A Note on Arc-transitive Circulants

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Abstract

Two new infinite classes of arc-transitive circulant graphs are described, along with a classification of all arc-transitive circulants on up to thirty vertices.

1 Introduction

The graphs considered in this paper are simple and undirected. A graph X is said to be *k-arc-transitive* if the automorphism group of X, denoted $\mathrm{Aut}(X)$, acts transitively on the k-arcs of X. A *1-arc-transitive* (or simply arc-transitive) graph is often described as symmetric.

Let n be a positive integer and $S \subseteq \mathbf{Z}_n \backslash \{0\}$ satisfy $i \in S$ if and only if $n - i \in S$. Define a graph $X(n;S)$ to have vertex set labelled with the elements of $\mathbf{Z}_n$, with an edge joining vertices i and j if and only if $i - j \in S$. The graph $X(n;S)$ is called a *circulant* and the set S is called the *symbol* of $X(n;S)$.

It is known that for any positive integer n and any even order subgroup of $\mathbf{Z}_n^*$, the graph $X(n;S)$ is an arc-transitive circulant. Berggren [2], and later Chao [4], described the situation for $n = p$ prime. Using different approaches they showed that the aforementioned construction exhausts the entire class of arc-transitive circulants of prime order. In [1] Alspach et al gave a classification of 2-arc-transitive circulants.

2 Results and Discussion

Two new infinite classes of arc-transitive circulants are described in the following two theorems. They are respectively a class of circulants with increasing odd degree and a class of circulants of degree four. In both cases the symbol is not necessarily a subgroup of $\mathbf{Z}_n^*$.

Theorem 1 *Let u and v be positive integers greater than 2, and let $n = uv$. Then $K_n \backslash uK_v$ is an arc-transitive circulant graph with degree $v(u-1)$ and automorphism group $S_v \mathrm{wr} S_u$.*

Proof. Let Γ be the graph uK_v on n vertices consisting of u copies of the complete graph K_v. If $\theta \in \mathrm{Aut}(\Gamma)$, then by definition θ maps edges to other edges and nonedges to other nonedges; consequently $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(K_n \backslash \Gamma)$. Since Γ is arc-transitive, it follows that $K_n \backslash \Gamma$ is also arc-transitive. Since $\mathrm{Aut}(\Gamma)$ is the wreath product $S_v \mathrm{wr} S_u$ it follows that also $\mathrm{Aut}(K_n \backslash \Gamma) = S_v \mathrm{wr} S_u$, of order $(v!)^u u!$.

Without loss of generality the symbol for $K_n \backslash \Gamma$ can be written as

$$S = \{1, 2, \ldots, u-1, u+1, \ldots, 2u-1, 2u+1, \ldots, (v-1)u+1, \ldots, vu-1\},$$

that is $\mathbf{Z}_n \backslash \{u, 2u, \ldots, (v-1)u, vu\}$, and therefore $K_n \backslash \Gamma$ is circulant with degree $(n-1) - (v-1) = v(u-1)$. ■

Note that if u and v are both even then the degree of the circulant $K_n \backslash uK_v$ is odd.

Theorem 2 *Let* $S_n = \{1, \frac{n}{2}-1, \frac{n}{2}+1, n-1\}$ *where* $n \geq 10$ *is an even integer, then* $X(n; S_n)$ *is an arc-transitive circulant graph with degree four and automorphism group* $C_2 \mathrm{wr} D_{\frac{n}{2}}$.

Proof. Let $S_n = \{1, \frac{n}{2}-1, \frac{n}{2}+1, n-1\}$ and Γ be the graph $X(n; S_n)$ with vertex set labelled with the elements of $\mathbf{Z}_n$, with an edge joining vertices i and j if and only if $i - j \in S_n$. Then clearly Γ is a circulant graph with symbol S_n and degree four.

It is a routine exercise to show that every element of the wreath product $C_2 \mathrm{wr} D_{\frac{n}{2}}$ (acting naturally on the set $\{1, 2, \ldots, n\}$ with blades of the form $\{i, i+\frac{n}{2}\}$ for $1 \leq i \leq \frac{n}{2}$) induces an automorphism of Γ. Since $C_2 \mathrm{wr} D_{\frac{n}{2}}$ is vertex transitive on Γ and the stabilizer of any vertex acts transitively on its four neighbours it follows that Γ is arc-transitive. Also by considering the stabilizer of an arc it is not difficult to show that $C_2 \mathrm{wr} D_{\frac{n}{2}}$ is the full automorphism group of Γ. ■

The symbol set S_n in Theorem 2 for those circulants with $n \equiv 0 \bmod 4$ is an even ordered-subgroup of $\mathbf{Z}_n^*$ which is isomorphic to V_4. For $n = 8$ $C_2 \mathrm{wr} D_4$ is a proper subgroup of $\mathrm{Aut}(\Gamma)$; additional symmetries exist because of the small size of this graph. For $n \equiv 2 \bmod 4$ the symbol set S_n is not a subgroup of $\mathbf{Z}_n^*$. When $n = 6$, $C_2 \mathrm{wr} D_3$ is isomorphic to $S_2 \mathrm{wr} S_3$ which was described in Theorem 1. This subclass of circulants for $n \equiv 2 \bmod 4$ has not been previously documented in the literature.

Using MAGMA [3] an exhaustive search was performed to determine mutually non-isomorphic circulants on up to $n = 30$ vertices. The degrees and cycle structure of these graphs are listed in the following table. Since K_n (of degree $n-1$) and C_n (of degree 2) are known to occur occur for each value of n, these graphs are omitted from the table. In each column the degree of the circulant is the number given before the parentheses. Those in the column headed '$K_n \backslash uK_v$' have the values for u and v stated in brackets as an ordered pair. In addition bipartite circulants in this column are marked with an asterisk. Circulants in the column headed 'Others(B)' are also bipartite. For circulants in this column and the remaining ones in the column headed 'Others' the lengths of the cycles whose union forms the graph are given.

The numeric superscripts denote the number of each type of cycle which occurs in the circulant. For example the circulant $8(4^5, 10^2, 2 \times 20)$, listed for $n = 20$, has degree eight, and is composed of five cycles of length four (contributing a total of two incident edges to each vertex), two cycles of length ten (contributing another two incident edges to each vertex), and two cycles of length twenty (each contributing two incident edges to a vertex).

The alphabetic superscripts, which appear on some entries, are explained in the remarks following the table.

n	$K_n \backslash uK_v$	Others(B)	Others
6	$3^*(2,3), 4(3,2)$		
7			
8	$4^*(2,4), 6(4,2)$		
9	$6(3,3)$		
10	$5^*(2,5), 8(5,2)$	$4^*(2 \times 10)^a$	$4(5^2, 10)^c$
11			
12	$6^*(2,6)10(6,2)$	$4^*(2 \times 12)^b$	$6(6^2, 2 \times 12)$
	$8(3,4), 9(4,3)$		
13			$4(2 \times 13)$
			$6(3 \times 13)$
14	$7^*(2,7), 12(7,2)$	$6^*(3 \times 14)^a$	$4(7^2, 14)^c$
15	$10(3,5), 12(5,3)$		$4(2 \times 15)$
			$6(5^3, 2 \times 15)$
			$8(4 \times 15)$
16	$8^*(2,8), 14(8,2), 12(4,4)$	$4^*(2 \times 16)^b$	
17			$4(2 \times 17)$
			$8(4 \times 17)$

n	$K_n \backslash uK_v$	Others(B)	Others
18	$9^*(2,9), 16(9,2)$	$6^*(3 \times 18)$	$4(9^2, 18)^c$
	$12(3,6), 15(6,3)$	$8^*(4 \times 18)^a$	
19			$6(3 \times 19)$
20	$10^*(2,10), 18(10,2)$	$4^*(2 \times 20)^b$	$6(10^2, 2 \times 20)$
	$15(4,5), 16(5,4)$	$8^*(4 \times 20)$	$8(5^4, 10^2, 2 \times 20)$
			$12(2 \times 10^2, 4 \times 20)$
21	$14(3,7), 18(7,3)$		$4(2 \times 21)$
			$6(3 \times 21)$
			$6(7^3, 2 \times 21)$
			$12(6 \times 21)$
22	$11^*(2,11), 20(11,2)$	$10^*(5 \times 22)^a$	$4(11^2, 22)^c$
23			
24	$12^*(2,12), 22(12,2)$	$4^*(2 \times 24)^b$	$12(2 \times 12^2, 4 \times 24)$
	$16(3,8), 21(8,3)$	$4^*(2 \times 24)$	$14(6^4, 2 \times 12^2, 4 \times 24)$
	$18(4,6), 20(6,4)$	$4^*(2 \times 24)$	
		$6^*(8^3, 2 \times 24)$	
		$8^*(4 \times 24)$	
25	$20(5,5)$		$4(2 \times 25)$
			$10(5 \times 25)$
26	$13^*(2,13), 24(13,2)$	$4^*(2 \times 26)$	$4(13^2, 26)^c$
		$6^*(3 \times 26)$	$8(2 \times 13^2, 2 \times 26)$
		$12^*(6 \times 26)^a$	$12(3 \times 13^2, 3 \times 26)$
27	$18(3,9), 24(9,3)$		$6(3 \times 27)$
28	$14^*(2,14), 26(14,2)$	$4^*(2 \times 28)^b$	$6(14^2, 2 \times 28)$
	$21(4,7), 24(7,4)$	$6^*(3 \times 28)$	$8(7^4, 14^2, 2 \times 28)$
		$12^*(6 \times 28)$	$18(3 \times 14^2, 6 \times 28)$

n	$K_n\backslash uK_v$	Others(B)	Others
29			$4(2\times 29)$
			$14(7\times 29)$
30	$15^*(2,15), 28(15,2)$	$4^*(2\times 30)$	$4(15^2,30)^c$
	$20(3,10), 27(10,3)$	$6^*(3\times 30)$	$8(2\times 15^2, 2\times 30)$
	$24(5,6), 25(6,5)$	$8^*(4\times 30)$	$10(10^3, 2\times 15^2, 2\times 30)$
		$10^*(6^5, 4\times 30)$	$12(5^6, 10^3, 2\times 15^2, 2\times 30)$
		$12^*(2\times 10^3, 4\times 30)$	$16(4\times 15^2, 4\times 30)$
		$14^*(6^5, 2\times 10^3, 4\times 30)^a$	$18(3\times 15^2, 5\times 30)$
			$20(3^{10}, 6^5, 4\times 15^2, 4\times 30)$

The following remarks are based on examination of the symbols (not given in the table) for the above listed circulants. Note that the circulant denoted by $4^*(2\times 24)$ is listed twice for $n=24$. Both circulants have automorphism groups of order 96, however one occurrence has the symbol set $\{1,5,19,23\}$ and the other the symbol set $\{1,7,17,23\}$. Their respective automorphism groups have been shown to be non-isomorphic, with the help of Magma.

• The cases $n=3,4$ and 5 are omitted since the only circulants for these values of n are K_n and C_n.

• In the column headed '$K_n\backslash uK_v$' the circulants are those described in Theorem 1. The bipartite graphs of degree $\frac{n}{2}$, n even, are the 2-arc-transitive circulants $K_{\frac{n}{2},\frac{n}{2}}$ given in [1].

• In the column headed 'Others(B)', the circulants (marked with superscript a) of degree $\frac{n}{2}-1$ for $n\geq 10$ and $n\equiv 2 \bmod 4$ are the 2-arc-transitive circulants $K_{\frac{n}{2},\frac{n}{2}}$ minus a 1-factor given in [1].

• In the column headed 'Others(B)' the circulants (marked with superscript b), with $n\geq 12$ and $n\equiv 0 \bmod 4$, are those described in Theorem 2. For $n=8$ the corresponding circulant is $4^*(2,4)$.

• In the column headed 'Others' the circulants (marked with superscript c), with $n \geq 10$ and $n \equiv 2 \bmod 4$, are those described in Theorem 2. For $n = 6$ the corresponding circulant is $4(3, 2)$.
• In the column headed 'Others', the circulants of prime order ($n = p$) agree with those given in [2,4], in that their symbol sets are precisely the subgroups of even order contained in $\mathbf{Z}_n^*$.

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