying on the unit circle, and w_M is the Muckenhoupt weight.

This last statement is exactly the particular case of the Theorem 2 from [1] for $p = 2$ and the absence of the "poles" of the weight on $\mathbb{T}$ (i.e. when $w \in L_1(\mathbb{T})$). In the terminology of [1] the outer function $h, w = |h|^2$, has a zero of order k at the point $\zeta \in \mathbb{T}$ if the function $1/h$ is not locally square summable at ζ, but an expression $(z - \zeta)^k/h$ is locally square summable at ζ and k is the smallest positive integer with such a property. The "pole" of h is defined as a zero of $1/h$ at the point on the unit circle.

The Theorem 1 from [1] relaxes the above condition 1 on the closedness of the sum and relates the dimension of the intersection with the number of poles and zeroes of h on the unit circle.

In particular in a case of summable weight (absence of poles) this theorem asserts that

1.the dimension α of the intersection $\{H^2_{w,+} \bigcap H^2_{w,-}\}$ is finite if and only if the total number n_+ of zeroes of h on $\mathbb{T}$ (taking into account the multiplicities) is finite

and that
2.$\alpha = n_+$. (condition 5)
This is exactly the statement which is contained in the published version of that paper (announcement), which appeared recently in Dokladi Akademii Nauk, V.353, no. 6 (1997), pp 717—719. The author should mention that at the beginning of 1997 he had got the draft version of this paper from one of the co-authors (K.Kazarjan) and has informed him about the counterexample which shows that without the assumption on the closedness of the sum of the weighted Hardy spaces the condition (5) is not true (we will discuss this example in the next section). Later K.Kazarjan informed the author that they had sent the letter to the editorial board of DAN having asked to improve the miss-print: to change the sign of equality in the condition (5) to the sign of less-or-equal, i.e. that $\alpha \leq n_+$. After that he also has confirmed that by having done this replacement they have managed to avoid the closedness condition in Theorem 1 of their paper.

In what follows we intend to show that even in this modified form the Theorem 1 from [1] is not true by producing the example of the outer function h with the infinite number of "zeroes" on $\mathbb{T}$ for which α is still finite.

2. Counterexample to the published version of Theorem 1 from [1]

The counterexample which shows that condition (5) from [1,Theorem 1]is not true is as follows. Let us consider the function $h(z) = \sqrt{1-z}$. h is an outer function from H^∞_+ and has one very simple property

$$\frac{\bar{h}}{h} = c\frac{1}{z}\frac{h}{\bar{h}}, \tag{1}$$

where c is a unimodular constant depending on a choice of the branch of the square root, which could be produced explicitly but plays no role in our reasoning. In the terminology of the paper [1] h has exactly one zero on $\mathbb{T}$ at the point 1 of degree one. Let us show that for $w = |1-z|$ the dimension of the intersection $H^2_{+,w} \bigcap H^2_{-,w}$ is equal to zero or which is the same that the dimension of $KerT_{\frac{\bar{h}}{h}}$ is equal to zero. Suppose that there is a nonzero element q from the last mentioned kernel. Then by (1)

$$\frac{1}{z}\frac{h}{\bar{h}}q = q_-,$$

where $q_- \in H^2_-$. Now, since $h \in H^\infty_+$, we have

$$\bar{h}q_- = hq\frac{1}{z}. \tag{2}$$

Note that, since $h \in H^\infty_+$, $\bar{h}q_- \in H^2_-, hq \in H^2_+$. Therefore (2) may hold only if hq is equal to constant, say c. But then $q = c/\sqrt{1-z} \notin H^2_+$. The last contradiction shows that $KerT_{\frac{\bar{h}}{h}} = \{0\}$.

The replacement of sign "=" by the sign "$\leq$" in condition (5) of [1] makes this simple counterexample invalid but before producing the example which will show that the first statement of Theorem 1 from [1] is not true (dimension of intersection is finite iff the total number of zeroes of h on $\mathbb{T}$ is finite), let us consider another simple example which will show that there exists a function h which has arbitrary many zeroes on $\mathbb{T}$ for which $KerT_{\frac{\bar{h}}{h}}$ is still zero space.

This example is a simple modification of the previous one. Consider the function $h = \sqrt{1-z^k}, k \in \mathbb{N}$. h is an outer function from H^∞_+ with k "zeroes" on $\mathbb{T}$ at the k-roots of the unit, each of which is of degree one. Therefore the total number of "zeroes" of h on $\mathbb{T}$ is equal to k. Let us show that $dimKerT_{\frac{\bar{h}}{h}} = 0$. Suppose again that there exits nonzero element q from this kernel. Note that h has a property

$$\frac{\bar{h}}{h} = c\frac{1}{z^k}\frac{h}{\bar{h}},$$

where c is again a unimodular constant depending on a choice of the branch of the square root. Therefore

$$\frac{1}{z^k}\frac{h}{\bar{h}}q = q_- \in H^2_-.$$

Now

$$\frac{1}{z^k}hq = \bar{h}q_-, hq \in H^2_+, \bar{h}q_- \in H^2_-.$$

The last condition may hold only if hq is a polynomial in z of degree less or equal to $k-1$. But for the function q to remain in H^2_+ this polynomial should have at least simple zeroes at all k-roots of unit and therefore should have a degree at least k. This contradiction shows that again $KerT_{\frac{\bar{h}}{h}}$ is a zero space.

3. The counterexample to the modified version of Theorem 1 from [1].

The main operator-theoretical idea will be just the same as in the the previous counterexamples. We want to construct the function $\chi \in H^\infty_+$, which is meromorphic in $\bar{\mathbb{C}} \setminus \{-1, 1\}$, has an infinite number of simple zeroes at the points $\zeta_j \in \bar{\mathbb{T}} \setminus \{-1, 1\}, j = 1, 2, \ldots$, the set of which has two accumulating points 1 and -1, and infinite number of simple poles at the points $1/\bar{\lambda}_j, \lambda_j \in D, j = 1, 2, \ldots$ with the same accumulating points. The additional property we require from χ is as follows:

$$\frac{\bar{\chi}}{\chi} = \frac{1}{B_\Lambda}, \tag{3}$$

where B_Λ is a Blaschke product with the simple zeroes at $\lambda_1, \lambda_2, \ldots$. Then for the outer function $h = \sqrt{\chi} \in H^\infty_+$ we have

$$\frac{\bar{h}}{h} = \frac{1}{B_\Lambda}\frac{h}{\bar{h}}, \tag{4}$$

and just as in the previous section, if there exists a nonzero function $q \in KerT_{\frac{\bar{h}}{h}}$ then

$$\frac{1}{B_\Lambda}\frac{h}{\bar{h}}q = q_- \in H^2_-,$$

that is

$$\frac{1}{B_\Lambda} qh = q_- \bar{h}, \tag{5}$$

where the function qh is from H^2_+ and $q_- \bar{h}$ is from H^2_-. As above, (3) holds only if

$$qh \in K_{B_\Lambda} = \bigvee_{j=1}^{\infty} \{\frac{1}{1 - \bar{\lambda}_j \zeta}\}. \tag{6}$$

It is well known (see for example [5]) that all functions from K_{B_Λ} are analytically continuable through any point on $\mathbb{T}$, which is not an accumulating point of the zeroes of the Blaschke product B_Λ. Hence all the functions from K_{B_Λ} are analytic not only in D but also on $\bar{\mathbb{T}} \setminus \{-1, 1\}$. This means that qh is analytic at all points $\zeta_j, j = 1, 2, \ldots$ and, by construction of function h, which has zeroes of degree $1/2$ at all these points, qh should have at least simple zeroes at these points.

To construct the desired function χ and to show that there could be at most one function in K_{B_Λ} with at least simple zeroes at all points $\lambda_j, j = 1, 2, \ldots,$ we will use some facts from the theory of character-automorphic (with respect to some discrete group of Möbius transformations of D) Hardy spaces which is closely related to harmonic analysis in finitely connected domains (and bordered Riemann surfaces)(see [6, 7, 8, 9]). Though in what follows we will not even mention the connection with harmonic analysis on the bordered Riemann surfaces, some facts we cite and use below, could be explained in the most simple way in frames of this theory. The facts we will use are related to the simplest case of the annulus or which is the same to the case of one parametric discrete group of Möbius transformations in D. Note that the basis of harmonic analysis in this case for automorphic functions has been developed in [10,11].

We should also mention here that the desired function χ could be produced using the unit disk theory only (see remarks in the concluding section) and our way of using the theory of character-automorphic functions is motivated by the wish to demonstrate the certain ideas of this theory which could be useful in other questions of the unit disk situation.

We will consider the simplest discrete group Σ of Möbius transformations of D generated by one transformation $\gamma = \frac{\zeta - \beta}{1 - \beta\zeta}, 0 < \beta < 1$. This group has two limiting points 1 and -1. If we use another conformal representation of D in a form of an infinite strip $S = \{w = x + iy | 0 < y < 1\}$, where the points 1 and -1 in D correspond to ∞ and $-\infty$ in S. Then in this strip the group Σ has a very simple form: it is a group of shifts on a period $\tau = \tau(\beta) > 0$, $\gamma(w) = w + \tau$. Let us also agree that the point at the origin in the disk corresponds to the point $\tau/2 + i1/2$ in S. The rectangle $F_0 = \{w = x + iy \in S | 0 \le x < 1\}$ and the corresponding set in D which contains the origin (and will also be denoted by F_0) are the so-called fundamental domains (rectangles) of Σ. The union of the inner boundary sides $\{0 \le x < 1, y = 0\}$ and $\{\{0 \le x < 1, y = 1\}$ (and the corresponding subset of $\mathbb{T}$) of this rectangle will be denoted by Γ_0, $\Gamma_j = \Gamma_0 + j\tau$ $(\Gamma_j = \gamma^j(\Gamma_0)), j \in \mathbb{Z}$. The rectangle which in the disk-model is symmetric to F_0 with respect to the unit circle in the strip-model is a rectangle of the strip $\bar{S} = \{w = x + iy | -1 < y < 0\}$ and is symmetric to F_0 with respect to the real axis. We will denote it by $J(F_0)$ in the both models (J is a reflection with respect to the unit circle in the disk-model and is reflection with respect to the real axis in the strip-model).

The most simple and well-known subspace of H^2_+, related to the group Σ is the subspace of all functions $f \in H^2_+$ which are automorphic with respect to Σ, i.e. for which $f(\gamma(z)) = f(z)$. Note that for any two functions f and g from this subspace

$$< f, g >= \frac{1}{2\pi i}\int_{\mathbb{T}} f\bar{g}\frac{d\zeta}{\zeta} =$$

$$\frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}\int_{\Gamma_0} f(\gamma^j(\zeta))\overline{g(\gamma^j(\zeta))}\frac{(\gamma^j)'(\zeta)}{\gamma^j(\zeta)}d\zeta =$$

$$\frac{1}{2\pi i}\int_{\Gamma_0} f\bar{g}\sum_{j=-\infty}^{\infty}\frac{(\gamma^j)'(\zeta)}{\gamma^j(\zeta)}d\zeta,$$

which means that this subspace may be considered as a subspace of a Hilbert space $L_2(\gamma_0)$ with respect to the measure $d\rho = \frac{1}{2\pi i}\sum\limits_{j=-\infty}^{\infty}\frac{(\gamma^j)'(\zeta)}{\gamma^j(\zeta)}d\zeta$ on Γ_0. Note (see [8]) that the last series converges uniformly on any compact set in $\bar{\mathbb{C}} \setminus \{1, -1\}$.

But there are other natural subspaces of H^2_+ which can be reduced to the subspaces of $L_2(\Gamma_0, d\rho)$. For any fixed $\kappa \in [0, 2\pi]$ consider all the functions from H^2_+ with the property $f(\gamma(\zeta)) = e^{i\kappa}f(\zeta)$. This space is called the character-automorphic Hardy space corresponding to the character κ and is denoted by $H^2_{+,\kappa}$. In these notations the discussed above space of automorphic functions is exactly the space $H^2_{+,0}$. In the same way as it was done above for $H^2_{+,0}$, we can show that $H^2_{+,\kappa}$ may also be treated as a subspace of $L_2(\Gamma_0, d\rho)$.In the same way we can consider the character-automorphic subspaces of H^2_- and L^2. The first one will be denoted by $H^2_{-,\kappa}$ and the restriction of the functions from the second one on Γ_0 will obviously give us $L_2(\Gamma_0, d\rho)$ for any character κ. Lastly note that since $H^2_{+,\kappa}$ is a functional Hilbert space, for any $\zeta \in F_0$ there exist a reproducing kernel at this point, which will be denoted by k^κ_ζ,

$$\int_{\Gamma_0} f\bar{k}^\kappa_\zeta d\rho = f(\zeta), \forall f \in H^2_{+,\kappa}.$$

The facts we need from the theory of character-automorphic Hardy spaces are as follows.

(1) For any $\kappa \in [0, 2\pi]$

$$L_2(\Gamma_0, d\rho) = H^2_{+,\kappa} \oplus H^2_{-,\kappa} \oplus \mathfrak{M}_\kappa,$$

where $\mathfrak{M}_\kappa$ is one-dimensional so-called κ-automorphic defect space. The function m_κ which generates $\mathfrak{M}_\kappa$ is meromorphic in $\bar{\mathbb{C}} \setminus \{1, -1\}$. The zero-pole divisor of this function is very simple: In the rectangle $F_0 \cup J(F_0) \cup [0, \tau) = \{w = x + iy | 0 \le x < \tau, -1 < y \le 1$ it has two poles and two zeroes. The poles are at the points $i1/2, -i1/2$ and one of the zeroes is in the point $\tau/2 - i1/2$ (point at ∞ in the disk-model). Note that this part of zero-pole divisor of m_κ does not depend on κ. Only the second zero depends on κ, but for all κ the corresponding point lies on the line $x = \tau/2$. The total zero-pole divisor of this function is obtained from the described one by translation by the elements of the group Σ.

(2) To get the character-automorphic function, meromorphic on $\bar{\mathbb{C}} \setminus \{1, -1\}$, with the "minimal" zero-pole divisor in the rectangle $R_0 = F_0 \cup J(F_0) \cup [0, \tau)$, i.e. with only one pole p and only one zero z in R_0, one should choose z and p in such a way, that $Rez = Rep$. If so, the corresponding to this function character κ will be modulo 2π equal to $\pi(Imz - Imp)$.

Now if we take $z \in [0, \tau)$ and $p = z - ir, 0 < r < 1$, the corresponding function $\chi = \chi_{z,p}$ will be bounded analytic function in $F_0 \cup [0, \tau)$ and therefore bounded analytic function in S. It will have a simple zero at the point z and its translates by Σ. In what follows it will be more convenient for us to use the disk-model and we will denote by ζ_0 the point on Γ_0 in the disk-model, corresponding to the point z and by λ_0 the point in D corresponding to the point $\bar{p} = x + ir$. Therefore χ will the meromorphic in $\bar{\mathbb{C}} \setminus \{1, -1\}$, bounded in D analytic on $\bar{\mathbb{T}} \setminus \{1, -1\}$ and will have simple zeroes at the points $\zeta_j = \gamma^j(\zeta_0), j \in \mathbb{Z}$. Moreover the function $\bar{\chi}/\chi$ will be the function of the same type, with the zeroes at $\lambda_j = \gamma^j(\lambda_0)$ and poles at $1/\overline{\lambda_j} = \gamma^j(\frac{1}{\overline{\lambda_0}})$. This function is unimodular on the boundary of the strip S and in the disk-model is a Blaschke product B_Λ with the zeroes at the points $\lambda_j = \gamma^j(J(\lambda_0))$, i.e.

$$\frac{\bar{\chi}}{\chi} = \frac{1}{B_\Lambda}.$$

(3) Let us denote by E_κ the operator of orthogonal projection in $L_2(\mathbb{T})$ onto the space L_2^κ of κ-automorphic functions from $L_2(\mathbb{T})$ (see [7, 8, 12, 13]). Following [8] we will call this operator the κ-automorphic conditional expectation. The following expression for E_κ in terms of Poincare θ-series (averaging with respect to the group) has been obtained in [8]:

$$E_\kappa f = \frac{\sum\limits_{j=-\infty}^{\infty} f(\gamma^j)\frac{(\gamma^j)'}{\gamma^j}e^{-ij\kappa}}{\sum\limits_{j=-\infty}^{\infty} \frac{(\gamma^j)'}{\gamma^j}} \quad \forall f \in L_2.$$

The series in the numerator converges in L_2 (as well as in L_1) and if $f \in H_+^2$, the term number j of it has only one simple pole at the point $\gamma^j(0)$. Therefore for the function zf each term is analytic in D and since for the functions from H_+^2 the L_2 convergence implies the uniform convergence on the compact subsets of D, and the series for zf and hence for f itself converges uniformly inside D.

The main properties of the operator E_κ are as follows:

(1) For any automorphic subset X of $\mathbb{T}$ ($\gamma(X) = X$) and $f \in L_1$

$$\int_X E_0 f d\varphi = \int_X f d\varphi.$$

(2)
$$\|E_\kappa f\|_{L_1} \le \|E|f|\|_{L_1} \qquad \forall f \in L_2$$

(3) For $f \in L_2$ and $g \in L_2^\mu$
$$E_\kappa fg = gE_{\kappa-\mu} f$$

(4)
$$E_\kappa H^2_+ = H^2_{+,\kappa} \oplus \mathfrak{M}_\kappa.$$

(5) For the reproducing kernel $k_z = 1/(1-\bar z\zeta), z \in F_j$
$$P^\kappa_+ E_\kappa k_z = e^{ij\kappa} k^\kappa_z.$$

Moreover, we will need the following property of this operator which, possibly is well known, but is not contained in the classic papers on this subject. We will present here only rather week statement on the continuous dependence of E_κ of κ, which is quite enough for our purposes.

Proposition 3.1. *Let $f(\zeta) \in L_1(\mathbb{T})$, then for almost every (with respect to Lebesgue measure) point $\zeta \in \mathbb{T}$ the function $E_\kappa f(\zeta)$ as a function of variable $t = e^{i\kappa}$ is continuous on the unit circle $|t| = 1$.*

Proof. By the properties (1) and (2) of the operator E_κ
$$\|E_0|f|\|_{L_1} = \int_{\mathbb{T}} \sum_{j=-\infty}^{\infty} |f(\gamma^j)\frac{(\gamma^j)'}{\gamma^j}|d\varphi = \int_{\mathbb{T}} |f| d\varphi.$$

This means that the series
$$\sum_{j=-\infty}^{\infty} |f(\gamma^j)\frac{(\gamma^j)'}{\gamma^j}|$$

converges in L_1 and hence it converges point wise for almost every point $\zeta \in \mathbb{T}$. But this implies the absolute convergence of the power series
$$\sum_{j=0}^{\infty} f(\gamma^j)\frac{(\gamma^j)'}{\gamma^j} t^j \quad \text{and} \quad \sum_{j=-\infty}^{-1} f(\gamma^j)\frac{(\gamma^j)'}{\gamma^j} t^j$$

on the unit circle $|t| = 1$ for almost every point $\zeta \in \mathbb{T}$ Therefore for such ζ the power series $\sum_{j=-\infty}^{\infty} f(\gamma^j)\frac{(\gamma^j)'}{\gamma^j} t^j$ is absolutely convergent on the unit circle and defines the continuous function on the unit circle.

Now in construction of the function χ we choose the point $z = z_\Gamma$ on $[0, \tau)$ not to be equal to $\tau/2$. Thus z_Γ is not a zero of the defect element m_κ for all κ. It is well known that for the Blaschke product B_Λ with the simple zeroes at the points $\{\gamma^j(\lambda_0)\}, j \in \mathbb{Z}$, the sequence $\{\gamma^j(\lambda_0)\}_{j=-\infty}^{\infty}$ satisfies the Carleson condition and hence the system
$$\left\{ \frac{\sqrt{1-|\gamma^j(\lambda_0)|^2}}{1-\overline{\gamma^j(\lambda_0)}\zeta} \right\}_{j=-\infty}^{\infty},$$

forms the Riesz basis in K_{B_Λ}. Therefore for any $f \in K_{B_\Lambda}$

$$f = \sum_{j=-\infty}^{\infty} c_j \frac{\sqrt{1-|\gamma^j(\lambda_0)|^2}}{1-\overline{\gamma^j(\lambda_0)}\zeta}, \quad \sum_{j=-\infty}^{\infty} |c_j|^2 \leq \infty. \tag{4}$$

Hence, by the property (5) of the operator E_κ

$$P_+^\kappa E_\kappa f = c(e^{i\kappa}) k_{\lambda_0}^\kappa,$$

where $c(e^{i\kappa}) = \sum\limits_{j=-\infty}^{\infty} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j e^{ij\kappa}$ and $c(t)$ is the continuous function on the unit circle $|t| = 1$. Note that $P_+^\kappa E_\kappa f$ could not be identically equal to zero for all κ, since otherwise the function $c(e^{i\kappa})$ is identically equal to zero on the unit circle but has nonzero sequence $\{\sqrt{1-|\gamma^j(\lambda_0)|^2} c_j\}_{j=-\infty}^{\infty}$ of Fourier coefficients. Now for the outer function $h = \sqrt{\chi}$ let us suppose that there are at least two linear independent elements in $KerT_{\frac{\bar h}{h}}$, then by the reasoning at the beginning of this section there are two linear independent nonzero functions $f_1, f_2 \in K_{B_\Lambda}$ both of which has at least simple zeroes at all points $\zeta_j = \gamma^j(\zeta_0), j \in \mathbb{Z}$. Note that we can suppose that at least one of these functions (say f_1) is not vanishing at the point $\zeta = 0$. Indeed, if the order of the zero of f_1 at $\zeta = 0$ is equal to $k > 0$ and is less or equal then the order of zero of f_2 at $\zeta = 0$ then, since K_{B_Λ} is the co-invariant subspace of the shift operator, the function $f_1/\zeta^k = P_+\{f_1/\zeta^k\}$ also belongs to K_{B_Λ} and has at least the simple zeroes at the points $\zeta_j = \gamma^j(\zeta_0),\ j \in \mathbb{Z}$ as well as the functions f_1 and f_2 themselves. Moreover this function is linear independent from f_2. Therefore we can suppose that our initial function f_1 is not vanishing at $\zeta = 0$, and hence from (4) we see that the corresponding function $c_1(e^{i\kappa})$ is not equal to zero at $\kappa = 0$. By the properties of E_κ, both functions $E_\kappa f_l, l = 1, 2$, have the zeroes at the points $\gamma^j(\zeta_0), j \in \mathbb{Z}$ and , as we have see above, the functions $c_k(e^{i\kappa}), k = 1, 2$, are continuous on the unit circle and both are not identically equal to zero. Therefore the function $c_2(e^{i\kappa})E_\kappa f_1 - c_1(e^{i\kappa})E_\kappa f_2$ belongs to $\mathfrak{M}_\kappa$ for all κ and has zero at the point $\zeta_0 \in \Gamma_0$. This means that this function is identically equal to zero for all κ.

But this means that

$$\Big(\sum_{j=-\infty}^{\infty} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j^{(2)} e^{ij\kappa}\Big) \frac{\sum\limits_{j=-\infty}^{\infty} f_1(\gamma^j) \frac{(\gamma^j)'}{\gamma^j} e^{-ij\kappa}}{\sum\limits_{j=-\infty}^{\infty} \frac{(\gamma^j)'}{\gamma^j}} =$$

$$\Big(\sum_{j=-\infty}^{\infty} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j^{(1)} e^{ij\kappa}\Big) \frac{\sum\limits_{j=-\infty}^{\infty} f_2(\gamma^j) \frac{(\gamma^j)'}{\gamma^j} e^{-ij\kappa}}{\sum\limits_{j=-\infty}^{\infty} \frac{(\gamma^j)'}{\gamma^j}}$$

Using the proposition 3.1 and integrating the both sides of the last equality in κ over $[0, 2\pi]$, we get that for almost all $\zeta \in \mathbb{T} \setminus \{1, -1\}$

$$\sum_{j=-\infty}^{\infty} f_2(\gamma^j) \frac{(\gamma^j)'}{\gamma^j} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j^{(1)} = \sum_{j=-\infty}^{\infty} f_1(\gamma^j) \frac{(\gamma^j)'}{\gamma^j} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j^{(2)}. \tag{5}$$

But the series in the right-hand side and the left-hand side of the last identity converge uniformly on any compact set inside the unit disk and hence represent the function meromorphic in the unit disk. Therefore, it follows that (5) is true also for all $\zeta \in D$. Let us compare the residues of the right-hand side and of the left-hand side at the points $\gamma^j(0), j \in \mathbb{Z}$. The j-th terms in the both sides of (5) have at most (in the unit disk) the simple poles at the point $\zeta = \gamma^{-j}(0)$. Evaluating the residues at these points with the help of (4), we get the series of identities

$$\sum_{n=-\infty}^{\infty} \sqrt{1-|\gamma^n(\lambda_0)|^2} c_n^{(2)} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j^{(1)} =$$

$$\sum_{n=-\infty}^{\infty} \sqrt{1-|\gamma^n(\lambda_0)|^2} c_n^{(1)} \sqrt{1-|\gamma^j(\lambda_0)|^2} c_j^{(2)}, \quad j \in \mathbb{Z}.$$

Since by our assumption $c_1(1) = \sum_{n=-\infty}^{\infty} \sqrt{1-|\gamma^n(\lambda_0)|^2} c_n^{(1)} \neq 0$, we see that $c_j^{(1)} = const \cdot c_j^{(2)}$, which means that f_1 and f_2 are linearly dependent. This contradiction with our initial assumption finishes the construction of desired counterexample.

4. Final remarks and acknowledgments

After constructing the above described counterexample the author has given two talks on this subject in the St.Petersburg Division of Steklov mathematical Institute of Russian Academy of Science. The author is grateful for Professor V.I Vasunin and Doctors D.Yakubovich and V.Kapustin for helpful discussions. The above mentioned persons have suggested the simplification which avoids the theory of character-automorphic Hardy spaces. The way of their reasoning was as follows:

Consider the infinite product in the unit disk

$$\prod_{j=1}^{\infty} \frac{1-\overline{\zeta_j}\zeta}{1-\overline{\lambda_j}\zeta},$$

with the points $\zeta_j, \lambda_j, |\zeta_j| = 1, |\lambda_j| = 1, j = 1, \ldots$ having only one point $\zeta = 1$ as an accumulation point, which represents the function χ, meromorphic in $\mathbb{C} \setminus \{1\}$ and bounded in D. This can be done by considering the Blaschke product for the "shifted" disk which contains D and whose circle is tangent to $\mathbb{T}$ at the point $\zeta = 1$. Then the function χ satisfies the condition (3), where B_Λ is a Blaschke product with the simple zeroes at the points $\lambda_j, j = 1, \ldots$. Then the function $h = \sqrt{\chi}$ satisfy the condition (4) and if $KerT_{\frac{\bar h}{h}}$ contains nonzero element conditions (5) and (6) hold. Then the same reasoning as at the beginning of section 3 shows that the functions qh and $q_-\bar h$ are analytic on $\mathbb{T} \setminus \{1\}$ and have at least simple zeroes at all the points $\zeta_j, j = 1, \ldots$. Then

$$\frac{q}{h} = \frac{q_-}{\bar h},$$

where the left-hand side represents the function analytic in $D \cup \mathbb{T} \setminus \{1\}$ and the right-hand side represents the function analytic in the exterior of the unit disk plus

$\mathbb{T}\setminus\{1\}$. therefore we get the function which is analytic in $\mathbb{C}\setminus\{1\}$ and equal to zero at infinity by conditions (5) and (6). After that by showing the the isolated singular point of this function at $\zeta = 1$ is removable (which itself is rather clever and tricky reasoning) we get the desired contradiction which shows that this construction also gives a counterexample to Theorem 1 from [1].

We want to emphasize once more that one may consider the theory of character-automorphic functions to be a too much heavy "weapon" used to fight the problem, but we hope that the technique demonstrated in section 3, which actually reduced the question about infinitely dimensional space K_{B_Λ} to the finite dimensional space (its character-automorphic projection) could be useful in another situations.

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