Separable determination of integrability and minimality of the Clarke subdifferential mapping

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Abstract

In this paper we show that the study of integrability and D-representability of Lipschitz functions defined on arbitrary Banach spaces reduces to the study of these properties on separable Banach spaces.

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1 Introduction.

One of the key tools of Non-smooth anaylsis is the generalized derivative and one of the first and best known generalized derivative is the Clarke generalized derivative [6]. However, inspite of having enjoyed widespread utility the Clarke derivative has attracted its fair share of detractors. Their main objection to this derivative is that in general the Clarke gradient mappings is just too large to reveal any fine structure. This critisim is of course not without some basis. Indeed, even on $\mathbb{R}$ there exist (uncountable many distinct, by more than an additive constant) Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ such that $\partial f(x) \equiv [0, 1]$ on $\mathbb{R}$. Clearly, for such functions the Clarke derivative yields very little information. On the other hand, one could argue that such pathological functions do not naturally arise. This argument is supported by the fact that it has recently been shown that there is a large class of locally Lipschitz functions for which the Clarke derivative is well-behaved. Indeed, in the papers [2], [3] and [4] the authors demonstrate that there is a large robust class of locally Lipschitz functions whose members are:

(*i*) D-representable, that is, they are Gâteaux differentiable on some dense subset D of their domain and their Clarke subdifferential mapping may be recovered by using the derivatives chosen from *any* dense subset of D;
(*ii*) Integrable, that is, they may be determined up to an additive constant from their Clarke sudifferential mapping - at least on the connected components of their domain.

In this paper we show that the study of integrability and D-representability reduces to the study of these properties on separable Banach spaces. In this way we may extend many of the results in [2] to arbitrary Banach spaces and make use of the Rademacher-type theorems that exist on separable Banach spaces. (Note that for a densely Gâteaux differentiable Lipschitz function, D-representability is equivalent to minimality of the Clarke subdifferential mapping, [2].)
We begin with some preliminary definitions. A real-valued function f defined on a non-empty open subset A of a Banach space X is said to be *locally Lipschitz* on A, if for each $x_0 \in A$ there exists a $K > 0$ and a $\delta > 0$ such that $|f(x) - f(y)| \leq K\|x - y\|$ for all $x, y \in B(x_0, \delta)$.

For functions in this class, it is sometimes instructive to consider the following directional derivatives:

(i) The *upper Dini derivative* at $x \in A$ *in the direction* y, given by:

$$f^+(x;y) \equiv \limsup_{\lambda \to 0^+} \frac{f(x+\lambda y) - f(x)}{\lambda}$$

(ii) The *Clarke generalized directional derivative* at $x \in A$ *in the direction* y, given by:

$$f^0(x;y) \equiv \limsup_{z \to x} f^+(z;y)$$

Associated with the Clarke generalized directional derivative is the *Clarke subdifferential mapping*, which is defined by:

$$\partial f(x) \equiv \{x^* \in X^* : \langle x^*, y\rangle \leq f^0(x;y) \text{ for each } y \in X\}.$$

The next definition required in order to formulate our two main theorems is that of a 'rich' family of separable subspaces. Let X be a normed linear space. We will call a family $\mathcal{F}$ of closed separable subspaces of X *rich* if:

$*_1$ For each increasing sequence of closed separable subspaces $\{Y_n : n \in I\!N\}$ in $\mathcal{F}$, $\overline{\bigcup\{Y_n : n \in I\!N\}} \in \mathcal{F}$;
$*_2$ For each separable subspace Y^0 of X there exists a $Y \in \mathcal{F}$ such that $Y^0 \subseteq Y$.

Proposition 1.1 *Let X be a normed linear space and let $\{\mathcal{F}_n : n \in I\!N\}$ be rich families of closed separable subspaces of X. Then $\mathcal{F} = \bigcap\{\mathcal{F}_n : n \in I\!N\}$ is also a rich family of closed separable subspaces of X.*

Proof: Clearly $\mathcal{F}$ satifies $*_1$ so it remains to show that $\mathcal{F}$ satisfies $*_2$. To this end let Y^0 be any separable subspace of X. We proceed from here by induction. At the first step we choose $Y_1^1 \in \mathcal{F}_1$ so that $Y^0 \subseteq Y_1^1$. Now after the first n-steps of the induction have been completed we will have constructed an increasing sequence of closed separable subspaces,

$$Y^0 \subseteq Y_1^1 \subseteq Y_1^2 \subseteq Y_2^2 \subseteq Y_1^3 \subseteq \ \ldots\ Y_1^n \subseteq Y_2^n \subseteq \ \ldots\ Y_n^n$$

so that $Y_j^m \in \mathcal{F}_j$ for each $1 \leq j \leq m \leq n$. At the next step we choose closed separable subspaces $Y_j^{n+1} \in \mathcal{F}_j,\ 1 \leq j \leq n+1$ such that:

$$Y_n^n \subseteq Y_1^{n+1} \subseteq Y_2^{n+1} \subseteq \ \ldots\ Y_{n+1}^{n+1}$$

This completes the induction. Let $Y \equiv \overline{\bigcup\{Y_n^n : n \in I\!N\}}$, then $Y^0 \subseteq Y$ and for each $j \in I\!N$, $Y = \overline{\bigcup\{Y_j^n : j \leq n\}} \in \mathcal{F}_j$. Therefore, $Y^0 \subseteq Y \in \bigcap\{\mathcal{F}_n : n \in I\!N\} = \mathcal{F}$; which shows that $\mathcal{F}$ is rich. ☺

For a real-valued locally Lipschitz function f defined on a non-empty open subset A of a Banach space X we shall denote by $\mathcal{F}_f$ the family of all those closed separable subspaces Y of X for which there exists a countable dense subset S of $(A \cap Y) \times Y$ with the property that for each $(x, y, m) \in S \times I\!N$, $\sup\{f^+(z; y) : z \in B(x, 1/m) \cap A\} = \sup\{f^+(z; y) : z \in B(x, 1/m) \cap A \cap Y\}$

Proposition 1.2 *Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then $\mathcal{F}_f$ is a rich family of closed separable subspaces of X.*

Proof: With a moments thought one can see that $\mathcal{F}_f$ satisfies $*_1$. Therefore it remains to show that $\mathcal{F}_f$ satisfies $*_2$. To this end let Y^0 be any separable subspace of X. (Note that by possibly making Y^0 larger we may assume $A \cap Y^0 \neq \emptyset$). We proceed by induction. At the first step we choose a countable dense subset S^1 of $(A \cap Y^0) \times Y^0$. Then for each $(x, y, m) \in S^1 \times I\!N$ we select a countable set $C^1_{(x,y,m)} \subseteq B(x, 1/m) \cap A$ so that:

$$\sup\{f^+(z; y) : z \in B(x, 1/m) \cap A\} = \sup\{f^+(z; y) : z \in C^1_{(x,y,m)}\}$$

and set $Y^1 \equiv \overline{sp} \bigcup\{C^1_{(x,y,m)} : (x, y, m) \in S^1 \times I\!N\}$. Now after the first n steps of the induction have been completed we will have constructed countable sets $S^1, S^2, \ldots S^n$ and closed separable subspaces $Y^0 \subseteq Y^1 \subseteq Y^2 \subseteq \ldots Y^n$ with the property that for each $1 \leq j \leq n$ and $(x, y, m) \in S^j \times I\!N$, S^j is dense in $(A \cap Y^{j-1}) \times Y^{j-1}$ and

$$\sup\{f^+(z; y) : z \in B(x, 1/m) \cap A\} = \sup\{f^+(z; y) : z \in B(x, 1/m) \cap A \cap Y^j\}$$

At the next step we choose a countable dense subset S^{n+1} of $(A \cap Y^n) \times Y^n$. Then for each $(x, y, m) \in S^{n+1} \times I\!N$ we select a countable set $C^{n+1}_{(x,y,m)} \subseteq B(x, 1/m) \cap A$ so that:

$$\sup\{f^+(z; y) : z \in B(x, 1/m) \cap A\} = \sup\{f^+(z; y) : z \in C^{n+1}_{(x,y,m)}\}$$

and set $Y^{n+1} \equiv \overline{sp} \bigcup\{C^{n+1}_{(x,y,m)} : (x, y, m) \in S^{n+1} \times I\!N\}$. This completes the induction. Let $Y \equiv \overline{\bigcup\{Y^n : n \in I\!N\}}$ and $S \equiv \bigcup\{S^n : n \in I\!N\}$. It is now easy to see that S is dense in $(A \cap Y) \times Y$ and that:

$$\sup\{f^+(z; y) : z \in B(x, 1/m) \cap A\} = \sup\{f^+(z; y) : z \in B(x, 1/m) \cap A \cap Y\}$$

for each $(x, y, m) \in S \times I\!N$. Therefore, $Y^0 \subseteq Y \in \mathcal{F}_f$; which shows that $\mathcal{F}_f$ is rich. ☺

Remark 1.1 *Note that for each $Y \in \mathcal{F}_f$ and each $(x, y) \in (A \cap Y) \times Y$ $f^0(x; y) = (f|_{A \cap Y})^0(x; y)$ and so we have that:*
$\partial(f|_{A \cap Y})(x_0) = \{y^* \in Y^* : y^* = x^*|_Y$ and $x^* \in \partial f(x_0)\}$ for each $x_0 \in A \cap Y$.

2 Separable reduction

Given a topological space A and a normed linear space X, a set-valued mapping Φ from A into non-empty subsets of the dual of X is called a *weak* cusco* on A if:
(i) For each $x \in A$, $\Phi(x)$ is weak* compact and convex;
(ii) For each weak* open subset W of X^*, $\{x \in A : \Phi(x) \subseteq W\}$ is open.

Moreover, Φ is called a *minimal weak* cusco* on A if its graph does not properly contain the graph of any other weak* cusco on A (see [7] for more information). A closely related notion for real-vauled functions is the following. A real-valued function f defined on A is said to be *quasi lower semi-continuous* on A if for each $r \in I\!R$, $f^{-1}((r, \infty))$ is semi-open in A. Recall that a subset B of a topological space is called *semi-open* if $B \subseteq \overline{\mathrm{int} B}$.

Lemma 2.1 *([2], Theorem 3.3) Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then $x \to \partial f(x)$, is a minimal weak* cusco on A if, and only if, for each $y \in X$, the mapping $x \to f^+(x; y)$ is quasi lower semi-continuous on A.*

We now present the first of our two main theorems.

Theorem 2.1 *Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then, $x \to \partial f(x)$, is a minimal weak* cusco on A if, and only if, there exists a rich family $\mathcal{F}$ of closed separable subspaces of X, such that, $x \to \partial(f|_{A \cap Y})(x)$, is a minimal weak* cusco on $A \cap Y$ for each $Y \in \mathcal{F}$ with $A \cap Y \neq \emptyset$.*

Proof: Let $\mathcal{F}_M$ be some rich family of closed separable subspaces of X, such that $x \to \partial(f|_{A \cap Y})(x)$ is a minimal weak* cusco on $A \cap Y$ for each $Y \in \mathcal{F}_M$ with $A \cap Y \neq \emptyset$. Let us suppose, for the purpose of obtaining a contradiction that $x \to \partial f(x)$ is not a minimal weak* cusco on A. That is, let us suppose that there is a weak* cusco $\Phi : A \to 2^{X^*}$ whose gragh is properly contained in that of ∂f. It follows then, via a separation argument in (X, weak^*), that we may find a point $x_0 \in A$ and a direction $y \in X$ so that:

$$\max \hat{y}(\Phi(x_0)) < \max \hat{y}(\partial f(x_0))$$

Let $Y^0 \equiv \text{sp}\{x_0, y\}$ and choose and $Y \in \mathcal{F}_M \cap \mathcal{F}_f$ so that $Y^0 \subseteq Y \subseteq X$. Then define $\Phi_Y : A \cap Y \to 2^{Y^*}$ by $\Phi_Y(x) \equiv \{y^* \in Y^* : y^* = x^*|_Y \text{ and } x^* \in \Phi(x)\}$. Clearly, Φ is a weak* cusco and $\Phi_Y(x) \subseteq \partial(f|_{A\cap Y})(x)$ for each $x \in A \cap Y$ because $Y \in \mathcal{F}_f$. In fact, $\Phi_Y = \partial(f|_{A\cap Y})$ since $Y \in \mathcal{F}_M$. This however is impossible because,

$$\max \hat{y}(\Phi_Y(x_0)) = \max \hat{y}(\Phi(x_0)) < \max \hat{y}(\partial f(x_0)) = \max \hat{y}(\partial f|_{A\cap Y}(x_0))$$

Therefore Φ must indeed be a minimal weak* cusco on A.

We now consider the converse. Let $\{r_n : n \in I\!N\}$ be an enumeration of the rational numbers. For each $(y, n) \in X \times I\!N$ let us denote by $A_{(y,n)} \equiv \{x \in A : f^+(x; y) > r_n\}$. We note that since $x \to \partial f(x)$ is a minimal weak* cusco on A, each set $A_{(y,n)}$ is semi-open in X. Let use denote by $\mathcal{F}_M$ the family of all those closed separable subspaces Y of X for which there is a countable dense subset S of $(A \cap Y) \times Y$ such that: $B(x, 1/m) \cap A_{(y,n)} \neq \emptyset$ if, and only if, $B(x, 1/m) \cap \text{int} A_{(y,n)} \cap Y \neq \emptyset$ for each $(x, y, n, m) \in S \times I\!N^2$. It follows in a similar manner to the proof of Proposition 1.2 that $\mathcal{F}_M$ is rich. So we are left with showing that for each $Y \in \mathcal{F}_M$, $x \to \partial(f|_{A\cap Y})(x)$ is a minimal weak* cusco on $A \cap Y$. Fix $Y \in \mathcal{F}_M$ and let S be the countable dense subset of $(A \cap Y) \times Y$ given by the definition of $\mathcal{F}_M$. It follows from Lemma 2.1 that we need to show that for each $(y, n) \in Y \times I\!N$, $A^Y_{(y,n)} \equiv A_{(y,n)} \cap Y$ is semi-open in Y. So let us fix $(y_0, n_0) \in Y \times I\!N$ and consider the set $A^Y_{(y_0,n_0)}$. If $A^Y_{(y_0,n_0)} = \emptyset$ then we are done. So let us suppose that $A^Y_{(y_0,n_0)} \neq \emptyset$. Let $x_0 \in A^Y_{(y_0,n_0)}$ and let V be any neighbourhood of x_0. We may, without loss of generality, assume that f is Lipschitz on V with Lipschitz constant K. Let $\varepsilon \equiv 1/2(f^+(x_0; y_0) - r_{n_0}) > 0$. Next we choose $(x, y, m) \in S \times I\!N$ so that $x_0 \in B(x, 1/m) \subseteq V$ and $||y - y_0|| < \varepsilon/K$. Then $f^+(x_0; y) > r_n > f^+(x_0; y_0) - \varepsilon$ for some $n \in I\!N$, and so $B(x, 1/m) \cap A^Y_{(y,n)} \neq \emptyset$. Therefore, by the definition of S, $B(x, 1/m) \cap \text{int} A^Y_{(y,n)} \neq \emptyset$. Now for each $z \in B(x, 1/m) \cap \text{int} A^Y_{(y,n)}$, $f^+(z; y_0) > f^+(z; y) - \varepsilon > r_n - \varepsilon > f^+(x_0; y_0) - 2\varepsilon = r_{n_0}$.

Hence, $\emptyset \neq B(x, 1/m) \cap \operatorname{int} A^Y_{(y,n)} \subseteq \operatorname{int} A^Y_{(y_0,n_0)} \cap V$; which shows that $A^Y_{(y_0,n_0)}$ is semi-open in Y. ☺

Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then we say that f is *integrable* on A if $\partial(f-g)(x) \equiv \{0\}$ for each real-valued locally Lipschitz function g defined on A with $\partial g(x) \subseteq \partial f(x)$ for all $x \in A$. It is easy to see that this is equivalent to saying that $f - g \equiv$ constant on each of the connected components of A, whenever $\partial g(x) \subseteq \partial f(x)$ for all $x \in A$. This of course implies that f is integrable on A if, and only if, the appropriate restriction of f is integrable on each of the connected components of A.

Lemma 2.2 *Let f be an integrable real-valued locally Lipschitz function defined on a non-empty open connected subset A of a Banach space X. Then for each $(x_0, y_0, \epsilon_0) \in A \times A \times (0, \infty)$ there is a finite dimensional subspace $Y_{(x_0,y_0,\epsilon_0)}$, containing the points x_0 and y_0, such that if g is any locally Lipschitz function defined on $A \cap Y_{(x_0,y_0,\epsilon_0)}$ with $g^0(x; y) \leq f^0(x; y)$ for each $(x, y) \in (A \cap Y_{(x_0,y_0,\epsilon_0)}) \times Y_{(x_0,y_0,\epsilon_0)}$ then $g(x_0) - g(y_0) < f(x_0) - f(y_0) + \epsilon_0$.*

Proof: Fix $(x_0, y_0, \varepsilon_0) \in A \times A \times (0, \infty)$ and let us suppose, for the purpose of obtaining a contradiction, that the conclusion of the lemma is false. Let D be the family of all the finite dimensional subspaces Y of X that contain the points x_0 and y_0. Then $(D, \subseteq)$ is an upwardly directed set. Now, by our assumption there exists, for each $Y \in D$, a locally Lipschitz function $g_Y : A \cap Y \to I\!\!R$ such that $g_Y^0(x; y) \leq f^0(x; y)$ for all $(x, y) \in (A \cap Y) \times Y$ while,

$$g_Y(x_0) - g_Y(y_0) \geq f(x_0) - f(v_0) + \varepsilon_0$$

(Note that we may, and do, assume $g_Y(x_0) = 0$ for all $Y \in D$). For each such function we consider the following extension, $\tilde{g}_Y : A \to I\!\!R$, defined by:

$$\tilde{g}_Y(x) = \begin{cases} g_Y(x) & \text{if } x \in A \cap Y \\ 0 & \text{otherwise} \end{cases}$$

Thus, $(\tilde{g}_Y : D)$ is a net in $(I\!\!R_e)^A$ - which is compact. Therefore, $(\tilde{g}_Y : D)$ has a convergent subnet which converges to some element $g \in (I\!\!R_e)^A$. It is now routine to check that g is real-valued and locally Lipschitz on A. In fact, one can show that $\partial g(x) \subseteq \partial f(x)$ for each $x \in A$. On the other hand,

$g(x_0) - g(y_0) \geq f(x_0) - f(y_0) + \varepsilon_0$; which is impossible since f is integrable on A. Hence the statement of the lemma must in fact be true. ☺

We now present the second of our two main theorems.

Theorem 2.2 *Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then f is integrable on A if, and only if, there exists a rich family $\mathcal{F}$ of closed separable subspaces of X, such that $f|_{A\cap Y}$ is integrable on $A \cap Y$ for each $Y \in \mathcal{F}$ with $A \cap Y \neq \emptyset$.*

Proof: Let $\mathcal{F}_I$ be some rich family of closed separable subspaces of X such that for each $Y \in \mathcal{F}_I$ with $A \cap Y \neq \emptyset$, $f_{A\cap Y}$ is integrable on $A \cap Y$. Furthermore, let $\mathcal{F} = \mathcal{F}_I \cap \mathcal{F}_f$ and let g be any real-valued locally Lipschitz function defined on A such that $\partial g(x) \subseteq \partial f(x)$ for each $x \in A$. It follows immediately from the definition of $\mathcal{F}_I$ and Remark 1.1 that for each $Y \in \mathcal{F}$, $f|_{A\cap Y} - g|_{A\cap Y} \equiv$ constant on each of the connected components of $A \cap Y$. Now with some thought it should eventually become clear that this implies $f - g \equiv$ constant on each of the connected components of A. This shows that f is integrable on A.

We now consider the converse. Let $\{A_\gamma : \gamma \in \Gamma\}$ denote the connected components of A and let $\mathcal{F}_I$ denote the family of all the closed separable subspace Y of X for which there is a countable subset S of:

$$\bigcup\{(A_\gamma \cap Y) \times (A_\gamma \cap Y) \times (0, \infty) : \gamma \in \Gamma \text{ and } A_\gamma \cap Y \neq \emptyset\}$$

such that $Y_{(x,y,\varepsilon)} \subseteq Y$ for each $(x, y, \varepsilon) \in S$. (Note: if Y is any separable subspace of X then $\{\gamma \in \Gamma : A_\gamma \cap Y \neq \emptyset\}$ is at most countable). As in Theorem 2.1, the proof that $\mathcal{F}_I$ is a rich family is similar to the proof of Proposition 1.2. It should also be clear that for each $Y \in \mathcal{F}_I$ with $A \cap Y \neq \emptyset$, $f|_{A\cap Y}$ is integrable on $A \cap Y$. ☺

We now present a few simple consequences of Theorems 2.1 and 2.2.

Let X be a separable Banach space, then a Borel subset N is a *Haar-null* set if there is a Borel probability measure p on X such that $x + N$ is p-null for all $x \in X$. It is well-known that the Haar-null sets are closed under translation and countable unions and moreover, it is known that the complement of any Haar-null set is dense in X, [5]. We shall call a real-vauled locally Lipschitz function f defined on a non-empty open subset A of X *essentially smooth* if

$\{x \in A : \partial f(x)$ is not a singleton$\}$ is a Haar-null set, and we shall denote by $\mathcal{S}_e(A)$ the set of all the essentially smooth locally Lipschitz functions on A. In [2] it is shown that the essentially smooth functions are both integrable and D-representable.

For a non-empty open subset A of an arbitrary Banach space X we shall denote by $\mathcal{R}_e(A)$ the family of all those locally Lipschitz functions f on A for which there is a rich family $\mathcal{F}$ (possibly depending on f) of closed separable subspaces Y of X such that $f|_{A\cap Y} \in \mathcal{S}_e(A \cap Y)$ for each $Y \in \mathcal{F}$ with $A \cap Y \neq \emptyset$. It follows then that each member of $\mathcal{R}_e(A)$ is integrable and possesses a minimal Clarke subdifferential mapping. In fact we have the following even stronger result.

Proposition 2.1 *Let f and g be real-valued locally Lipschitz functions defined on a non-empty open subset A of a Banach space X.*
(a) If $x \to \partial f(x)$ is a minimal weak cusco on A and $g \in \mathcal{R}_e(A)$, then $x \to \partial(f+g)(x)$ is a minimal weak* cusco on A.*
(b) If f is integrable on A and $g \in \mathcal{R}_e(A)$ then $f+g$ is integrable on A.

Proof: In light of Theorems 2.1 and 2.2 we may assume that X is separable. (a) follows from Proposition 8.2 part (i) in [2]. (b) Suppose that $h : A \to \mathbb{R}$ is locally Lipschitz on A and $\partial h(x) \subseteq \partial(f+g)(x)$ for all $x \in A$. Then:

$$\begin{aligned}\partial(h-g)(x) &\subseteq \partial h(x) - \partial g(x)\\ &\subseteq \partial(f+g)(x) - \partial g(x)\\ &\subseteq \partial f(x) + \partial g(x) - \partial g(x)\end{aligned}$$

Now, $\partial g(x) - \partial g(x) = \{0\}$ except on a Haar-null set. Therefore, by Proposition 2.2 in [8] we have that $\partial(h-g)(x) \subseteq \partial f(x)$ for all $x \in A$. The result should now be clear. ☺

Remark 2.1 *In contrast to Proposition 2.1 it is shown in [2] that neither integrability nor D-representability is closed under addition.*

Proposition 2.2 *Let A be a non-empty open subset of a Banach space X, then $\mathcal{R}_e(A)$ is closed under addition, subtraction, multiplication and division (when this is defined), as well as, both of the lattice operations. Moreover, $\mathcal{R}_e(A)$ contains all the pseudo-regular and semi-smooth locally Lipschitz functions defined on A.*

Proof: This follows directly from Theorems 2.1 and 2.2 and Theorem 3.12 in [1], see [2] for the definition of semi-smooth and pseudo-regular. ☺

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