CONTINUOUS BRANCHES OF INVERSES OF THE 12 JACOBI ELLIPTIC FUNCTIONS FOR REAL ARGUMENT

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Abstract

Continuous complex branches of the inverses of each of the 12 Jacobi elliptic functions, for real argument, are constructed in terms of real Incomplete Elliptic Integrals of the First Kind. These formulæ have been used for practical computation of complex inverses of Jacobi elliptic functions.

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Jacobi Elliptic Functions

1 Notation for Elliptic Integrals and Elliptic Functions

We shall use Milne–Thomson's notation for Legendre's elliptic integrals and Jacobi elliptic functions [Milne–Thomson].

1.1 Elliptic Integrals

An integral of the form $\int R(x, y)\,\mathrm{d}x$, where $R(x, y)$ is a rational function of x and y, and $y^2 = P(x)$ where P is a polynomial of degree 3 or 4, is called an *elliptic integral* [Milne–Thomson, **17.1**].

Legendre's Elliptic Integral of the First Kind, with amplitude φ and parameter m, is defined [Milne–Thomson, **17.2.7**] as

$$F(\varphi \mid m) \quad \stackrel{\mathrm{def}}{=} \quad \int_0^{\sin\varphi} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}}. \tag{1}$$

The parameter m can be taken as real with $0 \le m \le 1$, and the complementary parameter is

$$m_1 \quad \stackrel{\mathrm{def}}{=} \quad 1 - m. \tag{2}$$

(Earlier authors often used the modulus k, where $m = k^2$). $F(\varphi \mid m)$ is often abbreviated to $F(\varphi)$, when the parameter m is to be understood.

Legendre's Complete Elliptic Integral of the First Kind [Milne–Thomson, **17.3.1**] is

$$K(m) \quad \stackrel{\mathrm{def}}{=} \quad F(\tfrac{1}{2}\pi \mid m) \quad = \quad \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}}. \tag{3}$$

As $m \nearrow 1$, then $K(m) \nearrow \infty$.

Legendre's Complementary Complete Elliptic Integral of the First Kind is defined [Milne–Thomson, **17.3.5**] as

$$K'(m) \quad \stackrel{\mathrm{def}}{=} \quad K(m_1) \quad = \quad K(1-m). \tag{4}$$

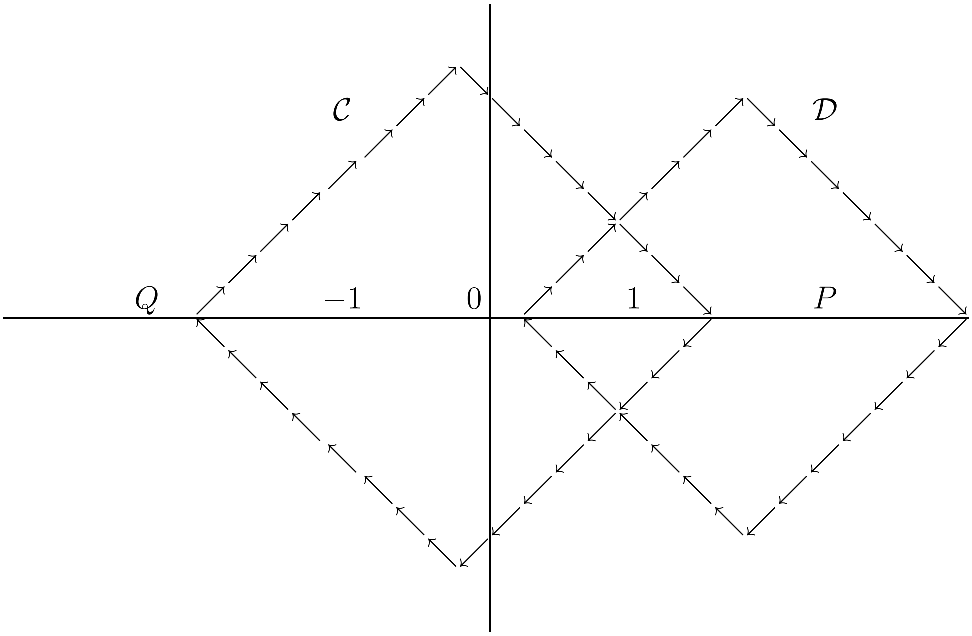
$K(m)$ and $K'(m)$ are often abbreviated to K and K', when the parameter m is to be understood.

In the definition (1) of $F(\varphi \mid m)$, the integrand has branch–points at $t = \pm 1$ and at $t = 1/\sqrt{m}$ (at P) and at $t = -1/\sqrt{m}$ (at Q)[1]. If the integral in (1) is taken as a Riemann integral on the real interval $(-1, 1)$ then it is single–valued; but with complex limit $\sin\varphi$ the Cauchy line integral has infinitely many values, depending on the number of loops made around each of the branch–points.

If $\sin\varphi$ is in the upper half t–plane (including the real axis) and the path of integration keeps t in the upper half–plane (including the real axis), then the imaginary part of $F(\varphi \mid m)$ is in the interval $[0, K']$. But if $\sin\varphi$ is in the

[1] The t–plane could be cut from 1 to $1/\sqrt{m}$ and from $-1/\sqrt{m}$ to -1, giving a 4-sheeted Riemann surface

lower half t–plane (including the real axis) and the path of integration keeps t in the lower half–plane (including the real axis) then the imaginary part of $F(\varphi \mid m$ is in the interval $[-K', 0]$. In each case, the real part of $F(\varphi \mid m)$ is in $[-K, K]$. [Jeffreys & Jeffreys, p.672].



(5)

But, for a path of integration taken (clockwise) around any loop $\mathcal{C}$ containing the branch–points -1 and 1 the integral equals $4K$, and around any loop $\mathcal{D}$ containing the branch–points 1 and $1/\sqrt{m}$ (at P) the integral equals $i2K'$ [Jeffreys & Jeffreys, p.672]. Hence any value of the integral (1) can have $4jK + i2kK'$ added to it, by splicing j clockwise circuits of $\mathcal{C}$ and k clockwise circuits of $\mathcal{D}$ into the path of integration.

1.2 Elliptic Functions

The theory of elliptic integrals, as developed by Fagnano, Euler and Legendre, was exceedingly complicated, involving infinitely many values for each elliptic integral. In 1827, Abel simplified the subject immensely by inverting elliptic integrals to get elliptic functions, and he shewed that elliptic functions are doubly–periodic single–valued functions [Abel, p.264].

If g is a doubly–periodic function with ξ a period of least modulus, and with χ a period of least modulus which is not an integral multiple of ξ, then the pair of periods (ξ, χ) are called fundamental periods of g [Jeffreys & Jeffreys, p.673], or its *primitive periods*.

The doubly periodic function $g(u)$ over the primitive period parallelogram which is generated by the vectors ξ and χ from 0 in the complex u–plane, gives a full representation of $g(u)$; for the entire complex plane could be tiled with copies of that parallelogram and the values of $g(u)$ over it. Indeed, *any* parallelogram with sides equal and parallel to those vectors, centred anywhere in the complex plane, could be taken as a basic period parallelogram for g, which could be copied to tile the entire complex plane with g.

As functions of the complex variable u, the Jacobi elliptic functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ are doubly–periodic single–valued functions of u.

1.2.1 Jacobi Elliptic Function sn(u)

The inverse function of the Legendre elliptic function F is $\varphi = F^{-1}(u)$, and the Jacobi elliptic function $\mathrm{sn}\, u \overset{\text{def}}{=} \sin\varphi$ (or $\mathrm{sn}(u \mid m)$) is single–valued for all complex parameters [Milne–Thomson, **16.1.3**], with

$$u \;=\; \int_0^{\mathrm{sn}\, u} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}}, \tag{6}$$

and $\mathrm{sn}(u)$ is an odd single–valued function of u.

For real u, the function sn has real period $4K(m)$ and range $[-1, 1]$, with $\mathrm{sn}(0) = 0$, $\mathrm{sn}(K) = 1$, $\mathrm{sn}(2K) = 0$, $\mathrm{sn}(3K) = -1$ and $\mathrm{sn}(4K) = 0$ [Milne–Thomson, **16.2**]. Let $\tau = \mathrm{sn}\, u$, so that

$$\mathrm{sn}^{-1}\tau \;=\; u \;=\; \int_0^{\tau} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}} \;=\; F\left(\sin^{-1}\tau \mid m\right), \tag{7}$$

using the principal branch of the function $\sin^{-1}\tau$, through the origin. On the real interval $-K \le u \le K$ the function sn increases monotonically from -1 to 1, and so for real $\tau \in [-1, 1]$ the function $\mathrm{sn}^{-1}\tau$ has a single value in the real interval $[-K, K]$.

In addition to the primitive real period $4K$, sn also has the primitive imaginary period $i\,2K'$. It follows that, for every complex u, every η satisfying the equation

$$\mathrm{sn}\,\eta \;=\; \mathrm{sn}\, u \;=\; \tau \tag{8}$$

is of the form

$$\eta \;=\; (-1)^j u + 2jK + i2kK' \tag{9}$$

for integers j, k [Dixon, p.32].

1.2.2 Jacobi Elliptic Function cn(u)

The Jacobi elliptic function cn is defined by

$$\operatorname{cn} u \;=\; \operatorname{cn}(u \mid m) \;\stackrel{\text{def}}{=}\; \cos\phi, \tag{10}$$

so that

$$\operatorname{cn} u \;=\; \sqrt{1-\operatorname{sn}^2 u}, \tag{11}$$

and $\operatorname{cn}(u)$ is an even single–valued function of u. The branch of the square root function in (11) is determined by (10).

For real u, the function cn has real period $4K(m)$ and range $[-1,1]$, with $\operatorname{cn}(0)=1$, $\operatorname{cn}(K)=0$, $\operatorname{cn}(2K)=-1$, $\operatorname{cn}(3K)=0$ and $\operatorname{cn}(4K)=1$ [Milne–Thomson, **16.2**]. On the real interval $0 \le u \le 2K$ the function cn decreases monotonically from 1 to -1, and so for real $r \in [-1,1]$ the function $\operatorname{cn}^{-1} r$ has a single value in the real interval $[0,2K]$.

In addition to the real primitive period $4K$, cn also has the primitive complex period $2K+i2K'$; and hence it also has the imaginary period $i\,4K' = 2(2K+i2K')-4K$, which is not a primitive period. It follows that, for every complex u, every η satisfying the equation

$$\operatorname{cn}\eta \;=\; \operatorname{cn} u \;=\; \tau \tag{12}$$

is of the form

$$\eta \;=\; \pm u + 4jK + k(2K + i\,2K') \tag{13}$$

for integers j,k [Dixon, p.33].

1.2.3 Jacobi Elliptic Function dn(u)

The Jacobi elliptic function dn is defined by

$$\operatorname{dn} u \;=\; \operatorname{dn}(u \mid m) \;\stackrel{\text{def}}{=}\; \sqrt{1-m\operatorname{sn}^2 u}, \tag{14}$$

and so $\operatorname{dn}(u)$ is an even single–valued function of u.

For real u, the function dn has real period $2K(m)$ and range $[\sqrt{m_1},1]$, with $\operatorname{dn}(0)=1$, $\operatorname{dn}(K)=\sqrt{m_1}$ and $\operatorname{dn}(2K)=1$ [Milne–Thomson, **16.2**]. On the real interval $0 \le u \le K$ the function dn decreases monotonically from 1 to $\sqrt{m_1}$, and so for real $r \in [\sqrt{m_1},1]$ the function $\operatorname{dn}^{-1} r$ has a single value in the real interval $[0,K]$.

In addition to the primitive real period $2K$, dn also has the primitive imaginary period $i\,4K'$. It follows that, for every complex u, every η satisfying the equation

$$\operatorname{dn}\eta \;=\; \operatorname{dn} u \;=\; \tau \tag{15}$$

is of the form

$$\eta \; = \; \pm u + 2jK + i\,4kK' \tag{16}$$

for integers j, k [Dixon, p.33].

2 Addition Formulæ for Elliptic Functions

For the extreme values $m = 0$ and $m = 1$, those elliptic functions reduce respectively [Milne–Thomson, **16.6**] to trigonometric and hyperbolic functions:

$$\begin{aligned} \operatorname{sn}(u \mid 0) &= \sin u, \qquad \operatorname{cn}(u \mid 0) = \cos u, \qquad \operatorname{dn}(u \mid 0) = 1, \\ \operatorname{sn}(u \mid 1) &= \tanh u, \qquad \operatorname{cn}(u \mid 1) \; = \; \operatorname{dn}(u \mid 1) \; = \; \operatorname{sech} u. \end{aligned} \tag{17}$$

Hereafter we shall consider $0 < m < 1$.

For all complex α and β [Milne–Thomson, **16.17**],

$$\begin{aligned} \operatorname{sn}(\alpha+\beta) &= \frac{\operatorname{sn}\alpha \cdot \operatorname{cn}\beta \cdot \operatorname{dn}\beta + \operatorname{sn}\beta \cdot \operatorname{cn}\alpha \cdot \operatorname{dn}\alpha}{1 - m\operatorname{sn}^2\alpha \cdot \operatorname{sn}^2\beta}, \\ \operatorname{cn}(\alpha+\beta) &= \frac{\operatorname{cn}\alpha \cdot \operatorname{cn}\beta - \operatorname{sn}\alpha \cdot \operatorname{dn}\alpha \cdot \operatorname{sn}\beta \cdot \operatorname{dn}\beta}{1 - m\operatorname{sn}^2\alpha \cdot \operatorname{sn}^2\beta}, \\ \operatorname{dn}(\alpha+\beta) &= \frac{\operatorname{dn}\alpha \cdot \operatorname{dn}\beta - m\operatorname{sn}\alpha \cdot \operatorname{cn}\alpha \cdot \operatorname{sn}\beta \cdot \operatorname{cn}\beta}{1 - m\operatorname{sn}^2\alpha \cdot \operatorname{sn}^2\beta}. \end{aligned} \tag{18}$$

2.1 Complex Amplitude and Parameter

Jacobi's imaginary transform [Milne–Thomson, **16.20**] gives the elliptic functions of imaginary argument $u = iy$:

$$\begin{aligned} \operatorname{sn}(iy \mid m) &= \frac{i\operatorname{sn}(y \mid m_1)}{\operatorname{cn}(y \mid m_1)} = i\operatorname{sc}(y \mid m_1), \\ \operatorname{cn}(iy \mid m) &= \frac{1}{\operatorname{cn}(y \mid m_1)} = \operatorname{nc}(y \mid m_1), \\ \operatorname{dn}(iy \mid m) &= \frac{\operatorname{dn}(y \mid m_1)}{\operatorname{cn}(y \mid m_1)} = \operatorname{dc}(y \mid m_1). \end{aligned} \tag{19}$$

Applying Jacobi's equations (19) to the addition formulæ (18), we get the Jacobi elliptic functions for complex $u = x + iy$ in terms of elliptic functions of real argument [Milne–Thomson, **16.21**]. For brevity, here we write s, c and d for sn, cn and dn:

$$\mathrm{s}(x+iy \mid m) = \frac{\mathrm{s}(x \mid m)\mathrm{d}(y \mid m_1) + i\,\mathrm{c}(x \mid m)\mathrm{d}(x \mid m)\mathrm{s}(y \mid m_1)\mathrm{c}(y \mid m_1)}{\mathrm{c}^2(y \mid m_1) + m\,\mathrm{s}^2(x \mid m)\mathrm{s}^2(y \mid m_1)},$$

$$
\begin{aligned}
\mathrm{c}(x+iy \mid m) &= \frac{\mathrm{c}(x \mid m)\mathrm{c}(y \mid m_1) - i\,\mathrm{s}(x \mid m)\mathrm{d}(x \mid m)\mathrm{s}(y \mid m_1)\mathrm{d}(y \mid m_1)}{\mathrm{c}^2(y \mid m_1) + m\,\mathrm{s}^2(x \mid m)\mathrm{s}^2(y \mid m_1)}, \\
\mathrm{d}(x+iy \mid m) &= \frac{\mathrm{d}(x \mid m)\mathrm{c}(y \mid m_1)\mathrm{d}(y \mid m_1) - im\,\mathrm{s}(x \mid m)\mathrm{c}(x \mid m)\mathrm{s}(y \mid m_1)}{\mathrm{c}^2(y \mid m_1) + m\,\mathrm{s}^2(x \mid m)\mathrm{s}^2(y \mid m_1)}.
\end{aligned}
\tag{20}
$$

These formulæ hold, even if x and y are not real.

Graphs of $\mathrm{sn}(u \mid m)$, $\mathrm{cn}(u \mid m)$ and $\mathrm{dn}(u \mid m)$ (and of the 9 other Jacobi elliptic functions ns, nc, nd, sc, sd, cs, cd, ds and dc) for real u (and $m = 0{\cdot}5$) are given in [Milne–Thomson, Figure 16.1]. Very effective graphs of F, E, sn, cn and dn for complex u (and $m = 0{\cdot}64$) are given in [Jahnke & Emde, pp. 91–93].

"In the complex domain $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$ are not essentially different from $\mathrm{sn}\,u$" [Jahnke & Emde, p.93], for it follows from (20) that

$$\mathrm{dn}(u \mid m) \;=\; \sqrt{m_1}\,\mathrm{sn}(K' - iK + iu \mid m_1) \tag{21}$$

and

$$\mathrm{cn}(u \mid m) = \mathrm{sn}\Big(K\sqrt{m_1} + u\sqrt{m_1} \mid -m/m_1\Big). \tag{22}$$

The evaluation of $\mathrm{sn}(w \mid -\lambda)$ with negative parameter $-\lambda$ (where $\lambda = m/m_1$ in (22)) can be done in terms of a positive parameter μ [Milne–Thomson **16.10**]. Define

$$\mu \;=\; \frac{\lambda}{1+\lambda}, \quad \mu_1 \;=\; \frac{1}{1+\lambda}, \quad v \;=\; \frac{w}{\sqrt{\mu_1}}. \tag{23}$$

Then,

$$\mathrm{sn}(w \mid -\lambda) \;=\; \sqrt{\mu_1}\,\mathrm{sd}(v \mid \mu) \;=\; \frac{\sqrt{\mu_1}\,\mathrm{sn}(v \mid \mu)}{\mathrm{dn}(v \mid \mu)}. \tag{24}$$

Inverse Elliptic Functions With Real Argument

3 Inverse Elliptic Functions In Terms Of Real $F(\varphi \mid m)$

The inverse of each elliptic function has infinitely many complex values; but it is sufficient to compute a single value u, since every value η of that inverse function can be expressed in terms of u. For computational purposes, any function being computed should be a continuous function of its argument, wherever possible. Accordingly, we shall construct algorithms for practical computation of the inverses of each of the 12 Jacobi elliptic functions for real argument, in each case constructing a single–valued complex function which gives a continuous branch of the infinitely–valued complex inverse function.

These algorithms have been used extensively, in testing numerically many novel congruence identities [Tee 1994] for integer sums of powers of Jacobi elliptic functions [Tee]. The computations were performed on a Macintosh computer using THINK PASCAL 4.0.2 with **extended** precision, rounding to about 19 significant decimal figures. The complete elliptic integrals and the elliptic functions (real and complex) were computed by procedures based on the Arithmetic-Geometric Mean [Milne–Thomson, **16.4, 16.21, 17.6**]. However, the similar method given [Milne–Thomson, **17.6.8**] for computing the incomplete elliptic integral $F(\varphi \mid m)$ is not workable, and hence that incomplete elliptic integral was evaluated by numerical integration. The version (1) of $F(\varphi \mid m)$ is not suitable for numerical evaluation, since for all m the integrand is infinite at $\varphi = \frac{1}{2}\pi$ $(t = 1)$. Accordingly, the Romberg algorithm for numerical integration was applied to (1), transformed by substituting $t = \sin\xi$:

$$F(\varphi \mid m) \;= \int_0^{\varphi} \frac{\mathrm{d}\xi}{\sqrt{1 - m\sin^2\xi}}, \tag{25}$$

in which (with $m < 1$) the integrand is a bounded function of ξ for all φ.

As $m \nearrow 1$ and $\varphi \nearrow \frac{1}{2}\pi$, then $F(\varphi \mid m) \nearrow +\infty$, and hence direct numerical evaluation of the above integral becomes difficult. If φ is nearly $\frac{1}{2}\pi$ and m is nearly 1, then methods given in [Milne–Thomson **17.4.13, 17.5**] should be used for evaluating $F(\varphi \mid m)$.

The computation of $F(\varphi \mid m)$ for complex amplitude φ was considered in [Milne–Thomson, **17.4.11**]. The method given there could be used to determine the moduli of the real and imaginary parts of one value of $F(\varphi \mid m)$ — but the information provided is not sufficient for determining the signs of those real and imaginary parts. Carlson's method [Carlson, (4.5)] could be used to compute a value of $F(\varphi \mid m)$ for complex amplitude φ. However, it would probably be simpler to evaluate the integral (25) for complex φ numerically, by constructing a simple integration path from 0 to φ in the complex plane, composed of intervals parallel to the real or imaginary axes (avoiding points ξ for which $1 - m\sin^2\xi = 0$). Over each such interval, the complex path integral could readily be evaluated to high accuracy by Romberg integration, and the complete path integral (25) is the sum of such integrals over all intervals in the integration path.

3.1 $\mathrm{sn}^{-1}\tau$, for real τ

As was noted after (7), as x increases from $-K$ to K, $\mathrm{sn}\, x$ increases monotonically from -1 to 1; and hence for real $\tau \in [-1, 1]$, $x = \mathrm{sn}^{-1}\tau$ has a single real value in $[-K, K]$:

$$x \;=\; \mathrm{sn}^{-1}\tau \;=\; F\left(\sin^{-1}\tau \mid m\right). \tag{26}$$

For $u = K + iy$, it follows from (20) that

$$\mathrm{sn}(K+iy) \;=\; \frac{\mathrm{dn}(y \mid m_1) + i0}{\mathrm{cn}^2(y \mid m_1) + m\,\mathrm{sn}^2(y \mid m_1)} \;=\; \frac{1}{\mathrm{dn}(y \mid m_1)}, \tag{27}$$

which is real (for real y); and as y increases from 0 to K', $\mathrm{sn}(K+iy)$ increases monotonically from 1 to $1/\sqrt{m}$. Hence, for $\tau \in [1, \sqrt{m}]$, $\mathrm{sn}^{-1}\tau$ has a single value of the form $K + iy$, where $0 \le y \le K'$.

$$\tau \;=\; \mathrm{sn}(K+iy) \;=\; \frac{1}{\mathrm{dn}(y \mid m_1)}, \tag{28}$$

and hence

$$\tau^{-2} \;=\; \mathrm{dn}^2(y \mid m_1) \;=\; 1 - m_1\,\mathrm{sn}^2(y \mid m_1), \tag{29}$$

and so

$$\mathrm{sn}(y \mid m_1) \;=\; \pm\sqrt{\frac{1-\tau^{-2}}{m_1}}. \tag{30}$$

Therefore, for $\tau \in [1, 1/\sqrt{m}]$

$$\mathrm{sn}^{-1}\tau \;=\; K \;+\; i\,F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^{-2}}{m_1}}\right) \mid m_1\right). \tag{31}$$

It follows from (20) that

$$\mathrm{sn}(u \pm i\,K') \;=\; \frac{\mathrm{sn}\,u\sqrt{m} + i0}{0 + m\,\mathrm{sn}^2 u} \;=\; \frac{1}{\sqrt{m}\;\mathrm{sn}\,u}, \tag{32}$$

and so, as real x decreases from K to 0, $\mathrm{sn}(x + iK')$ increases monotonically from $1/\sqrt{m}$ to $+\infty$. Hence, for $\tau \ge 1/\sqrt{m}$, $\mathrm{sn}^{-1}\tau$ has a single value of the form $x + iK'$, where $0 \le x \le K$, with

$$\tau \;=\; \mathrm{sn}(x + iK') \;=\; \frac{1}{\sqrt{m}\;\mathrm{sn}\,x}, \tag{33}$$

so that

$$\mathrm{sn}\,x \;=\; \frac{1}{\tau\sqrt{m}}. \tag{34}$$

Therefore, for $\tau \geq 1/\sqrt{m}$,

$$\mathrm{sn}^{-1}\tau \;=\; F\left(\sin^{-1}\left(\frac{1}{\tau\sqrt{m}}\right) \mid m\right) \;+\; iK'. \tag{35}$$

And sn is an odd function, so that $\mathrm{sn}^{-1}(-\tau) \;= -\mathrm{sn}^{-1}\tau$.

Thus, for real τ, a continuous branch of $u = \mathrm{sn}^{-1}\tau$ is represented by the following diagram in the plane of u, with arrows indicating increasing τ:

$$\begin{array}{ccccccc}
 & & (u = iK') & F & \longleftarrow\longleftarrow\longleftarrow\longleftarrow & E & \\
 & & & & & \uparrow & \\
 & & & & & \uparrow & \\
(u = -K) & C & \longrightarrow\longrightarrow\longrightarrow\longrightarrow & 0 & \longrightarrow\longrightarrow\longrightarrow\longrightarrow & D & (u = K) \\
 & \uparrow & & & & & \\
 & \uparrow & & & & & \\
 & B & \longleftarrow\longleftarrow\longleftarrow\longleftarrow & A & (u = -iK') & &
\end{array} \tag{36}$$

As τ increases from $-\infty$ to $-1/\sqrt{m},\;$ u moves from A to B. As τ increases from $-1/\sqrt{m}$ to $-1,\;$ u moves from B to C. As τ increases from -1 to $1,\;$ u moves from C through the origin 0 to D. As τ increases from 1 to $1/\sqrt{m},\;$ u moves from D to E. As τ increases from $1/\sqrt{m}$ to $+\infty,\;$ u moves from E to F.

From the value u found on this continuous branch of the function sn^{-1}, every value $\eta = \mathrm{sn}^{-1}\tau$ is of the form (9).

3.2 $\mathrm{ns}^{-1}\sigma$, for real σ

If

$$\tau \;=\; \mathrm{sn}\,u, \tag{37}$$

then

$$\sigma \;\stackrel{\text{def}}{=}\; \frac{1}{\tau} \;=\; \frac{1}{\mathrm{sn}\,u} \;=\; \mathrm{ns}\,u, \tag{38}$$

so that

$$u \;=\; \mathrm{ns}^{-1}\sigma \;=\; \mathrm{ns}^{-1}(1/\tau). \tag{39}$$

The function sn has period $i2K'$, and hence so does the function ns. Thus, from the diagram (36), for real σ a continuous branch of $u = \mathrm{ns}^{-1}\sigma$ is represented by the following diagram in the plane of u, with arrows indicating decreasing σ:

$$
\begin{array}{lllll}
H & \longrightarrow\longrightarrow\longrightarrow\longrightarrow & I & (u = i2K') & \\
\uparrow & & & & \\
\uparrow & & & & \\
G & \longleftarrow\longleftarrow\longleftarrow\longleftarrow & F & \longleftarrow\longleftarrow\longleftarrow\longleftarrow & E \\
 & & & & \uparrow \\
 & & & & \uparrow \\
 & & 0 & \longrightarrow\longrightarrow\longrightarrow\longrightarrow & D \ (u = K)
\end{array}
\tag{40}
$$

For $\sigma > 0$, as σ decreases from $+\infty$ to 0, u moves from 0 ($\sigma = +\infty$) to D ($\sigma = 1$) to $E(\sigma = \sqrt{m})$ to F ($\sigma = 0$), with

$$u \;=\; \mathrm{ns}^{-1}\sigma \;=\; \mathrm{sn}^{-1}(1/\sigma) \qquad (\sigma > 0). \tag{41}$$

For $\sigma = 0$,

$$u \;=\; \mathrm{ns}^{-1}\sigma \;=\; iK'. \qquad (\sigma = 0). \tag{42}$$

For $\sigma < 0$, as σ decreases from 0 to $-\infty$, u moves from F to G ($\sigma = -\sqrt{m}$) to H ($\sigma = -1$) to $I(\sigma = -\infty)$, with

$$u \;=\; \mathrm{ns}^{-1}\sigma \;=\; \mathrm{sn}^{-1}(1/\sigma) + i2K' \qquad (\sigma < 0). \tag{43}$$

In both (41) and (43), the branch of sn^{-1} given in (36) is to be used.

From the value u found on this continuous branch of the function ns^{-1}, <u>every</u> value η can be generated from (9).

3.3 $\mathrm{cn}^{-1}\tau$, for real τ

Let

$$\tau \;=\; \mathrm{cn}\, u, \tag{44}$$

so that

$$\tau^2 \;=\; \mathrm{cn}^2 u \;=\; 1 - \mathrm{sn}^2 u, \tag{45}$$

and hence

$$\mathrm{sn}\, u \;=\; \pm\sqrt{1-\tau^2}. \tag{46}$$

As was noted after (11), as x increases from 0 to $2K$, $\mathrm{cn}\, x$ decreases monotonically from 1 to -1; and hence for real $\tau \in [0, 1]$, $x = \mathrm{cn}^{-1}\tau$ has a single real value in $[0, K]$:

$$x \;=\; \mathrm{cn}^{-1}\tau \;=\; F\left(\sin^{-1}\left(\sqrt{1-\tau^2}\right) \mid m\right). \tag{47}$$

For all u, $\mathrm{cn}(2K-u) = -\mathrm{cn}\, u$ in view of (18), and hence for real $\tau \in [-1,0]$, $x = \mathrm{cn}^{-1}\tau$ has a single real value in $[K, 2K]$:

$$x \;=\; \mathrm{cn}^{-1}\tau \;= 2K \;-\; F\left(\sin^{-1}\left(\sqrt{1-\tau^2}\right) \mid m\right). \tag{48}$$

For $u = iy$, it follows from (19) that as y increases from 0 to K', $\mathrm{cn}(iy)$ increases monotonically from 1 to $+\infty$. Hence, for $\tau \geq 1$, $\mathrm{cn}^{-1}\tau$ has a single value of the form $u = iy$, where $0 \leq y \leq K'$.

$$\tau \;=\; \mathrm{cn}(iy) \;=\frac{1}{\mathrm{cn}(y \mid m_1)}, \tag{49}$$

and hence

$$\tau^{-2} \;=\; \mathrm{cn}^2(y \mid m_1) \;=\; 1 - \mathrm{sn}^2(y \mid m_1), \tag{50}$$

and so

$$\mathrm{sn}(y \mid m_1) \;=\; \pm\sqrt{1-\tau^{-2}}. \tag{51}$$

Therefore, for $\tau \geq 1$

$$\mathrm{cn}^{-1}\tau \;=\; i\,F\left(\sin^{-1}\left(\sqrt{1-\tau^{-2}}\right) \mid m_1\right). \tag{52}$$

It follows from (20) that

$$\mathrm{cn}(2K - iy) \;=\; \frac{-1}{\mathrm{cn}(y \mid m_1)}, \tag{53}$$

and so, as real y increases from 0 to K', $\mathrm{cn}(2K-iy)$ decreases monotonically from -1 to $-\infty$. Hence, for $\tau \leq -1$, $\mathrm{cn}^{-1}\tau$ has a single value of the form $2K - iy$, where $0 \leq y < K'$, with

$$\tau^2 \;=\; \mathrm{cn}^2(2K - iy) \;=\; \frac{1}{\mathrm{cn}^2(y \mid m_1)} \;=\; \frac{1}{1-\mathrm{sn}^2(y \mid m_1)}, \tag{54}$$

so that

$$\mathrm{sn}(y \mid m_1) \;= \pm\sqrt{1-\tau^{-2}}. \tag{55}$$

Therefore, for $\tau \leq -1$,

$$\mathrm{cn}^{-1}\tau \;=\; 2K \;-\; i\,F\left(\sin^{-1}\left(\sqrt{1-\tau^{-2}}\right) \mid m_1\right)\;. \tag{56}$$

Thus, for real τ, a continuous branch of $u = \mathrm{cn}^{-1}\tau$ is represented by the following diagram in the plane of u, with arrows indicating increasing τ:

$$
\begin{array}{lcccl}
(u = iK') & D & & & \\
 & \uparrow & & & \\
 & \uparrow & & & \\
 & 0 & \longleftarrow\!\!\longleftarrow \; C \; \longleftarrow\!\!\longleftarrow & B & (u = 2K) \\
 & & & \uparrow & \\
 & & & \uparrow & \\
 & & & A & (u = 2K - iK')
\end{array}
\tag{57}
$$

As τ increases from $-\infty$ to -1, u moves from A to B. As τ increases from -1 to 0, u moves from B to C ($u = K$). As τ increases from 0 to 1, u moves from C to the origin 0. As τ increases from 1 to ∞, u moves from the origin 0 to D.

From the value u found on this continuous branch of the function cn^{-1}, every value $\eta = \mathrm{cn}^{-1}\tau$ is of the form (13).

3.4 $\mathrm{nc}^{-1}\sigma$, for real σ

If

$$\tau \; = \; \mathrm{cn}\, u, \tag{58}$$

then

$$\sigma \; \stackrel{\mathrm{def}}{=} \; \frac{1}{\tau} \; = \; \frac{1}{\mathrm{cn}\, u} \; = \; \mathrm{nc}\, u, \tag{59}$$

so that

$$u \; = \; \mathrm{nc}^{-1}\sigma \; = \; \mathrm{nc}^{-1}(1/\tau). \tag{60}$$

The function cn has primitive periods $4K$ and $2K + i2K'$, so that cn also has the period $-2K + i2K'$, and hence so does the function nc. Thus, from the diagram (57), for real σ a continuous branch of $u = \mathrm{nc}^{-1}\sigma$ is represented by the following diagram in the plane of u, with arrows indicating decreasing σ:

$$
\begin{array}{rccl}
(u = -K + i2K') \; F & \longleftarrow\!\!\longleftarrow & E & (u = i2K') \\
 & & \uparrow & \\
 & & \uparrow & \\
 & & D & \\
 & & \uparrow & \\
 & & \uparrow & \\
 & & 0 & \longleftarrow\!\!\longleftarrow \; C \; (u = K)
\end{array}
\tag{61}
$$

For $\sigma > 0$, as σ decreases from $+\infty$ to 0, u moves from C ($\sigma = +\infty$) to 0 ($\sigma = 1$) to D ($\sigma = 0$), with

$$u = \mathrm{nc}^{-1}\sigma = \mathrm{cn}^{-1}(1/\sigma) \qquad (\sigma > 0). \tag{62}$$

For $\sigma = 0$,

$$u = \mathrm{nc}^{-1}\sigma = iK' \qquad (\sigma = 0). \tag{63}$$

For $\sigma < 0$, as σ decreases from 0 to $-\infty$, u moves from D to E ($\sigma = -1$) to F ($\sigma = -\infty$), with

$$u = \mathrm{nc}^{-1}\sigma = \mathrm{cn}^{-1}(1/\sigma) - 2K + i2K' \qquad (\sigma < 0). \tag{64}$$

In both (62) and (64), the branch of cn^{-1} given in (57) is to be used.

From the value u found on this continuous branch of the function nc^{-1}, every value η can be generated from (13).

3.5 $\mathrm{dn}^{-1}\tau$, for real τ

Let

$$\tau = \mathrm{dn}\, u, \tag{65}$$

so that

$$\tau^2 = \mathrm{dn}^2 u = 1 - m\,\mathrm{sn}^2 u, \tag{66}$$

and hence

$$\mathrm{sn}\, u = \pm\sqrt{\frac{1-\tau^2}{m}}. \tag{67}$$

As was noted after (14), as x increases from 0 to K, $\mathrm{dn}\, x$ decreases monotonically from 1 to $\sqrt{m_1}$; and hence for real $\tau \in [\sqrt{m_1}, 1]$, $x = \mathrm{dn}^{-1}\tau$ has a single real value in $[0, K]$:

$$x = \mathrm{dn}^{-1}\tau = F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^2}{m}}\right) \mid m\right). \tag{68}$$

For $u = iy$, it follows from (19) that as y increases from 0 to K', $\mathrm{dn}(iy)$ increases monotonically from 1 to $+\infty$. Hence, for $\tau \geq 1$, $\mathrm{dn}^{-1}\tau$ has a single value of the form $u = iy$, where $0 \leq y \leq K'$.

$$\tau = \mathrm{dn}(iy) = \frac{\mathrm{dn}(y \mid m_1)}{\mathrm{cn}(y \mid m_1)}, \tag{69}$$

and hence

$$\tau^2 = \frac{\mathrm{dn}^2(y \mid m_1)}{\mathrm{cn}^2(y \mid m_1)} = \frac{1 - m_1\mathrm{sn}^2(y \mid m_1)}{1 - \mathrm{sn}^2(y \mid m_1)}, \tag{70}$$

and so

$$\operatorname{sn}(y \mid m_1) \;=\; \pm\sqrt{\frac{1-\tau^{-2}}{1-m_1\tau^{-2}}}. \tag{71}$$

Therefore, for $\tau \geq 1$

$$\operatorname{dn}^{-1}\tau \;=\; i\,F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^{-2}}{1-m_1\tau^{-2}}}\right) \mid m_1\right). \tag{72}$$

It follows from (20) that

$$\operatorname{dn}(K+iy) \;=\; \frac{\sqrt{m_1}\left(\frac{\operatorname{dn}(y|m_1)}{\operatorname{cn}(y|m_1)-i0}\right)}{1+\frac{m_1\operatorname{sn}^2(y|m_1)}{\operatorname{cn}^2(y|m_1)}} \;=\; \frac{\sqrt{m_1}\,\operatorname{cn}(y \mid m_1)}{\operatorname{dn}(y \mid m_1)}; \tag{73}$$

and so, as y increases from 0 to $2K'$, $\operatorname{dn}(K+iy)$ decreases monotonically from $\sqrt{m_1}$ to $-\sqrt{m_1}$. Hence, for $\tau \in [-\sqrt{m_1}, +\sqrt{m_1}]$, $\operatorname{dn}^{-1}(K+iy)$ has a single value of the form $K+iy$, where $0 \leq y \leq 2K'$.

Now,

$$\tau^2 \;=\; \operatorname{dn}^2(K+iy) \;=\; \frac{m_1\operatorname{cn}^2(y \mid m_1)}{\operatorname{dn}^2(y \mid m_1)} \;=\; \frac{m_1(1-\operatorname{sn}^2(y \mid m_1))}{1-m_1\operatorname{sn}^2(y \mid m_1)}, \tag{74}$$

and so

$$\operatorname{sn}(y \mid m_1) \;=\; \pm\sqrt{\frac{1-\tau^2/m_1}{1-\tau^2}}. \tag{75}$$

Therefore, for $\tau \in [0, \sqrt{m_1}]$,

$$\operatorname{dn}^{-1}\tau \;=\; K+i\,F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^2/m_1}{1-\tau^2}}\right) \mid m_1\right). \tag{76}$$

Now, from (20), $\operatorname{sn}(i(2K'-y) \mid m_1) = \operatorname{sn}(i\,y \mid m_1)$; and therefore, for $\tau \in [-\sqrt{m_1}, 0]$,

$$\operatorname{dn}^{-1}\tau \;=\; K+i\left(2K'-F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^2/m_1}{1-\tau^2}}\right) \mid m_1\right)\right). \tag{77}$$

It follows from (20) that for all u,

$$\operatorname{dn}(2K-u+i\,2K') \;=\; \operatorname{dn}(-u+i\,2K') \;=\; -\operatorname{dn}(-u) \;=\; -\operatorname{dn}u; \tag{78}$$

and so as x increases from 0 to K, $\operatorname{dn}(2K-x+i\,2K')$ increases monotonically from $-\sqrt{m_1}$ to -1. Hence, for $\tau \in [-1, -\sqrt{m_1}]$, $\operatorname{dn}^{-1}\tau$ has a single value of the form $2K-x+i\,2K'$, where $0 \leq x \leq K$, with

$$\tau \;=\; \operatorname{dn}(2K-x+i\,2K') \;=\; -\operatorname{dn}x, \tag{79}$$

so that

$$\operatorname{sn} x \;=\; \pm\sqrt{\frac{1-\tau^2}{m}}. \tag{80}$$

Therefore, for $\tau \in [-1, -\sqrt{m_1}]$,

$$\operatorname{dn}^{-1}\tau \;=\; 2K - F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^2}{m}}\right) \mid m\right) + i\,K'. \tag{81}$$

It follows from (19) that for all y

$$\operatorname{dn}(2K + i\,2K' - iy) \;=\; \operatorname{dn}(i(2K'-y)) \;= -\operatorname{dn}(iy) \;=\; \frac{-\operatorname{dn}(y \mid m_1)}{\operatorname{cn}(y \mid m_1)}; \tag{82}$$

and so, as $y$increases from 0 to K', $\operatorname{dn}(2K+2K'-iy)$ decreases monotonically from -1 to $-\infty$. Further,

$$\tau^2 \;= \operatorname{dn}^2(2K + i(2K'-y)) \;=\; \frac{\operatorname{dn}^2(y \mid m_1)}{\operatorname{cn}^2(y \mid m_1)} \;=\; \frac{1 - m_1 \operatorname{sn}^2(y \mid m_1)}{1 - \operatorname{sn}^2(y \mid m_1)}, \tag{83}$$

whence

$$\operatorname{sn}(y \mid m_1) \;=\; \pm\sqrt{\frac{1-\tau^{-2}}{1-m_1\tau^{-2}}}. \tag{84}$$

Hence, for $\tau \leq -1$, $\operatorname{dn}^{-1}\tau$ has a single value of the form $2K + i\,2K' - iy$, where $0 \leq y \leq K'$, with

$$\operatorname{dn}^{-1}\tau \;=\; 2K + i\left(2K' - F\left(\sin^{-1}\left(\sqrt{\frac{1-\tau^{-2}}{1-m_1\tau^{-2}}}\right) \mid m_1\right)\right). \tag{85}$$

Thus, for real τ, a continuous branch of $u = \operatorname{dn}^{-1}\tau$ is represented by the following diagram in the plane of u, with arrows indicating increasing τ:

$$\begin{array}{ccccc}
 & (u = K + i\,2K') & C & \longleftarrow\!\!\longleftarrow\!\!\longleftarrow\!\!\longleftarrow & B\ (u = 2K + i\,2K') \\
 & & \downarrow & & \uparrow \\
 & & \downarrow & & \uparrow \\
(u = iK')\ F & & D & & A\ (u = 2K + i\,K') \\
\uparrow & & \downarrow & & \\
\uparrow & & \downarrow & & \\
0 \longleftarrow\!\!\longleftarrow\!\!\longleftarrow\!\!\longleftarrow & & E & (u = K) &
\end{array} \tag{86}$$

As τ increases from $-\infty$ to -1, u moves from A to B. As τ increases from -1 to $-\sqrt{m_1}$, u moves from B to C. As τ increases from $-\sqrt{m_1}$ to 0, u

moves from C to D. As τ increases from 0 to $\sqrt{m_1}$, u moves from D to E. As τ increases from $\sqrt{m_1}$ to 1, u moves from E to the origin 0. As τ increases from 1 to ∞, u moves from the origin 0 to F.

From the value u found on this continuous branch of the function dn^{-1}, <u>every</u> value $\eta = \mathrm{dn}^{-1}\tau$ is of the form (16).

Note the close relation between the diagram (36) for $\mathrm{sn}^{-1}\tau$ and the diagram (86) for $\mathrm{dn}^{-1}\tau$, which follows from (21) and from the fact that dn is an even function.

3.6 $\mathrm{nd}^{-1}\sigma$, for real σ

If

$$\tau \;=\; \mathrm{dn}\, u, \tag{87}$$

then

$$\sigma \;\stackrel{\text{def}}{=}\; \frac{1}{\tau} \;=\; \frac{1}{\mathrm{dn}\, u} \;=\; \mathrm{nd}\, u, \tag{88}$$

so that

$$u \;=\; \mathrm{nd}^{-1}\sigma \;=\; \mathrm{nd}^{-1}(1/\tau). \tag{89}$$

The function dn has primitive periods $2K$ and $i4K'$, and hence so does the function nd. Thus, from the diagram (86), for real σ a continuous branch of $u = \mathrm{nd}^{-1}\sigma$ is represented by the following diagram in the plane of u, with arrows indicating decreasing σ:

$$\begin{array}{rcrlcl}
 & H & \longleftarrow\!\longleftarrow\!\longleftarrow\!\longleftarrow & G \;\; (u = i2K') & & \\
 & \downarrow & & \uparrow & & \\
 & \downarrow & & \uparrow & & \\
(u = -K + iK') & I & (u = iK') & F & & D \;\; (u = K + iK') \\
 & & & \uparrow & & \downarrow \\
 & & & \uparrow & & \downarrow \\
 & & & 0 & \longleftarrow\!\longleftarrow\!\longleftarrow\!\longleftarrow & E \;\; (u = K)
\end{array} \tag{90}$$

For $\sigma > 0$, as σ decreases from $+\infty$ to 0, u moves from D ($\sigma = +\infty$) to E ($\sigma = 1/\sqrt{m}$) to 0 ($\sigma = 1$) to F ($\sigma = 0$), with

$$u \;=\; \mathrm{nd}^{-1}\sigma \;=\; \mathrm{dn}^{-1}(1/\sigma) \qquad (\sigma > 0). \tag{91}$$

For $\sigma = 0$,

$$u \;=\; \mathrm{nd}^{-1}\sigma \;=\; iK' \qquad (\sigma = 0). \tag{92}$$

For $\sigma < 0$, as σ decreases from 0 to $-\infty$, u moves from F to G $(\sigma = -1)$ to H $(\sigma = -1/\sqrt{m})$ to $(\sigma = -\infty)$, with

$$u \;=\; \mathrm{nd}^{-1}\sigma \;=\; \mathrm{dn}^{-1}(1/\sigma) - 2K \qquad (\sigma < 0). \tag{93}$$

In both (91) and (93), the branch of dn^{-1} given in (86) is to be used.

From the value u found on this continuous branch of the function nd^{-1}, every value η can be generated from (16).

3.7 Inverses of sc and cs

If

$$\tau \;=\; \mathrm{sc}\,u \;=\; \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u}, \tag{94}$$

then

$$\tau^2 \;=\; \frac{\mathrm{sn}^2 u}{\mathrm{cn}^2 u} \;=\; \frac{\mathrm{sn}^2 u}{1 - \mathrm{sn}^2 u}, \tag{95}$$

and so

$$\mathrm{sn}\,u \;=\; \frac{\pm\tau}{\sqrt{1+\tau^2}}. \tag{96}$$

For $u \in (-K, K)$,

$$\mathrm{sn}\,u \;=\; \frac{\tau}{\sqrt{1+\tau^2}}. \tag{97}$$

As u increases from $-K$ to K, $\mathrm{sc}\,u$ increases monotonically and continuously from $-\infty$ to $+\infty$. Hence, for real τ, a continuous branch of the function $\mathrm{sc}^{-1}\tau$, which increases from $-K$ to K as τ increases from $-\infty$ to $+\infty$, is given (cf. (7)) by

$$u \;= \mathrm{sc}^{-1}\tau \;=\; \mathrm{sn}^{-1}\left(\frac{\tau}{\sqrt{1+\tau^2}}\right) \;=\; F\left(\sin^{-1}\left(\frac{\tau}{\sqrt{1+\tau^2}}\right) \mid m\right). \tag{98}$$

Thus, for all real τ, a continuous branch of $\mathrm{sc}^{-1}\tau$ is given by

$$u \;=\; \mathrm{sc}^{-1}\tau \;=\; F\left(\tan^{-1}\tau \mid m\right), \tag{99}$$

using the principal branch of $\tan^{-1}\tau$, through the origin.

Jacobi's imaginary transform (19) shews that

$$\mathrm{sc}(y \mid m) \;=\; -i\,\mathrm{sn}(iy \mid m_1). \tag{100}$$

Hence the function sc has the primitive periods $2K'$ and $i4K'$, and the equation

$$\mathrm{sc}(\eta \mid m) \;=\; \mathrm{sc}(u \mid m) \;=\; \tau \tag{101}$$

is equivalent to

$$\operatorname{sn}(i\eta \mid m_1) \;=\; \operatorname{sn}(iu \mid m_1), \tag{102}$$

whose general solution is (cf. (9))

$$i\eta \;=\; (-1)^j iu + 2jK' + i\,2kK, \tag{103}$$

Thus, from the value (99) of u found on this continuous branch of the function sc^{-1}, <u>every</u> value η satisfying (101) is of the form

$$\eta \;=\; (-1)^j u + 2kK - i\,2jK' \tag{104}$$

for integers j, k . And that can be rewritten as

$$\eta \;=\; (-1)^j u + 2kK + i\,2jK' \tag{105}$$

for integers j, k .

With

$$\sigma \;\stackrel{\text{def}}{=}\; \frac{1}{\tau} \;=\; \frac{1}{\operatorname{sc} u} \;=\; \operatorname{cs} u, \tag{106}$$

so that

$$u \;=\; \operatorname{cs}^{-1}\sigma \;=\; \operatorname{sc}^{-1}(1/\sigma). \tag{107}$$

The function sc has primitive periods $2K$ and $i4K'$, and hence so does the function cs. Hence, for real σ a continuous branch of $\operatorname{cs}^{-1}\sigma$, which increases from 0 to $2K$ as σ decreases from $+\infty$ to $-\infty$, is given (cf. (99)) by:

$$\operatorname{cs}^{-1}\sigma \;=\; \begin{cases} F\left(\tan^{-1}(1/\sigma) \mid m\right) & (\sigma > 0) \\ K & (\sigma = 0) \\ F\left(\tan^{-1}(1/\sigma) \mid m\right) + 2K & (\sigma < 0) \end{cases} \tag{108}$$

From the value of u found on this continuous branch of the function cs^{-1}, <u>every</u> value η can be generated as in (105).

3.8 Inverses of cd, dc, sd, ds

From the addition formulæ (20), we get the identity in u:

$$\operatorname{sn}(u+K) \;=\; \operatorname{cd} u \;=\; \frac{\operatorname{cn} u}{\operatorname{dn} u}. \tag{109}$$

Hence, if

$$\tau \;=\; \operatorname{cd} u \;= \operatorname{sn}(u+K), \tag{110}$$

so that

$$u \;=\; \operatorname{cd}^{-1}\tau, \tag{111}$$

then also

$$u + K \;=\; \mathrm{sn}^{-1}\tau. \tag{112}$$

Therefore, a continuous branch of $\mathrm{cd}^{-1}\tau$, for real τ, can be computed as

$$u \;= \mathrm{cd}^{-1}\tau \;=\; \mathrm{sn}^{-1}\tau - K, \tag{113}$$

using the continuous branch of sn^{-1} given in (36).

Similarly, if

$$\sigma \;=\; \mathrm{dc}\,u \;=\; \mathrm{ns}(u + K), \tag{114}$$

then a continuous branch of $\mathrm{dc}^{-1}\sigma$, for real σ, can be computed as

$$u \;= \mathrm{dc}^{-1}\sigma \;=\; \mathrm{ns}^{-1}\sigma - K, \tag{115}$$

using the continuous branch of ns^{-1} given in (40).

Equation (109) shews that the functions cd and dc have the same primitive periods as sn; and hence from the values of u generated on the continuous branch of cd^{-1} (or of dc^{-1}), the general solution η is given by (9).

Likewise, from the addition formulæ (20), we get the identity in u:

$$\mathrm{cn}(u - K) \;=\; \sqrt{m_1}\,\mathrm{sd}\,u \;=\; \sqrt{m_1}\,\frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}. \tag{116}$$

Hence, if

$$\tau \;=\; \mathrm{sd}\,u \;= \frac{\mathrm{cn}(u - K)}{\sqrt{m_1}}, \tag{117}$$

so that

$$u \;=\; \mathrm{sd}^{-1}\tau, \tag{118}$$

then also

$$u - K \;=\; \mathrm{cn}^{-1}(\tau\sqrt{m_1}). \tag{119}$$

Therefore, a continuous branch of $\mathrm{cd}^{-1}\tau$, for real τ, can be computed as

$$u \;= \mathrm{sd}^{-1}\tau \;=\; \mathrm{cn}^{-1}(\tau\sqrt{m_1}) + K, \tag{120}$$

using the continuous branch of cn^{-1} given in (57).

Similarly, if

$$\sigma \;=\; \mathrm{ds}\,u \;=\; \sqrt{m_1}\,\mathrm{nc}(u - K), \tag{121}$$

then a continuous branch of $\mathrm{ds}^{-1}\sigma$, for real σ, can be computed as

$$u \;= \mathrm{ds}^{-1}\sigma \;=\; \mathrm{nc}^{-1}(\sigma/\sqrt{m_1}) + K, \tag{122}$$

using the continuous branch of nc^{-1} given in (61).

Equation (109) shews that the functions sd and ds have the same primitive periods as cn; and hence from the values of u generated on the continuous branch of sd^{-1} (or of ds^{-1}), the general solution η is given by (13).

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