1-factorizations of Cayley graphs on solvable groups

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Abstract

A well known unresolved conjecture states that every Cayley graph on a solvable group G has a 1-factorization. We show that if the commutator subgroup of of such a group is an elementary abelian p-group, then every quartic Cayley graph on G has a 1-factorization.

1 Introduction

Let G be a finite group and $S \subset G$ satisfy $1 \notin S$ and $s \in S$ if and only if $s^{-1} \in S$. The *Cayley Graph* $X = X(G;S)$ is the graph with vertex set G and $ab \in E(G)$ if and only if $b = as$ for some $s \in S$. A Cayley graph is a simple vertex transitive graph. Let C be a cycle in $X = X(G;S)$ with vertex set $\{a, ax_1, ax_1x_2, ax_1x_2x_3, \cdots, ax_1x_2\cdots x_k\}$ where $x_1, x_2, \cdots, x_k \in S$. We denote C by $C = [a](x_1, x_2, \cdots, x_k)$ where $[a]$ indicates the cycle passing through the vertex a. If $a = 1$, we simply write $C = (x_1, x_2, \cdots, x_k)$. For an integer n, we also use $n * x$ to indicate $(x, x, \cdots, x)$ (there are n terms). So for example, $C = (2 * (a, 3 * b), c)$ is just $(a, b, b, b, a, b, b, b, c)$.

A *j*-factor of a graph is a spanning subgraph which is regular of valence j. In particular, a *1-factor* of a graph is a collection of vertices such that each vertex is incident with exactly one edge. A *1-factorization* of a regular graph is a partition of the edge set of the graph into disjoint 1-factors. A *1-factorization* of a regular graph of valence k is equivalent to a colouring of the edges in k colours where each 1-factor is coloured a different colour.

By Vizing's Theorem [2, p245-248] any simple regular graph of valence k can be edge coloured in either k or $k+1$ colours. Therefore, if a simple regular graph of valence k cannot be 1-factorised then it can be edge coloured in $k+1$ and not fewer colours. Such graphs include all Cayley graphs of groups of odd order.

Previous results on Cayley graphs of groups of even order include those of Stong [4] and Alspach et al [1]. The former showed that every connected graph on a finite abelian group of even order has a 1-factorization and the latter that every cubic Cayley graph on an even solvable group has a 1-factorization. A well known unresloved conjecture states that every Cayley graph on an even solvable group has a 1-factorization. We investigate this conjecture for quartic Cayley graphs.

Conjecture: Every quartic Cayley graph on an even solvable group has a 1-factorization.

Theorem 1 *Let G be a solvable group of even order. If the commutator subgroup G' is an elementary abelian p-group, then every quartic Cayley graph on G has a 1-factorization.*

We note that this rather restricted condition concerning the the commutator subgroup is actually only needed in the proof of one specific subcase.

2 Proof of the Main Theorem for Quartic Graphs

Throughout this section G always denotes a solvable group of even order whose commutator subgroup G' is an elementary abelian p-group where p is a prime.

Let $X = X(G; S)$ be a Cayley graph of degree four. By Stong's result [4] we need only consider nonabelian groups. Induction is used in some parts of the proof, we assume that every quartic Cayley graph $X(H; S)$ with H having order less than G has a 1-factorization. We also assume that X is connected, for if it isn't then the induction assumption may be applied to each component.

There are three possible forms for the generating set S.

Form 1: $S = \{a, b, c, d\}$ where $a^2 = b^2 = c^2 = d^2 = 1$.

Each element of order two generates a 1-factor, hence X has a 1-factorization.

Form 2: $S = \{a, b, c, c^{-1}\}$ where $a^2 = b^2 = c^s = 1,\ s > 2$.

Let $\overline{S} = \{a, c, c^{-1}\}$, then $\overline{X} = \overline{X}(G : \overline{S})$ is a cubic Cayley graph on the even solvable group G and hence by [1] is 3-edge colourable. Since X is just $\overline{X}$ with a 1-factor added it follows that X is 4-edge colourable and hence has a 1-factorization.

Form 3: $S = \{a, a^{-1}, b, b^{-1}\}$ where $a^s = b^t = 1,\ s, t > 2$.

Let N be a minimal normal subgroup of G. Since G is a finite solvable group, N is an elementary abelian subgroup [3, p112]. We now consider all possible cases of the Cayley graphs that arise as the intersection of this minimal normal subgroup N with S varies in content.

CASE A: Assume that $N \cap S = \emptyset$.

For the case $N \cap S = \emptyset$ consider the quotient graph $\overline{X}$ obtained by first contracting every coset of N to a single vertex. Since N is a normal subgroup if some vertex of a coset gN is adjacent to d vertices of another coset hN then every vertex of gN is adjacent to d vertices of hN and vice-versa. Put an edge of multiplicity d between the vertices corresponding to the cosets gN and hN in $\overline{X}$.

Note that since $N \cap S = \emptyset$ no two adjacent vertices in X can be in the same coset of $\overline{X}$. There are five possibilities for the quotient graph $\overline{X}$ thus formed.

Possibility (a): $\overline{X}$ is $4K_2$ (that is, two vertices joined by an edge of multiplicity 4).

Possibility (b): Every vertex of $\overline{X}$ is incident with an edge of multiplicity two. This means that the quotient graph looks like a cycle where each edge in the cycle has multiplicity two.

Possibility (c): Every vertex of $\overline{X}$ is incident with an edge of multiplicity one and another edge of multiplicity three. This means that the quotient graph looks like an even length cycle in which every other edge around the cycle has multiplicity three.

Possibility (d): $\overline{X}$ is a quartic multigraph, each vertex has two single edges and one double edge.

Possibility (e): $\overline{X}$ is a quartic graph.

If p is odd then $|G/N|$ is even (since $|G|$ is even and $|N|$ is odd). Possibilities (a), (b), (c), and (d) are verified by using the argument that each edge of $\overline{X}$ corresponds to a regular bipartite subgraph of X. Possibility (e) follows since G/N is a solvable even group of degree less than G so $\overline{X}$ is 4-edge colourable by induction. Each colour class lifts to a 1-factor of X so that X is 4-edge colourable.

If p is even then $p = 2$ and $|G/N|$ may be odd or even. If $|G/N|$ is even then everything follows through as for $p \neq 2$. If $|G/N|$ is odd then Possibilities (a) and (c) obviously can't occur. Using a result from Weilandt [4, Proposition 7.2] it follows that Possiblity (d) can't occur because the group G/N acts transitively on this graph and each double edge must be mapped to a double edge so there are an even number of vertices which contradicts the fact that $|G/N|$ odd. This leaves Possibilities (b) and (e) to be settled for $|G/N|$ odd.

Let $G = 2^\ell m$ where $\gcd(2^\ell, m) = 1$ and $|N| = 2^k$, then $k = \ell$ since $|G/N|$ is odd. Let $|G/N| = 2r + 1$.

Case (i): Possibility (b) when $|G/N|$ is odd.
The following facts can be verified:

(a) Since the multigraph $\overline{X}$ is cyclic there must be a cycle through the $2r + 1$ cosets, one such cycle is $(ab)^r a$ so $(ab)^r a \in N$.

(b) We can assume $(ab)^r a \neq 1$. For suppose $(ab)^r a = 1$, then $a^{-1} = (ab)^r$, so $G = \langle a, b\rangle = \langle ab, a\rangle = \langle ab\rangle$ which means that G is cyclic (and hence abelian) so by Stong [4] has a 1-factorisation.

(c) Since $(ab)^r a \in N$ and every element in N has order 2 we have $[(ab)^r a]^2 = 1$.

(d) Steps (a) through (c) can be repeated to show that $[(ba)^r b]^2 = 1$.

(e) $\langle (ab)^r a\rangle$ is a normal subgroup of N (since N is abelian) and N is normal in G, let $x_1, x_2, \cdots, x_\ell \in N$ such that $\overline{x_1}, \overline{x_2}, \cdots, \overline{x_\ell}$ are all the cosets of $\langle (ab)^r a\rangle$ in N. Then

$$[x_i]\Big(2 * (r * (a, b), a)\Big),$$

$1 \le i \le \ell$, are vertex-disjoint cycles of length $2(2r + 1)$. Similarly, if we let $y_1, y_2, \cdots, y_\ell \in N$ such that $\overline{y_1}, \overline{y_2}, \cdots, \overline{y_\ell}$ are all the cosets of $\langle (ba)^r b\rangle$, then

$$[y_i]\Big(2 * (r * (b, a), b)\Big),$$

$1 \le i \le \ell$, are vertex-disjoint cycles of length $2(2r + 1)$.

(f) The cycles described in (e) are edge disjoint and the edges of X are the union of these disjoint cycles of even length, thus X is 4-edge colourable and has a 1-factorization.

Case (ii): Possibility (e) when $|G/N|$ is odd.

In this case either $G' = N$ or $G' \cap N = \{1\}$. In the later case we can find a minimal normal subgroup of G in G' instead of N and use the previous arguments. So we suppose $G' = N$ in the following.

Subcase (1): At least one of a and b has even order.

Without loss of generality suppose that a has even order. Let $M = < N \cup \{a\} >$. Since $N = G'$ we have G' is a subgroup of M and hence M is normal in G. (Let $m \in M$ and $x \in G$, then $xmx^{-1}m^{-1} \in G'$ so $xmx^{-1} \in M$.) Now consider the quotient graph $\overline{Y}$ obtained by contracting every coset of M to a single vertex. Then $\overline{Y}$ is a cyclic graph in which every edge around the cycle has multiplicity s which is even. These are the b-edges. The a-edges (within M) form disjoint cycles of even length and hence a 1-factor can be removed from each of these cycles. Now the remaining graph is cubic and a 3-edge colouring for it can obtained as follows.

Let $\overline{x_1}, \overline{x_2}, \cdots, \overline{x_{2r+1}}$ be the cosets of M in G. After removing one 1-factor from the subgraph of X induced by each coset $\overline{x_i}$, $1 \leq i \leq 2r+1$, we can edge colour the remaining cubic subgraph properly as follows:

- Colour all the a-edges in $\overline{x_1}$ and those b-edges between $\overline{x_{2j}}$ and $\overline{x_{2j+1}}$ for $1 \leq j \leq r$ RED;
- Colour all the a-edges in $\overline{x_{2r+1}}$ and those b-edges between $\overline{x_{2j-1}}$ and $\overline{x_{2j}}$ for $1 \leq j \leq r$ BLUE;
- Colour all the remaining edges in the cubic subgraph YELLOW.

Subcase (2): Both a and b have odd order. (This is the only case where we require the condition that G' is an elementary p-group, for then $N = G'$.) Since G' is a 2-group we make the following claims.
Claim 1: For any integers i, j such that i and $o(a)$ are coprime and j and $o(b)$ are coprime it follows that $a^i b^j \neq b^j a^i$. (For otherwise $\langle a^i \rangle$ and $\langle b^j \rangle$ are both odd order normal subgroups of G and thus we can form a minimal normal subgroup of odd order which may be used to replace N.)

For each cycle $\overline{C} = [\overline{y}](\overline{x_1}, \overline{x_2}, \cdots, \overline{x_i})$ in $\overline{X} = X(G/G'; \overline{S})$, we call the product $x_1 x_2 \cdots x_i$ the *end vertex* of $\overline{C}$ (in $X(G; S)$). (Note that the cycle which forms the end vertex may pass through the identity without causing a problem.)

Claim 2: If the end vertex of $\overline{C}$ is not equal to the identity, the cycles lifted from $\overline{C}$ to $X(G; S)$ have even length.
Claim 3: $o(a) = o(\overline{a}) = s$ and $o(b) = o(\overline{b}) = t$.

The proof of this subcase is now presented in two parts depending on the content of $\langle \overline{a} \rangle \cap \langle \overline{b} \rangle$. We find a decomposition of G/G' into 2-factors and then lift each 2-factor to obtain cycles of even length and consequently a 1-factorization.

(1) $\langle\overline{a}\rangle \cap \langle\overline{b}\rangle \neq \{\overline{1}\}$.

Let r be the smallest power of $\overline{b}$ so that $\overline{b}^r = \overline{a}^j$ for some j where $1 \leq j \leq s$. We know that both r and t are odd. We can suppose j is even (use a^{-1} instead of a if necessary). Then $s-1-j$ is even and $r \geq 3$. Three subcases are now considered, first $s-j-1 \geq 4$, then $s-j-1 = 2$ and finally $s-j-1 = 0$.

(1.1) $s-j-1 \geq 4$.

Let

$$\overline{C}_{0,0} = \overline{C}_1 = (j/2 * (\overline{a}, 2 * \overline{b}, \overline{a}, 2 * \overline{b^{-1}}), r * \overline{b}),$$

$$\overline{C}_{0,j+\ell} = [\overline{a}^{j+\ell}](2 * \overline{b}, \overline{a}, 2 * \overline{b^{-1}}, \overline{a^{-1}})$$

$$\overline{C}_{0,s-4} = [\overline{a}^{s-4}](\overline{a}, \overline{b}, \overline{a}, \overline{b^{-1}}, \overline{a}, 2 * \overline{b}, 3 * \overline{a^{-1}}, 2 * \overline{b^{-1}}),$$

$$\overline{C}_{i,s-4} = [\overline{b}^i \overline{a}^{s-4}](3 * \overline{a}, \overline{b}, 3 * \overline{a^{-1}}, \overline{b^{-1}})$$

and

$$\overline{C}_{i,\ell} = [\overline{b}^i \overline{a}^\ell](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}}),$$

where i, ℓ are odd integers such that $1 \leq \ell \leq s-6$ and $3 \leq i \leq r-2$ and $j + \ell < s - 4$ in $\overline{C}_{0,j+\ell}$. (See Figure 1).

Let $\overline{C}$ be the union of all these cycles and C be the 2-factor of X lifted from $\overline{C}$.

First we suppose the end vertex of $\overline{C}_1$ is not the identity. By Claim 2, C consists of only even cycles of X, which can be factored into two 1-factors of X.

Note that $\overline{X} - \overline{C}$ consists of even cycles (of length four) except for one odd cycle which is $\overline{C}_2 = [\overline{b}\overline{a}^{s-3}]((s-1) * \overline{a^{-1}}, (r-1) * \overline{b}, \overline{a^{-1}}, (r-1) * \overline{b^{-1}})$ (this can be easily checked, see Figure 1). By Claim 3, we know the end vertex of $\overline{C}_2$ is $ab^{r-1}a^{-1}b^{-(r-1)}$, which is not the identity by Claim 1. Therefore the 2-factor lifted from $X - C$ consists of only even cycles.

Suppose the end vertex of $\overline{C}_1$ is the identity. Let

$$\overline{C'}_1 = ((j/2-1) * (\overline{a}, 2 * \overline{b}, \overline{a}, 2 * \overline{b^{-1}}), 2 * \overline{a}, (r-1) * \overline{b^{-1}}).$$

Since the end vertex of $\overline{C}_1$ is the identity, by Claim 1 we know the end vertex of $\overline{C'}_1$ is not the identity. We break $\overline{C}_1$ into two cycles, one is $\overline{C'}_1$, the other is the cycle $[\overline{b}\overline{a}^{j-1}](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$ and keep all the other cycles of $\overline{C}$. The resulting 2-factor of $\overline{X}$ is denoted by $\overline{C'}$. Let C' be the 2-factor lifted from $\overline{C'}$. We know that C' consists of only even cycles.

Consider $\overline{X} - \overline{C'}$. The only odd cycle in $\overline{X} - \overline{C'}$ is

$$\overline{C'}_2 = [\overline{b}\overline{a}^{(j-1)}]((j+1) * \overline{a^{-1}}, (r-1) * \overline{b}, \overline{a^{-1}}, (r-1) * \overline{b^{-1}},$$

$$(s-j-3) * \overline{a^{-1}}, \overline{b^{-1}}, \overline{a}, \overline{b^{-1}}, 2 * \overline{a^{-1}}, 2 * \overline{b}).$$

If the end vertex of $\overline{C'}_2$ is not the identity, we are done. If it is the identity, then we break the cycle $\overline{C}_{0,s-4}$ in $\overline{C'}$ into two cycles, one is $\overline{C}_{0,s-2} = [\overline{a}^{s-2}](\overline{a}, 2 * \overline{b}, \overline{a^{-1}}, 2 * \overline{b^{-1}})$, the remaining vertices form another 6-cycle, and keep all the other cycles of $\overline{C'}$. The resulting 2-factor of $\overline{X}$ is denoted by $\overline{C''}$. We know the 2-factor C'' lifted from $\overline{C''}$ consists of only even cycles. The only odd cycle in $\overline{X} - \overline{C''}$ is $\overline{C''}_2 = [\overline{b}\overline{a}^{j-1}]((s-1) * \overline{a^{-1}}, \overline{b^{-1}}, \overline{a}, \overline{b^{-1}}, 2 * \overline{a^{-1}}, 2 * \overline{b})$. Since the end vertex of $\overline{C'}_2$ is the identity, by Claim 1, the identity of $\overline{C''}_2$ is not the identity. That is, $X - C''$ is also the union of even cycles.

(1.2) $s - j - 1 = 2$.

Let

$$\overline{C}_{0,s-2} = [\overline{a}^{s-2}](\overline{a}, 2 * \overline{b}, 3 * \overline{a^{-1}}, \overline{b^{-1}}, 2 * \overline{a}, \overline{b^{-1}}),$$

$$\overline{C}_{i,s-4} = [\overline{b}^{i}\overline{a}^{s-4}](3 * \overline{a}, \overline{b}, 3 * \overline{a^{-1}}, \overline{b^{-1}})$$

and

$$\overline{C}_{i,\ell} = [\overline{b}^{i}\overline{a}^{\ell}](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}}),$$

where i, ℓ are odd integers such that $3 \le i \le r-2$ and $1 \le \ell \le s-6$. Also let $\overline{C'}_1$ be the same as defined in the case $s-j-1>2$, (see Figure 2).

Again, let $\overline{C}$ to be the union of these cycles and C be the 2-factor lifted from $\overline{C}$. Let $\overline{C'} = \overline{X} - \overline{C}$. Then $(\overline{C}, \overline{C'})$ is a pair of edge-disjoint 2-factors.

Now keep all the cycles in $\overline{C}$ but break $\overline{C}_{0,s-2}$ into two cycles, one is $\overline{C}_{1,s-4} = [\overline{b}\overline{a}^{s-4}](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$, the other is $\overline{C}_{0,s-2} = [\overline{a}^{s-2}](\overline{a}, 2{*}\overline{b}, \overline{a^{-1}}, 2{*}\overline{b^{-1}})$. Let $\overline{D}$ be the resulting 2-factor of $\overline{X}$ and D be the 2-factor of X lifted from $\overline{D}$. Let $\overline{D'} = \overline{X} - \overline{D}$. Then $(\overline{D}, \overline{D'})$ is another pair of edge-disjoint 2-factors.

If the end vertex of $\overline{C'}_1$ is not the identity, by the same argument as above, we are done.

If the end vertex of $\overline{C'}_1$ is the identity, since $b^r a^{-j} \neq 1$, $j \geq 4$.

Let

$$\overline{C''}_{0,s-2} = [\overline{a}^{s-2}](\overline{a}, 2 * \overline{b}, \overline{a^{-1}}, 2 * \overline{b^{-1}}),$$

$$\overline{C''}_{1,j-3} = [\overline{b}\overline{a}^{j-3}](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$$

and

$$\overline{C''}_1 = ((r+2) * \overline{b}, \overline{a^{-1}}, 2 * \overline{b^{-1}}, 3 * \overline{a^{-1}}, (j-4)/2 * (2 * \overline{b}, \overline{a^{-1}}, 2 * \overline{b^{-1}}, \overline{a^{-1}})).$$

Let $\overline{F}$ be the union of $\overline{C''}_1$, $\overline{C''}_{0,s-2}$ and $\overline{C''}_{1,j-3}$ and all those cycles in $\overline{D}$ which are vertex-disjoint with these three cycles.

Now we claim that the end vertex of $\overline{C''}_1$ cannot be the identity (note that it is the only odd cycle in $\overline{F}$). Otherwise, because of the construction of $\overline{C'}_1$ and $\overline{C''}_1$, we have $b^2ab^{-2}a^2 = a^2b^2ab^{-2}$, which implies a^2 is commutative with b^2ab^{-2}. But $o(a)$ is odd, so $\langle a^2 \rangle = \langle a \rangle$ and hence a^{-1} is commutative with b^2ab^{-2}, that is,

$$a^{-1}b^2ab^{-2} = b^2ab^{-2}a^{-1}. \tag{1}$$

Since $a^{-1}b^2ab^{-2} \in G'$, we have

$$a^{-1}b^2ab^{-2} = b^2a^{-1}b^{-2}a. \tag{2}$$

By equations (1) and (2) we have $a^2b^{-2} = b^{-2}a^2$, which contradicts Claim 1.

Therefore the 2-factor F of X lifted from $\overline{F}$ is a vertex-disjoint union of even cycles. Now consider $\overline{F'} = \overline{X} - \overline{F}$. The only odd cycle in $\overline{F'}$ is $[\overline{b}\overline{a}^{j-3}]((s-1)*\overline{a^{-1}}, 2*\overline{b^{-1}}, \overline{a^{-1}}, 2*\overline{b})$, whose end vertex cannot be the identity.

(1.3) $s - j - 1 = 0$.

Let

$$\overline{C}_1 = (\overline{b}, (s-1)*\overline{a^{-1}}, 2*\overline{b^{-1}}, (s-1)*\overline{a^{-1}}, \overline{b}, (s-2)*\overline{a})$$

and

$$\overline{C}_i = [\overline{b}^i](\overline{b}, (s-1)*\overline{a^{-1}}, \overline{b^{-1}}, (s-1)*\overline{a})$$

for each even i, $2 \leq i \leq r-3$ (see Figure 3).

Let $\overline{C}$ be the union of these cycles. $\overline{C}_1$ is the only odd cycle in $\overline{C}$. Since the only odd cycle in $\overline{X} - \overline{C}$ is $(2*\overline{a}, (s-2)r*\overline{b})$, whose end vertex can not be the identity, if the end vertex of $\overline{C}_1$ is not the identity, then the partition lifted from $(\overline{C}, \overline{X} - \overline{C})$ satisfies our requirement.

Suppose the end vertex of $\overline{C}_1$ is the identity, that is,

$$bab^{-2}aba^{-2} = 1. \tag{3}$$

Note that $o(b) > 3$. So $o(a) = s > 3$. For otherwise (3) would lead to $b^3a = ab^3$.

Let

$$\overline{D}_1 = ((2r-1)*\overline{b}, \overline{a}, (r-1)*\overline{b^{-1}}, 3*\overline{a}, (s-5)/2*((r-1)*\overline{b}, \overline{a}, (r-1)*\overline{b^{-1}}, \overline{a}))$$

and

$$\overline{D}_i = [\overline{b}^i \overline{a}^3](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$$

for each odd i, $1 \leq i \leq r-2$.

Let $\overline{D}$ be the union of these cycles. $\overline{D}_1$ is the only odd cycle in $\overline{D}$. If the end vertex of $\overline{D}_1$ is not the identity, since in $\overline{X} - \overline{D}$ the only odd cycle

$$[\overline{b}\overline{a}^3]((s-1) * \overline{a^{-1}}, 2 * \overline{b^{-1}}, \overline{a^{-1}}, 2 * \overline{b})$$

has an element which is not the identity as its end vertex, the partition $(\overline{D}, \overline{X} - \overline{D})$ is suitable.

Suppose the end vertex of $\overline{D}_1$ is the identity. Define

$$\overline{F}_1 = (r * \overline{b}, 2 * \overline{a}, (s-3)/2 * ((r-1) * \overline{b}, \overline{a}, (r-1) * \overline{b^{-1}}, \overline{a}))$$

and

$$\overline{F}_i = [\overline{b}^i \overline{a}](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$$

for each odd i, $1 \leq i \leq r-2$.

Let $\overline{F}$ be the union of all these cycles. Using the same method that was used to prove the case where $s - j - 1 = 2$, the end vertex of $\overline{F}_1$ cannot be the identity. The odd cycle in $\overline{X} - \overline{F}$ is

$$(\overline{a}, \overline{b}, (s-1) * \overline{a^{-1}}, 2 * \overline{b^{-1}}, 2 * \overline{a^{-1}}, \overline{b}),$$

whose end vertex is $abab^{-2}a^{-2}b$.

If

$$abab^{-2}a^{-2}b = 1,$$

combined with (3), we get $ab^{-3}a^{-1}b^3 = 1$, which is impossible. Therefore $(\overline{F}, \overline{X} - \overline{F})$ satisfies our requirement.

(2) $\langle \overline{a} \rangle \cap \langle \overline{b} \rangle = \{\overline{1}\}$.

(2.1) $o(a) = s > 3$ and $o(b) = r > 3$.

In this subcase, we construct two 2-factors as follows.

Let

$$\overline{C}_1 = ((r-1) * \overline{b}, \overline{a}, (r-1) * \overline{b^{-1}}, (s-1) * \overline{a}),$$

$$\overline{C}_{i,\ell} = [\overline{b}^i \overline{a}^\ell](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$$

and

$$\begin{aligned}\overline{C}_2 \quad = \quad & [\overline{b}\overline{a}^2]((s-3) * \overline{a}, \overline{b}, (r-3)/2 * (\overline{a^{-1}}, \overline{b}, \overline{a}, \overline{b}), (s-3) * \overline{a^{-1}}, (r-5)/2 * \\ & (\overline{b^{-1}}, 2 * \overline{a}, \overline{b^{-1}}, 2 * \overline{a^{-1}}), 3 * \overline{b^{-1}}),\end{aligned}$$

where i is even and ℓ is odd such that $2 \leq i \leq r-3$, $5 \leq \ell \leq s-4$ or $i = 2$, $\ell = 3$ (see Figure 4).

Let $\overline{C}$ be the union of these cycles which is a 2-factor of $\overline{X}$.

The other 2-factor is $\overline{X} - \overline{C}$, which is denoted by $\overline{C'}$.

Note that in $\overline{C}$, the only odd cycle is $\overline{C}_1$ whose end vertex is not the identity by Claim 1. The only odd cycle in $\overline{C'}$ is

$$[\overline{b}^2 \overline{a}^3](4 * \overline{a^{-1}}, \overline{b}, 4 * \overline{a}, (r-1) * \overline{b}).$$

By Claim 1 the end vertex of this cycle is not the identity either. Therefore, we have factored X into two 2-factors each of which consists of only even cycles.

(2.2) $r = 3, s > 3$.

Let

$$\overline{C}_1 = (2 * \overline{b}, 2 * \overline{a}, \overline{b^{-1}}, \overline{a^{-1}}, \overline{b^{-1}}, (s-1) * \overline{a})$$

and

$$\overline{C}_\ell = [\overline{b}\overline{a}^\ell](\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b^{-1}})$$

for each odd number ℓ, $3 \leq \ell \leq s-2$ (see Figure 5).

Let $\overline{C}$ be the union of these cycles and C be the 2-factor lifted from $\overline{C}$. Note that $\overline{C}_1$ is the only odd cycle in $\overline{C}$. By Claim 3, the end vertex of $\overline{C}_1$ is $b^{-1}a^2b^{-1}a^{-1}b^{-1}a^{-1}$. If this vertex is the identity, then we have

$$(b^{-1}a^2b^{-1}a^{-1})(b^{-1}a^{-1}) = (b^{-1}a^{-1})(b^{-1}a^2b^{-1}a^{-1}) = 1,$$

which implies $a^3b^{-1} = b^{-1}a^3$, a contradiction with Claim 1 as $a^3 \neq 1$. Therefore the end vertex of $\overline{C}_1$ is not the identity.

Let $\overline{C'} = \overline{X} - \overline{C}$ and C' be the 2-factor lifted from $\overline{C'}$. Let

$$\overline{C'}_1 = (\overline{b^{-1}}, \overline{a^{-1}}, 2 * \overline{b}, 2 * \overline{a}, 2 * \overline{b}, \overline{a^{-1}}),$$

which is the only odd cycle in $\overline{C'}$. The end vertex of $\overline{C'}_1$ is $b^{-1}a^{-1}b^{-1}a^2b^{-1}a^{-1}$, which is the same as $\overline{C}_1$ and is not the identity.

(2.3) $r = s = 3$.

Let

$$\overline{C}_1 = (\overline{a}, 2 * \overline{b}, \overline{a^{-1}}, \overline{b^{-1}}, \overline{a^{-1}}, 2 * \overline{b}, \overline{a})$$

and

$$\overline{C}_2 = (\overline{a}, \overline{b}, \overline{a^{-1}}, \overline{b}, 2 * \overline{a}, 2 * \overline{b^{-1}}, \overline{a}).$$

If the end vertices of $\overline{C}_1$ and $\overline{C}_2$ are each the identity, then $a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1} = a^{-1}ba^{-1}ba^{-1}b = 1$. Since $a^3 = b^3 = 1$, $bab^{-1}a^{-1} = 1$, that is, $ab = ba$, a contradiction. (To see that $bab^{-1}a^{-1} = 1$ note that $ba^{-1}b^{-1}a^{-1}b = a^{-1}ba^{-1}$. Then $(ba^{-1}b^{-1}a)(ab) = a^{-1}ba^{-1}$. So we have $ba^{-1}b^{-1}a = a^{-1}ba^{-1}b^{-1}a^{-1} = (a^{-1}bab^{-1})\ (bab^{-1}a^{-1})$. So $bab^{-1}a^{-1} = (ba^{-1}b^{-1}a)(ba^{-1}b^{-1}a) = 1$.))

Therefore one of the end vertices of $\overline{C}_1$ and $\overline{C}_2$ is not the identity.

Let

$$\overline{C'}_1 = \overline{X} - \overline{C}_1 = [\overline{b}](2 * \overline{a}, \overline{b^{-1}}, \overline{a^{-1}}, \overline{b^{-1}}, 2 * \overline{a}, 2 * \overline{b}).$$

The end vertex of $\overline{C'}_1$ is $(a^{-1}b^{-1})^3$, which is the same as the end vertex of $\overline{C}_1$.

Let

$$\overline{C'}_2 = \overline{X} - \overline{C}_2 = [\overline{a}^2](\overline{a^{-1}}, 2 * \overline{b^{-1}}, 2 * \overline{a}, 2 * \overline{b^{-1}}, \overline{a^{-1}}, \overline{b}).$$

The end vertex of $\overline{C'}_2$ is $(a^{-1}b)^3$, which is the same as the end vertex of $\overline{C}_2$.

Therefore one of the factorizations of $\overline{C}_1 \cup \overline{C'}_1$ and $\overline{C}_2 \cup \overline{C'}_2$ would satisfy our requirement.

CASE B: Assume that $N \cap S \neq \emptyset$

Recall that N is a minimal normal subgroup in G and abelian. Moreover $|N| = p^k$ and every element in N has order p.

We can assume that if $a \in N \cap S$ then $b \notin N \cap S$. (For if a and b are both in N then $N = G$ which is a contradiction since N is a nontrivial subgroup of G.) So without loss of generality assume $a \in N$ and $b \notin N$.

Case (i): If $p = 2$ then there is an element x in $N \cap S$ with order two. The edges corresponding to x form a 1-factor, so if they are removed from X then the resulting graph is a cubic Cayley graph and is 3-edge colourable [1]. Thus X is 4-edge colourable and hence has a 1-factorization.

Case (ii): If $p \neq 2$ then G/N is cyclic (since $a \in N$ then edges between cosets are generated by b's, cosets are $N, bN, \ldots b^{t-1}N$).

Subcase(a): $N = \langle a \rangle$.

Since $p \neq 2$, $|G/N|$ has to be even.

Since G/N is cyclic, the commutator subgroup G' is contained in $\langle a \rangle$. By Stong [4, Theorem 2.9] X is 1-factorizable.

Subcase (b): $N \neq \langle a \rangle$.

Since G has even order and the order of N is odd it follows that $|G/N|$ is even. We show there are disjoint Hamilton cycles each of length $2p^2$ through the vertices in $N \cup bN$ which use a-edges in N and bN, and the connecting b-edges between these two cosets. Since these Hamilton cycles have even length they can be 2-edge coloured. The remainder of the edges in this subgraph form a 1-factor and can be 1-coloured. Then this subgraph is joined to the subgraph $b^2N \cup b^3N$ by another 1-factor, etc.

Since G has even order and the order of N is odd it follows that $|G/N|$ is even. Also since G/N is generated by $\overline{b}$, let $G/N = \{\overline{1}, \overline{b}, \overline{b^2}, \cdots, \overline{b^m}\}$. where m is odd. We have a 1-factor consisting all those b-edges between $\overline{b^i}$ and $\overline{b^{i+1}}$ for odd i, $1 \le i \le m$.

Next we want to factor each of the subgraphs induced by $\overline{b^j} \cup \overline{b^{j+1}}$ for those even j, $0 \le j \le m-1$. Since all these cubic subgraphs are isomorphic to each other, we consider the subgraph induced by $\overline{1} \cup \overline{b}$, that is, the subgraph induced by $N \cup bN$.

Since N is abelian we know that the group $M = \langle a, bab^{-1} \rangle$ is normal in N. We can assume that $\langle a \rangle \cap \langle bab^{-1} \rangle = \{1\}$, for otherwise since $|a| = |bab^{-1}| = p$ we have $\langle a \rangle$ and $\langle bab^{-1} \rangle$ are both cyclic groups of order p so $\langle a \rangle = \langle bab^{-1} \rangle$ and consequently we have $N = \langle a \rangle$ and are back in subcase (a). Note that since M is abelian the elements a and bab^{-1} will commute in G.

Let $M, x_1M, x_2M, \cdots, x_\ell M$ be all the cosets of M in N. In the subgraph induced by $M \cup bM$, we have a cycle $p * (b, a, b^{-1}, a^{-1})$. (To see this is a cycle, note the fact that a and bab^{-1} commute in G.) Now replace edges $(1, a), (bab^{-1}, bab^{-1}a^{-1}), \cdots, (ba^{-(p-1)}b^{-1}a^{-(p-2)}, ba^{-(p-1)}b^{-1}a^{-(p-1)})$ by paths $(p-1) * a^{-1}$, $[bab^{-1}](p-1)*a$, $\cdots$, $[ba^{-(p-1)}b^{-1}a](p-1)*a^{-1}$, respectively. Then we get a Hamilton cycle of length $2p^2$ for this induced subgraph.

We repeat this for each subgraph induced by $x_iM \cup bx_iM$, $1 \le i \le \ell$. Since for $i \ne j$, $(x_iM \cup bx_iM) \cap (x_jM \cup bx_jM) = \emptyset$, we get p^{k-2} vertex-disjoint even length cycles such that the union of their vertex sets is $N \cup bN$. That is, we get a 1-factorization for the subgraph induced by $N \cup bN$. By our remark above, we get a 1-factorization for the entire graph.

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