Certain cyclically presented groups are isomorphic

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Abstract

We present two pairs of infinite families of cyclically presented groups, where all groups in each pair have the same Alexander polynomials, and prove that the corresponding groups in each pair are isomorphic.

1 Introduction

Takahashi (1989) considers a family of closed 3-dimensional manifolds whose fundamental groups are cyclically presented. The polynomials associated with the cyclic presentations (see Johnson, 1974) coincide with the Alexander polynomials of the corresponding knots.

A natural question arising from this study is the following: are all cyclically presented groups with the same Alexander polynomial isomorphic?

While we expect a negative answer to this question in most cases, here we present two pairs of infinite families of cyclically presented groups and show that the corresponding groups in each pair are isomorphic.

The infinite families of groups are for all $n \geq 1$:

$$\begin{aligned} G_1(n) &= \langle\, x_i \mid x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_ix_{i+1}^{-1}x_i = 1 \,\rangle_n, \\ G_2(n) &= \langle\, y_i \mid y_{i+1}^{-1}y_{i+2}y_iy_{i+1}^{-1}y_{i+2}y_{i+1}^{-1}y_i = 1 \,\rangle_n \end{aligned} \qquad \text{(I)}$$

and

$$
\begin{aligned}
H_1(n) &= \langle\, x_i \mid x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_{i+1}^{-2}x_ix_{i+1}^{-1}x_i = 1 \,\rangle_n, \\
H_2(n) &= \langle\, y_i \mid y_{i+1}^{-1}y_{i+2}y_{i+1}^{-1}y_iy_{i+1}^{-1}y_{i+2}y_{i+1}^{-2}y_i = 1 \,\rangle_n.
\end{aligned} \tag{II}
$$

Each group has n generators and n relations, which are obtained from the given one by applying powers of the cycle $(1, 2, \ldots, n)$ to the subscripts and reducing these modulo n to lie in the set $\{1, 2, \ldots, n\}$. See Johnson (1990, p. 95) for further discussion of such presentations.

The first pair (I) has the Alexander polynomial $f(t) = 2t^2 - 3t + 2$ of the knot with 5 crossings denoted by 5_2 in Rolfsen (1976). The second pair (II) has the Alexander polynomial $f(t) = 2t^2 - 5t + 2$ of the knot with 6 crossings denoted by 6_1.

Our central result is:

Theorem 1.1 *For all $n \geq 1, G_1(n) \cong G_2(n)$ and $H_1(n) \cong H_2(n)$.*

The clue to the proof given in §3 is provided by the machine computations described in §2, which also suggest some of the remarks and open problems discussed in §4.

A detailed study of the connections between such presentations and closed 3-dimensional manifolds appeared in Kim and Vesnin (1997).

2 Investigating the presentations

As a first step towards understanding these groups, we investigated the presentations $G_k(3)$ for $k = 1, 2$.

While the general problem of deciding whether or not two finitely-presented groups are isomorphic is insoluble, O'Brien (1994) describes a practical algorithm which can decide whether or not two given finite p-groups are isomorphic. We used his implementation of this algorithm to establish that the class 20 5-quotients, each of order 5^{30}, of $G_1(3)$ and $G_2(3)$ are isomorphic.

Holt and Rees (1992) describe an approach that attempts to decide whether or not two finitely-presented groups are isomorphic. In particular, they attempt to prove isomorphism by using the Knuth-Bendix procedure to generate a word-reduction algorithm for words in the generators. If the attempt is successful, then it enables the program to verify that a particular map from one group to another is in fact an isomorphism. They generate candidate isomorphisms by an exhaustive search procedure. Their implementation, TestIsom, of this technique is distributed as part of the Quotpic package (see Holt and Rees, 1993).

TestIsom produced the following isomorphism from $G_1(3)$ to $G_2(3)$:

$$\begin{aligned} x_1 &\mapsto y_1y_2^{-1} \\ x_2 &\mapsto y_3y_1^{-1} \\ x_3 &\mapsto y_2y_3^{-1} \end{aligned}$$

It produced the following isomorphism from $H_1(3)$ to $H_2(3)$:

$$\begin{aligned} x_1 &\mapsto y_1y_2^{-1} \\ x_2 &\mapsto y_2y_3^{-1} \\ x_3 &\mapsto y_3y_1^{-1} \end{aligned}$$

TestIsom also produced isomorphisms for the pairs $G_k(4)$ and $H_k(4)$ for $k = 1, 2$, but failed to decide whether or not the pairs are isomorphic for $n = 5$.

These isomorphisms determined the form of the general proof outlined in §3.

3 Establishing the isomorphisms

We begin with the second pair since they are somewhat easier to handle, and perform the following operations on the given presentation for $H_1(n)$:

- isolate the sixth letter of the relation;
- adjoin new generators by Tietze transformations;
- adjust the dummy variable in the relations.

Applying this strategy, we obtain:

$$\begin{aligned} H_1(n) &\cong \langle\, x_i \mid x_{i+1} = (x_ix_{i+1}^{-1})^2(x_{i+2}x_{i+1}^{-1})^2 \,\rangle_n \\ &\cong \langle\, x_i,\ a_i \mid x_{i+1} = a_{i+1}^2a_{i+2}^{-2},\ a_i = x_{i-1}x_i^{-1} \,\rangle_n \\ &\cong \langle\, x_i,\ a_i \mid x_i = a_i^2a_{i+1}^{-2},\ a_i = x_{i-1}x_i^{-1} \,\rangle_n. \qquad (1) \end{aligned}$$

Operating on $H_2(n)$ in a similar way, we obtain:

$$\begin{aligned}
H_2(n) &\cong \langle\, y_i \mid y_{i+1} = (y_i y_{i+1}^{-1} y_{i+2} y_{i+1}^{-1})^2 \,\rangle_n \\
&\cong \langle\, y_i,\ b_i \mid y_{i+1} = (b_i b_{i+1}^{-1})^2,\ b_i = y_i y_{i+1}^{-1} \,\rangle_n \\
&\cong \langle\, y_i,\ b_i \mid y_i = (b_{i-1} b_i^{-1})^2,\ b_i = y_i y_{i+1}^{-1} \,\rangle_n.
\end{aligned} \tag{2}$$

The map $\alpha : x_i \mapsto b_i$ between the presentation (1) and (2) sends a_i to $b_{i-1}b_i^{-1}$ and a_i^2 to y_i. The relations in (1) are thus preserved and α induces a homomorphism. The same is true for the map $\beta : y_i \mapsto a_i^2,\ b_i \mapsto x_i$ and, since this is clearly the inverse of α, it follows that α in an isomorphism.

The procedure for the first pair is similar, except that we need to "reverse the orientation" in $G_2(n)$. That is, we replace i by $-i$ in the relation and then y_i by y_{-i}.

$$\begin{aligned}
G_2(n) &\cong \langle\, y_i \mid y_{i-1}^{-1} y_{i-2} y_i y_{i-1}^{-1} y_{i-2} y_{i-1}^{-1} y_i = 1 \,\rangle_n \\
&\cong \langle\, y_i \mid y_{i-2} = y_{i-1} y_i^{-1} y_{i-1} y_{i-2}^{-1} y_{i-1} y_i^{-1} \,\rangle_n \\
&\cong \langle\, y_i \mid y_i = y_{i+1} y_{i+2}^{-1} y_{i+1} y_i^{-1} y_{i+1} y_{i+2}^{-1} \,\rangle_n \\
&\cong \langle\, y_i,\ x_i \mid y_i = x_{i+1}^{-1} x_i x_{i+1}^{-1},\ x_i = y_{i+1} y_i^{-1} \,\rangle_n \\
&\cong \langle\, x_i \mid x_i = x_{i+2}^{-1} x_{i+1} x_{i+2}^{-1} x_{i+1} x_i^{-1} x_{i+1} \,\rangle_n \\
&\cong G_1(n).
\end{aligned}$$

4 Some remarks

Remark 4.1 The presentation (1) shows that the relations

$$x_1 x_2 \dots x_n = 1,\ a_1 a_2 \dots a_n = 1$$

hold in the groups $H_k(n)$. It also yields a third presentation for these groups: namely,

$$H_3(n) = \langle\, a_i \mid a_i = a_{i-1}^2 a_i^{-2} a_{i+1}^{-2} a_i^{-2} \,\rangle_n.$$

Remark 4.2 A relation matrix for $H_k(n)$ is the $n \times n$ circulant matrix associated with the polynomial

$$2t^2 - 5t + 2 = 2(t - 1/2)(t - 2),$$

and it follows from Johnson (1974) that the order of the derived quotient of $H_k(n)$ is

$$2^n(1-(1/2)^n)(2^n-1) = (2^n-1)^2.$$

Since the corresponding polynomial $2t^2-3t+2$ for $G_k(n)$ does not factorize, we have no such nice formula for the order of the derived quotient of $G_k(n)$; all we can say is that the derived quotient is finite.

Question 4.3 Which of the $G_k(n), H_k(n)$ are finite groups?

Remark 4.4 There is some evidence (based on an investigation of p-quotients) to suggest that the $G_k(n)$ and $H_k(n)$ are 2-generator groups. However, we cannot prove this claim in general.

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